Computer Formulation of the Equations of Motion Using Tensor Notation

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A means is described for extending the area of application of digital computers beyond the numerical data processing stage and reducing the need for human participation in the formulation of certain types of computer problems. By the use of tensor calculus and a computer language designed to facilitate symbolic mathematical computation, a method has been devised whereby a digital computer can be used to do non-numeric work, that is, symbolic algebraic manipulation and differentiation.

To illustrate the techniques involved, a digital computer has been used to derive the equations of motion of a point mass in a general orthogonal curvilinear coordinate system. Since this operation involves a formulation in terms of first- and second-order differential coefficients, it provides a good demonstration of a computer's capability to do non-numeric work and to assist in the formulation process which normally precedes the numerical data processing stage. Moreover, this particular problem serves to illustrate the advantages of the mathematical techniques employed. With the program prepared for this purpose the computer will derive the equations of motion in any coordinate system requested by the user. Results are presented for the following coordinate systems: cylindrical polar, spherical polar, and prolate spheroidal.

Introduction

The extensive logic and storage capabilities of digital computers, combined with the evolution of new computer languages, enable them to be used for a wide range of non-numeric operations. The author is aware of only two previous attempts to use computers in this manner: (a) In [1] an interesting technique is described whereby a digital computer was used to derive equations of motion. The technique, as described, was not completely satisfactory in that part of the operation had to be performed manually. (b) In [2] and [3] an IBM 7094 computer equipped with a Formac compiler was used to obtain the Christoffel symbols of the first and second kind for 12 orthogonal curvilinear coordinate systems.

If the extensive logic and storage capabilities of these computers are to be used to full advantage, a departure from conventional techniques of formulation may be necessary. For example, when conventional methods are used, the form which the equations of motion and of mathematical physics assumes depends on the coordinate system used to describe the problem. This dependence, which is due to the practice of expressing vectors in terms of their physical components, can be removed by the simple expedient of expressing all vectors in terms of their tensor components.

As a consequence of the geometrical simplification inherent in the tensor method, the operations involved in formulating problems in unfamiliar curvilinear coordinate systems can be reduced to routine computer operations. It is this aspect of the tensor method which is so attractive for the types of computer applications contemplated. It is the purpose of this report to show that digital computers can be used to do non-numeric work. With this object in mind, a computer program was written to demonstrate the effectiveness of the proposed technique. This program, in the Formac computer language, was used to derive the equations of motion of a point mass in a variety of curvilinear coordinate systems. To derive the equations of motion of a particle by this method, the user need only know the coordinate transformation equations relating the curvilinear coordinates to an orthogonal Cartesian triad. When this program is used and the coordinate transformation equations are supplied as input, the computer will derive the equations of motion. The equations of motion obtained will be relative to the curvilinear coordinate system specified by the coordinate transformation equations used as input. The computer presents the results in Fortran language. However, for the convenience of readers, the Fortran statements are translated to conventional mathematical symbolism.

Nomenclature

$\mathbf{A}$	vector
$A^i(x)$	contravariant vector components in the x-coordinate system
$A_j(x)$	covariant vector components in the x-coordinate system
$A,^i_j$	covariant derivative of a contravariant vector
$A_{i,j}$	covariant derivative of a covariant vector
$\mathbf{a}_i(x)$	system of base vectors in the x-coordinate system
$\hat{a}_i(x)$	system of unit base vectors in the direction of $\mathbf{a}_i(x)$
$\mathbf{a}^i(x)$	system of base vectors reciprocal to $\mathbf{a}_i(x)$
$\hat{a}^i(x)$	system of unit base vectors in the direction of $\mathbf{a}^i(x)$
$B^i(y)$	contravariant vector components in the y-coordinate system
$B_j(y)$	covariant vector components in the y-coordinate system
$\mathbf{b}_j(y)$	system of base vectors in the y-coordinate system
$\mathbf{b}^j(y)$	system of base vectors reciprocal to $\mathbf{b}_j(y)$
g_{ij}	$\mathbf{a}_i \cdot \mathbf{a}_j$
g^{ij}	$\mathbf{a}^i \cdot \mathbf{a}^j$
M	mass of particle
$P(j)$	$\frac{d^2x^i}{dt^2}$
$R(i)$	$\frac{dx^i}{dt}$
$\mathbf{T}$	thrust vector
T_i	covariant component of the thrust vector
T^i	contravariant component of the thrust vector
$\mathbf{V}$	velocity vector
x^i	system coordinates
$x^i(y^1y^2y^3)$	functional form of the transformation from the y-coordinate system to the x-coordinate system
y^i	system coordinates
$y^i(x^1x^2x^3)$	functional form of the transformation from the x-coordinate system to the y-coordinate system

$[ij, k]$	Christoffel symbol of the first kind
$\begin{Bmatrix} i \\ jk \end{Bmatrix}$	Christoffel symbol of the second kind
$\delta_j{}^i$	Kronecker delta
τ^i	physical component of the thrust vector
φ	potential function
$\nabla\varphi$	gradient of potential function

Subscripts

i, j, k, l	indices of covariance

Superscripts

α, i, j, k, l	indices of contravariance

Analysis

Components. In order to facilitate the processing of vectors, bi-vectors and general n-vectors, it is convenient from the point of view of non-numeric operations to express all such entities in terms of their tensor components, rather than in terms of their physical components. When referred to a general curvilinear coordinate system, a vector $\mathbf{A}$ may be expressed in the alternative forms (see [4]):

$$\mathbf{A} = A^i\mathbf{a}_i = A_j\mathbf{a}^j. \tag{1}$$

If in some expression a certain index occurs twice, this means that the expression is to be summed with respect to that index for all admissible values of the index.

Components denoted by superscripts, as in A^i, are termed contravariant components, whereas components denoted by subscripts, as in A_i, are termed covariant components. The base vectors $\mathbf{a}^i$ and $\mathbf{a}_j$ are related as follows

$$\mathbf{a}^i \cdot \mathbf{a}_j = \delta_j{}^i$$

where $\delta_j{}^i$ is the Kronecker delta (see [5]).

$$\delta_j{}^i = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

Because of this relationship, the base vector $\mathbf{a}^i$ is termed the reciprocal of the base vector $\mathbf{a}_i$. The variance of vector components is determined by the transformation law which the components obey. For a coordinate transformation from a coordinate system x to a coordinate system y given by

$$y^i = y^i(x^1x^2x^3), \tag{2}$$

the transformation law for the components of a contravariant vector A^i is given by (see [4])

$$B^j(y) = \frac{\partial y^j}{\partial x^i} A^i(x) \tag{2a}$$

where $A^i(x)$ are the contravariant components in the x-coordinate system and $B^j(y)$ are the components when referred to the y-coordinate system. For the same transformation of coordinates, which is assumed to be reversible and one-to-one, the transformation law for the covariant components A_i is

$$B_j(y) = \frac{\partial x^i}{\partial y^j} A_i(x). \tag{3}$$

In this case $A_i(x)$ are the covariant components in the x-coordinate system and $B_j(y)$ are the covariant components when referred to the y-coordinate system. The distinction between these two transformation laws vanishes when the transformation is orthogonal Cartesian. This explains why there is no preoccupation with these vectors in the study of ordinary vector analysis.

Base Vectors. The base vectors associated with the tensor components have a variance consistent with the index assigned to them. As in the case of tensor components, a base vector which is characterized by a subscript is covariant, whereas a base vector which is characterized by a superscript is contravariant. That is, if $\mathbf{a}(x)$ and $\mathbf{b}(y)$ are the base vectors in the x- and y-coordinate systems, respectively, and the coordinate transformation is assumed given by eq. (2), then

$$\mathbf{b}_j(y) = \frac{\partial x^i}{\partial y^j} \mathbf{a}_i(x) \tag{4}$$

and

$$\mathbf{b}^j(y) = \frac{\partial y^j}{\partial x^i} \mathbf{a}^i(x). \tag{5}$$

Vector Derivatives and the Christoffel Symbols. For a digital computer programmed for non-numeric operations, it is desirable to have formulas for the derivative of a vector which are sufficiently general to apply in all coordinate systems, and yet are amenable to routine computer determinations. Research indicates that the intrinsic derivative of a vector meets these requirements. The intrinsic derivative differs from the total derivatives of the differential calculus by a series of terms involving the Christoffel symbols of the second kind. These symbols may be obtained in terms of the scalar products of the base vectors by the following procedure. The scalar product of any two base vectors $\mathbf{a}_i$ and $\mathbf{a}_j$ may be defined as follows:

$$\mathbf{a}_i \cdot \mathbf{a}_j = g_{ij} = \mathbf{a}_j \cdot \mathbf{a}_i. \tag{6}$$

Likewise, the scalar product of the reciprocal base vectors $\mathbf{a}^i$ and $\mathbf{a}^j$ may be defined as

$$\mathbf{a}^i \cdot \mathbf{a}^j = g^{ij} = \mathbf{a}^j \cdot \mathbf{a}^i. \tag{7}$$

The symmetry of g_{ij} and g^{ij} follows from the nature of the scalar product. Certain combinations of the partial derivatives of these scalar products with respect to the system coordinates are useful in obtaining the derivative of a vector, in formulating the equations of motion, or in writing the equations of mathematical physics in a general curvilinear coordinate system. The definitions that follow are ascribed to Christoffel and are called Christoffel symbols [6]. There are two of these symbols, the first of which is defined as

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right). \tag{8}$$

The Christoffel symbol of the second kind is defined as follows:

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kl}[ij, l]. \tag{9}$$

These symbols may also be obtained from the coordinate transformation equations. It transpires that this method of derivation is more suitable for the computer applications contemplated in this report. Assuming that the curvilinear coordinates are related to a system of Cartesian coordinates by the transformation of eq. (2), it is easily shown that

$$[ij, k] = \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} \frac{\partial y^\alpha}{\partial x^k} \tag{10}$$

and

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = \frac{\partial^2 y^\alpha}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial y^\alpha}. \tag{11}$$

Likewise, the metric tensor g_{ij} is given by

$$g_{ij} = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\alpha}{\partial x^j}\,. \tag{12}$$

Covariant and Intrinsic Derivatives. Christoffel symbols play an important role in the operations of covariant and intrinsic differentiation. When a vector $\mathbf{A}$ is expressed in terms of its contravariant components and associated base vectors, covariant differentiation with respect to any coordinate x^k gives the following [6]:

$$\frac{d\mathbf{A}}{\partial x^k} = \left(\frac{\partial A^i}{dx^k} + \begin{Bmatrix} i \\ jk \end{Bmatrix} A^j\right) \mathbf{a}_i = A,^i_k \mathbf{a}_i. \tag{13}$$

Likewise, intrinsic differentiation with respect to some parameter t yields

$$\frac{d\mathbf{A}}{dt} = \left(\frac{dA^i}{dt} + \begin{Bmatrix} i \\ jk \end{Bmatrix} A^j \frac{dx^k}{dt}\right) \mathbf{a}_i\, A,^i_k \frac{dx^k}{dt}\, \mathbf{a}_i \tag{14}$$

where $A,^i_k$ is the covariant derivative of the contravariant form of the vector $\mathbf{A}$, and $A,^i_k(dx^k/dt)$ is the corresponding intrinsic derivative.

When the vector $\mathbf{A}$ is given in terms of its covariant components and reciprocal base vectors, covariant and intrinsic differentiation yields the following:

$$\frac{\partial \mathbf{A}}{\partial x^k} = \left(\frac{\partial A^i}{dx^k} - \begin{Bmatrix} j \\ ik \end{Bmatrix} A_j\right) \mathbf{a}^i = A_{i,k}\, \mathbf{a}^i \tag{15}$$

$$\frac{d\mathbf{A}}{dt} = \left(\frac{dA_i}{dt} - \begin{Bmatrix} j \\ ik \end{Bmatrix} A_j \frac{dx^k}{dt}\right) \mathbf{a}^i = A_{i,k} \frac{dx^k}{dt}\, \mathbf{a}^i \tag{16}$$

where $A_{i,k}$ is the covariant derivative of the covariant form of the vector $\mathbf{A}$ and $A_{i,k}(dx^k/dt)$ is the corresponding intrinsic derivative.

In a general space of three dimensions, each of eqs. (13)–(16) contains 27 Christoffel symbols. However, because of the symmetry of these symbols

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = \begin{Bmatrix} k \\ ji \end{Bmatrix} \tag{17}$$

and the number of independent Christoffel symbols reduces to 18.

The large number of terms appearing in the expanded form of eqs. (13)–(16) is due to the generality of these equations, which are applicable to any space of three dimensions. Fortunately, for the three-dimensional spaces most commonly used, these equations reduce to a more manageable size.

Computer Applications

As a consequence of the geometrical simplification inherent in the tensor method, the operations involved in obtaining derivatives and formulating equations of motion in unfamiliar curvilinear coordinate systems can be reduced to routine operations. It is this aspect of the tensor method which makes it so attractive for computer applications. Because of their logic and storage capabilities, digital computers are well suited to such routine operations. If the functional form given by eq. (2) is known, the Christoffel symbols may be obtained from eqs. (10) and (11). Moreover, given the Christoffel symbols, there are two formulas for finding the intrinsic derivative of a vector. Equation (14) gives the intrinsic derivative in terms of the contravariant components, whereas eq. (16) gives the same result in terms of the covariant components. Either of these equations may be used. However, in order to avoid the necessity of transforming covariant components into contravariant components and vice versa, it is better to match the formula to the variance of the vectors being used. It will be evident what the variance of the vectors is during the course of the analysis.

Acceleration Vector. If, in order to formulate the equations of motion of a particle, the acceleration vector were required, the velocity vector $\mathbf{V}$ would be substituted for the vector $\mathbf{A}$ in the equation for the intrinsic derivative. Since the velocity vector is contravariant, the tensor components of velocity must be substituted for the components of $\mathbf{A}$ in eq. (14). Hence, in a general curvilinear coordinate system the acceleration vector assumes the following form:

$$\frac{d\mathbf{V}}{dt} = \left(\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{dt} \frac{dx^k}{dt}\right) \mathbf{a}_i. \tag{18}$$

This equation gives the acceleration in any coordinate system provided the Christoffel symbols are appropriate to the coordinate system chosen to describe the motion of the particle.

Equations of Motion: Formulation in Terms of Contravariant Components. In using tensor methods to derive equations of motion, it is important to remember that the acceleration and force vectors must always be expressed in terms of their tensor components, rather than their physical components. Hence, the two sides of every equation must balance with respect to their covariant or contravariant properties before applying Newton's second law of motion. In this connection it is worth noting that, although the acceleration vector is expressed in contravariant form in eq. (18), the force vector may appear in the form of a covariant vector. The force vector assumes the covariant form in situations where it appears as the gradient of a scalar point function. This occurs in deriving the equations of motion of a space vehicle which, in addition to the thrust force, is subject to gravitational forces. If, as is usual, the gravitational forces are expressed in the form of the gradient of a gravitational potential function, the force vector is given by

$$\mathbf{F} = \nabla\varphi + \mathbf{T} \tag{19}$$

where φ is the gravitational potential function, which may include the influence of oblateness and extraterrestrial gravitational forces, and $\mathbf{T}$ is the thrust vector (see [7, 8]). The tensor form of the gradient of a scalar point function assumes the form (see [9]):

$$\nabla\varphi = \frac{\partial\varphi}{\partial x^i}\,\mathbf{a}^i. \tag{20}$$

The components of the gradient function in the y-coordinate system are related to those in the x-coordinate system by the equation:

$$\frac{\partial\varphi}{\partial y^i} = \frac{\partial\varphi}{\partial x^j}\frac{\partial x^j}{\partial y^i}. \tag{21}$$

This is seen to be the covariant transformation as defined in eq. (3).

In the general case, the equation of motion of a space vehicle, which is subject to gravitational and thrust forces, is obtained by combining eqs. (18) and (19). Newton's second law of motion requires that

$$M\left(\frac{d^2x^i}{dt^2} + \left\{ {i \atop jk} \right\} \frac{dx^j}{dt}\frac{dx^k}{dt}\right)\mathbf{a}_i = \nabla\varphi + \mathbf{T} \tag{22}$$

where M is the mass of the vehicle.

The acceleration components represented by the left side of this equation are all contravariant. The thrust vector, on the other hand, is usually given in terms of its physical components; and as already indicated in eq. (21), the gravitational forces assume the form of covariant vectors. In order to have a force system which is compatible with the acceleration, it is necessary to convert all the force terms to the contravariant form. The covariant and contravariant components are related as follows:

$$A^i = g^{ij}A_j. \tag{23}$$

Likewise, the physical components of the thrust vector are related to the contravariant components by the equation:

$$T^i = \frac{1}{(g_{(ii)})^{\frac{1}{2}}}\,\tau^i \tag{24}$$

where the parentheses imply suspension of the summation convention. By substitution from eqs. (23) and (24) in eq. (22), the equation of motion assumes the form

$$M\left(\frac{d^2x^i}{dt^2} + \left\{ {i \atop jk} \right\} \frac{dx^j}{dt}\frac{dx^k}{dt}\right) = \left(g^{ij}\frac{\partial\varphi}{\partial x^j} + \frac{\tau^i}{(g_{(ii)})^{\frac{1}{2}}}\right). \tag{25}$$

In orthogonal coordinate systems,

$$g^{ij} = 0 \qquad \text{for} \quad i \neq j$$

and

$$g^{(ii)} = \frac{1}{g_{(ii)}}.$$

Substitution of these values in eq. (25) gives for orthogonal systems

$$M\left(g_{(ii)}\frac{d^2x^i}{dt^2} + [jk, i]\frac{dx^j}{dt}\frac{dx^k}{dt}\right) = \frac{\partial\varphi}{\partial x^i} + (g_{(ii)})^{\frac{1}{2}}\,\tau^i. \tag{26}$$

From the point of view of non-numeric computer operations, it is expedient to eliminate the Christoffel symbols and the metric tensors from eq. (26). These are related to the coordinate transformation equations by eqs. (10) and (12). Substitution from eqs. (10) and (12) in eq. (26) gives

$$\begin{aligned} M\left[\left(\frac{\partial y^\alpha}{\partial x^{(i)}}\frac{\partial y^\alpha}{\partial x^{(i)}}\right)\frac{d^2x^i}{dt^2} + \left(\frac{\partial^2 y^\alpha}{\partial x^j\partial x^k}\frac{dy^\alpha}{\partial x^i}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}\right] \\ = \left[\frac{\partial\varphi}{\partial x^i} + \left(\frac{\partial y^\alpha}{\partial x^{(i)}}\frac{\partial y^\alpha}{\partial x^{(i)}}\right)^{\frac{1}{2}}\tau^i\right]. \end{aligned} \tag{27}$$

This equation is in a form well suited to routine non-numeric computer operations. The large number of terms appearing in eq. (27) is due to the generality of this equation, which is applicable to any space of three dimensions. Moreover, since this equation is applicable to any space of three dimensions, it may be permanently stored in the computer. Hence, in order to obtain the equations of motion in any system of coordinates, the only information required is the special form of eq.(2) relating that system of coordinates to the orthogonal Cartesian coordinates y^i. For example, consider a transformation of coordinates specifying the relation between the spherical polar coordinates x^i and the orthogonal Cartesian coordinates y^i (Figure 1). In this case, eq. (2) becomes:

$$\begin{aligned} y^1 &= x^1 \sin x^2 \cos x^3 \\ y^2 &= x^1 \sin x^2 \sin x^3 \\ y^3 &= x^1 \cos x^2. \end{aligned}$$

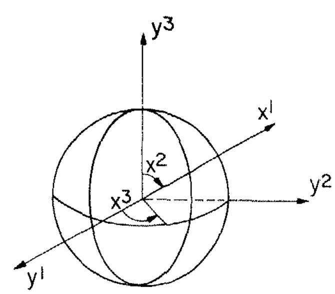

FIG. 1

These coordinate transformation equations were supplied as input to an IBM 7094 computer, which was equipped with a FORMAC Compiler. When the computer was programmed to perform the operations involved in eq. (27), the output was obtained in FORTRAN language as shown in Figure 2.

In interpreting these FORTRAN statements, it must be remembered that:

$$R(i) = \frac{dx^i}{dt}, \qquad P(i) = \frac{d^2x^i}{dt^2}.$$

In terms of conventional mathematical symbolism, these equations assume the following form:

$$M\left[\frac{d^2x^1}{dt^2} - x^1\left(\frac{dx^2}{dt}\right)^2 - x^1\left(\sin x^2 \frac{dx^3}{dt}\right)^2\right] = \frac{\partial \varphi}{\partial x^1} + \tau^1$$

$$M\left[(x^1)^2\frac{d^2x^2}{dt^2} + 2x^1\frac{dx^1}{dt}\frac{dx^2}{dt} - (x^1)^2 \sin x^2 \cos x^2\left(\frac{dx^3}{dt}\right)^2\right] = \left(\frac{\partial \varphi}{\partial x^2} + x^1\tau^2\right)$$

$$M\left[(x^1 \sin x^2)^2\frac{d^2x^3}{dt^2} + 2x^1 \sin^2 x^2\frac{dx^1}{dt}\frac{dx^3}{dt} + 2(x^1)^2 \sin x^2 \cos x^2 \frac{dx^2}{dt}\frac{dx^3}{dt}\right] = \left(\frac{\partial \varphi}{\partial x^3} + x^1 \sin x^2\, \tau^3\right).$$

Because of its generality, eq. (27) is applicable in all coordinate systems. Therefore, in order to obtain the equations of motion in any other coordinate system, all that is required is to supply the computer with the appropriate coordinate transformation equations. As a further illustration of the procedure involved, consider the equations of motion in a cylindrical polar system of coordinates. In this case, the coordinate transformation equations are (see Figure 3):

$$y^1 = x^1 \cos x^2$$
$$y^2 = x^1 \sin x^2$$
$$y^3 = x^3.$$

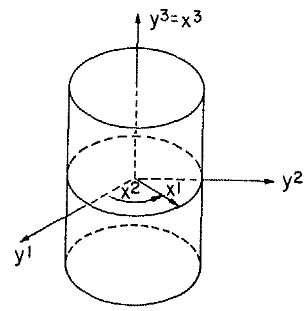

FIG. 3

When these coordinate transformation equations were used to evaluate the terms of eq. (27), the following computer output was obtained.

COMPUTER OUTPUT

```
FOR I = 1
The expression input for Y(I) is given below.
X(1)*FMCSIN(X(2))*FMCCOS(X(3))$
FOR I = 2
The expression input for Y(I) is given below.
X(1)*FMCSIN(X(2))*FMCSIN(X(3))$
FOR I = 3
The expression input for Y(I) is given below.
X(1)*FMCCOS(X(2))$
                EQUATIONS OF MOTION
FOR I = 1
The equation for I = 1 is given below.
M*(P(1)-R(2)**2.0*X(1)-R(3)**2.0*X(1)*FMCSIN(X(2))**
  2.0)$
=DPHI(1)+TAU(1)$
FOR I = 2
The equation for I = 2 is given below.
M*(P(2)*X(1)**2.0+R(1)*R(2)*X(1)*2.0-R(3)**2.0*X(1)**2.0*
  FMCSIN(X(2))*FMCCOS(X(2))$
=DPHI(2)+TAU(2)*X(1)$
FOR I = 3
The equation for I = 3 is given below.
M*(P(3)*X(1)**2.0*FMCSIN(X(2))**2.0+R(1)*R(3)*X(1)*
  FMCSIN(X(2))**2.0*2.0+R(2)*R(3)*
X(1)**2.0*FMCSIN(X(2))*FMCCOS(X(2))*2.0)$
= DPHI(3)+TAU(3)*X(1)*FMCSIN(X(2))$
```

JOB ACCOUNTING

	COMP/LOAD	EXECUTIVE
TIME	TIME	TIME
ON	MIN.	MIN.
950.09	.78	.41

FIG. 2

Translating these equations from FORTRAN language to conventional mathematical symbolism yields the following:

$$M\left[\frac{d^2x^1}{dt^2} - x^1\frac{dx^2}{dt}\frac{dx^2}{dt}\right] = \frac{\partial \varphi}{\partial x^1} + \tau^2$$

$$M\left[(x^1)\frac{d^2x^2}{dt^2} + 2x^1\frac{dx^1}{dt}\frac{dx^2}{dt}\right] = \frac{\partial \varphi}{\partial x^2} + x^1\tau^2$$

$$M\left[\frac{d^2x^3}{dt^2}\right] = \frac{\partial \varphi}{dx^3} + \tau^3.$$

PROLATE SPHEROIDAL COORDINATES. Another interesting system of orthogonal curvilinear coordinates are the prolate spheroidal coordinates. Coordinate surfaces are obtained by rotating a family of confocal ellipses and hyperbolae about their major axes. Rotation of these conic sections gives rise to a system of prolate spheroids and hyperboloids of two sheets. A family of planes through the axis of rotation completes the system of orthogonal surfaces. The curvilinear coordinate systems generated by the curves of intersection of these surfaces are useful in certain quantum-mechanical problems [10, 11]. The transformation equations relating this system of coordinates to the orthogonal Cartesian system are as follows:

$$y^1 = a \sinh x^1 \sin x^2 \cos x^3$$
$$y^2 = a \sinh x^1 \sin x^2 \sin x^3$$
$$y^3 = a \cosh x^1 \cos x^2$$

In order to obtain the equations of motion relative to a prolate spheroidal system of coordinates, these transformation equations were substituted for eq. (2) in the computer program. Execute time was 1.63 minutes. Omitting

COMPUTER OUTPUT

FOR I = 1
The expression input for Y(I) is given below.
X(1)*FMCCOS(X(2))$
FOR I = 2
The expression input for Y(I) is given below.
X(1)*FMCSIN(X(2))$
FOR I = 3
The expression input for Y(I) is given below.
X(3)$

EQUATIONS OF MOTION

The equation for I = 1 is given below.
M*(P(1)-R(2)**2.0*X(1))$
=DPHI(1)+TAU(1)$
The equation for I = 2 is given below.
M*(P(2)*X(1)**2.0+R(1)*R(2)*X(1)*2.0)$
=DPHI(2)+TAU(2)*X(1)$
The equation for I = 3 is given below.
M*P(3)$
=DPHI(3)+TAU(3)$

JOB ACCOUNTING

TIME ON	COMP/LOAD TIME MIN.	EXECUTIVE TIME MIN.
037.78	1.15	.14

FIG. 4

the printout in FORTRAN language, the equations of motion were obtained as follows:

$$M\left[a^2(\sin^2 x^2+\sinh^2 x^1)\frac{d^2x^1}{dt^2}+2a^2\sin x^2\cos x^2\frac{dx^1}{dt}\frac{dx^2}{dt}\right.$$
$$+a^2\sinh x^1\cosh x^1\frac{dx^1}{dt}\frac{dx^1}{dt}-a^2\sinh x^1\cosh x^1\frac{dx^2}{dt}\frac{dx^2}{dt}$$
$$\left.-a^2\sin^2 x^2\sinh x^1\cosh x^1\frac{dx^3}{dt}\frac{dx^3}{dt}\right]$$
$$=a(\sin^2 x^2+\sinh^2 x^1)^{\frac{1}{2}}\tau^1+\frac{\partial\varphi}{\partial x^1}$$

$$M\left[a^2(\sin^2 x^2+\sinh^2 x^1)\frac{d^2x^2}{dt^2}-a^2\sin x^2\cos x^2\frac{dx^1}{dt}\frac{dx^1}{dt}\right.$$
$$+2a^2\sinh x^1\cosh x^1\frac{dx^1}{dt}\frac{dx^2}{dt}+a^2\sin x^2\cos x^2\frac{dx^2}{dt}\frac{dx^2}{dt}$$
$$\left.-a^2\sin x^2\cos x^2\sinh^2 x^1\frac{dx^3}{dt}\frac{dx^3}{dt}\right]$$
$$=a(\sin^2 x^2+\sinh^2 x^1)^{\frac{1}{2}}\tau^2+\frac{\partial\varphi}{\partial x^2}$$

$$M\left[a^2\sin^2 x^2\sinh^2 x^1\frac{d^2x^3}{dt^2}+2a^2\sin^2 x^2\sinh x^1\cosh x^1\frac{dx^1}{dt}\frac{dx^3}{dt}\right.$$
$$\left.+2a^2\sin x^2\cos x^2\sinh^2 x^1\frac{dx^2}{dt}\frac{dx^3}{dt}\right]$$
$$=a\sin x^2\sinh x^1\tau^3+\frac{\partial\varphi}{\partial x^3}.$$

Conclusions

Digital computers can be used to perform a wide range of non-numeric operations, if use is made of new computer languages now available. For the class of problems considered, the results indicate that computers can be used more effectively for this purpose if all vector quantities are expressed in terms of their tensor components rather than in terms of their physical components. Because of the geometrical simplification inherent in the tensor method, and the invariant nature of the formulation with respect to coordinate transformations, the formulation of problems in curvilinear coordinate systems can be reduced to routine computer operations. The evolution of new computer languages, which can perform symbolic algebraic manipulation and differentiation, made it possible to write a program which could implement these ideas. This program was used successfully to derive equations of motion and perform other non-numeric operations. The exploitation and extension of these techniques should lead to a substantial reduction in the man hours required to formulate and process engineering and scientific problems.

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