Chebyshev Interpolation and Quadrature Formulas of Very High Degree

HERBERT E. SALZER*

All the zeros $x_{2^m,i}$, $i = 1(1)2^m$, of the Chebyshev polynomials $T_{2^m}(x)$, $m = 0(1)n$, are found recursively just by taking $n2^{n-1}$ real square roots. For interpolation or integration of $f(x)$, given $f(x_{2^m,i})$, only $x_{2^m,i}$ is needed to calculate (a) the $(2^m - 1)$-th degree Lagrange interpolation polynomial, and (b) the definite integral over $[-1, 1]$, either with or without the weight function $(1 - x^2)^{-\frac{1}{2}}$, the former being exact for $f(x)$ of degree $2^{m+1} - 1$.

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From $T_m(x) \equiv \cos(m \arccos x)$ and the well-known $T_{mn}(x) = T_m(T_n(x))$, we have

$$T_{2^n}(x) = T_{2^{n-1}}(T_2(x)). \tag{1}$$

The zeros of $T_m(x)$ in increasing order are

$$x_{m,i} = -\cos \frac{2i-1}{2m}\pi, \qquad i = 1, \cdots, m.$$

If y is a root of

$$T_2(y) \equiv 2y^2 - 1 = x_{2^{n-1},i}, \tag{2}$$

from (1) and (2), $T_{2^n}(y) = T_{2^{n-1}}(x_{2^{n-1},i}) = 0$; so y is a root $x_{2^n,i}$. The two roots of (2) for each $x_{2^{n-1},i}$, $i = 1, \cdots, 2^{n-1}$, give all the roots $x_{2^n,i}$, $i = 1, \cdots, 2^n$. Solving (2) in succession for $n = 1, 2, \cdots, n$, where $x_{2^0,i} = 0$, we get

$$x_{2^1,i} = \pm\left(\frac{1}{2}\right)^{\frac{1}{2}}, x_{2^2,i} = \pm\left(\frac{1 \pm \left(\frac{1}{2}\right)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}, \cdots, x_{2^n,i} = \pm\left(\frac{1 + x_{2^{n-1},i}}{2}\right)^{\frac{1}{2}}.$$

The $(2^n - 1)$-th degree Lagrange interpolation polynomial $L_{2^n-1}(x)$ which equals $f(x)$ at the 2^n points $x_{2^n,i}$, $i = 1, \cdots, 2^n$, is expressible as

$$L_{2^n-1}(x) = \sum_{i=1}^{2^n} \frac{A_{2^n,i}\, f(x_{2^n,i})}{x - x_{2^n,i}} \Big/ \sum_{i=1}^{2^n} \frac{A_{2^n,i}}{x - x_{2^n,i}} \tag{3}$$

where

$$A_{2^n,i} = (-1)^i(1 - x_{2^n,i}^2)^{\frac{1}{2}}. \tag{4}$$

Also from (2),

$$A_{2^n,i} = (-1)^i\left(\frac{1 - x_{2^{n-1},i}}{2}\right)^{\frac{1}{2}}. \tag{5}$$

But neither (4) nor (5) is needed in the computation of $L_{2^n-1}(x)$, because, from elementary trigonometry,

$$A_{2^n,i} = (-1)^{i-1}x_{2^n,2^{n-1}-i+1} = (-1)^i\,|x_{2^n,i+2^{n-1}}|, \quad i = 1, \cdots, 2^{n-1}, \tag{6a}$$

$$A_{2^n,i} = (-1)^{i-1}x_{2^n,i-2^{n-1}} = (-1)^i\,|x_{2^n,i-2^{n-1}}|, \quad i = 2^{n-1}+1, \cdots, 2^n. \tag{6b}$$

For x in the interval $[-1, 1]$,

$$f(x) - L_{2^n-1}(x) = T_{2^n}(x)f^{(2^n)}(\xi)/2^{2^n-1}(2^n)!, \quad -1 \leqq \xi \leqq 1. \tag{7}$$

For quadrature using $x_{2^n,i}$,

$$\int_{-1}^{1} \frac{f(x)}{(1-x^2)^{\frac{1}{2}}}\,dx = \frac{\pi}{2^n}\sum_{i=1}^{2^n} f(x_{2^n,i}) + R_{2^n}, \tag{8a}$$

where

$$R_{2^n} = \pi f^{(2^{n+1})}(\xi)/2^{2^{n+1}-1}(2^{n+1})!, \quad -1 < \xi < 1. \tag{8b}$$

When (8) is applied to

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \frac{(1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}} f(x)\,dx,$$

from (4) and (6a) we obtain

$$\int_{-1}^{1} f(x)\,dx = \frac{\pi}{2^n}\sum_{i=1}^{2^{n-1}} |x_{2^n,i+2^{n-1}}| \cdot [f(x_{2^n,i}) + f(-x_{2^n,i})] + \bar{R}_{2^n}, \tag{9}$$

where $\bar{R}_{2^n}$ is found by replacing $f(x)$ by $(1 - x^2)^{\frac{1}{2}}f(x)$ in (8b).

Only n square roots for each i are required for $x_{2^n,i}$, for a total of $n2^{n-1}$. For $n = 10$, interpolation by (3) gives 1023rd degree accuracy, while quadrature by (8) is exact up to the 2047th degree in $f(x)$; $n = 20$ gives 1,048,575th degree accuracy in (3) and 2,097,151th degree accuracy in (8).

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* Author's address: 941 Washington Ave., Brooklyn, NY 11225