$0 \Rightarrow x_{i1} = x_{i2}$. Therefore, from the first part of (4), $x_{i1} = x_{i2} = y_i/2$. The factor $\frac{1}{2}$ need not be used because eigenvectors are obtained only up to a constant multiplier.

(ii) $\mu_i \in \lambda(Q) \Rightarrow \mu_i = \eta_j$, $(j \in I2) \Rightarrow r_i = cz_j = x_{i1} - x_{i2}$, where c is an arbitrary constant. Therefore, $cz_j = x_{i1} - x_{i2}$ and $y_i = x_{i1} + x_{i2}$ give $x_{i1} = (y_i + cz_j)/2$ and $x_{i2} = (y_i - cz_j)/2$. Again the constant factor need not be used.

A proof for the case $i \in I2$ can be similarly constructed.

3. A Numerical Example

As an illustration of the use of Algorithm 1 we consider a 4×4 matrix

$$S = \begin{bmatrix} 0.25 & 3.25 & -1.25 & -1.25 \\ -1.25 & 0.75 & -1.75 & 3.25 \\ -1.25 & -1.25 & 0.25 & 3.25 \\ -1.75 & 3.25 & -1.25 & 0.75 \end{bmatrix}. \qquad (9)$$

From (9) we have

$$P = A + B = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix},$$
$$Q = A - B = \begin{bmatrix} 1.5 & 4.5 \\ 0.5 & -2.5 \end{bmatrix}. \qquad (10)$$

Solving for the eigenvalues and eigenvectors of P and Q gives $\lambda(P) = (\lambda_1, \lambda_2) = (\mu_1, \mu_2) = (1, 2)$; $\lambda(Q) = (\lambda_3, \lambda_4) = (\eta_3, \eta_4) = (2, -3)$, $y_1' = (1, 1)$, $y_2' = (\frac{2}{3}, 1)$, $z_3' = (9, 1)$, and $z_4' = (-1, 1)$.

By use of Algorithm 1 the eigenvector of S belonging to $\lambda_1 = 1$ can be formed by noting that $\lambda_1 \notin \lambda(Q)$. Therefore, $x_1' = (y_1', y_1') = (1, 1, 1, 1)$. Similarly, to form x_2 belonging to $\lambda_2 = 2$ we note that $\lambda_2 \in \lambda(Q)$. In particular $\lambda_2 = \lambda_3$. Therefore, $x_2' = (y_2' + cz_3', y_2 - cz_3') = (\frac{2}{3} + 9c, 1 + c, \frac{2}{3} - 9c, 1 - c)$, where c is an arbitrary constant.

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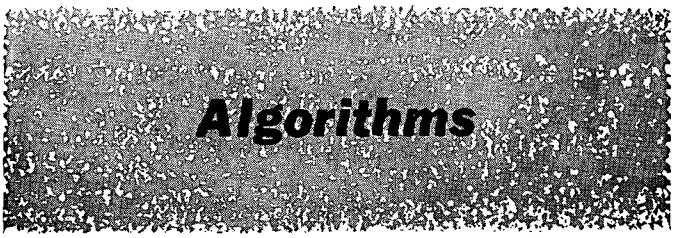

J. G. HERRIOT, Editor

ALGORITHM 318
CHEBYSCHEV CURVE-FIT (REVISED) [E2]
J. Boothroyd (Recd. 15 May 1967)
University of Tasmania, Hobart, Tas., Australia

```
procedure chebfit(x, y, n, a, m); value n, m;
  array x, y, a; integer n, m;
comment evaluates, in a[0] through a[m] of a[0:m+1], the co-
  efficients of an mth order polynomial P(x) = a0 + a1x + ··· amx^m
  such that the maximum error abs(P(xi)−yi)) is a minimum over
  the n(n>m+1) sample points x, y[1:n]. The x[i] must form a
  strictly monotonic sequence.
    This procedure is an extensive revision of Algorithm 91 (Albert
  Newhouse, Chebyshev Curve-Fit, Comm. ACM 5 (May 1961),
  281). The polynomial P(x) is a best-fit polynomial in the Cheby-
  shev sense as described by Stiefel (Numerical Methods of Tcheby-
  cheff Approximation), in Langer (Ed.), On Numerical Approxi-
  mation, U. of Wisconsin Press, 1959, pp. 217–232. Stiefel (p. 221)
  shows that the procedure must terminate after a finite number
  of steps. This is not always so with imperfect arithmetic, where
  roundoff errors may cause cycling of the chosen reference sets.
  This condition is detected by checking that the reference devia-
  tion is always raised monotonically. At exit the absolute value
  of a[m+1] yields the final reference deviation. Negative a[m+1]
  indicates that the procedure has been terminated following the
  detection of cycling;
begin
  integer i, j, k,mplus1, ri, i1, imax, rj, j1;
  real d, h, ai1, rhi1, denom, ai, rhi, xj, hmax, himax, xi, hi, abshi,
    nexthi, prevh;
  integer array r[0:m+1]; array rx, rh[0:m+1];
  mplus1 := m + 1; prevh := 0;
  comment index vector for initial reference set;
  r[0] := 1; r[mplus1] := n;
  d := (n−1)/mplus1; h := d;
  for i := 1 step 1 until m do
  begin r[i] := h + 1; h := h + d end;
start: h := −1.0;
  comment select m + 2 reference pairs and set alternating
    deviation vector;
  for i := 0 step 1 until mplus1 do
  begin
    ri := r[i];
    rx[i] := x[ri]; a[i] := y[ri];
    rh[i] := h := −h
  end i;
  comment compute m + 1 leading divided differences;
  for j := 0 step 1 until m do
  begin
    i1 := mplus1; ai1 := a[i1];
    rhi1 := rh[i1];
    for i := m step −1 until j do
    begin
      denom := rx[i1] − rx[i−j];
      ai := a[i]; rhi := rh[i];
      a[i1] := (ai1−ai)/denom;
```

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```
      rh[i1] := (rhi1−rhi)/denom;
      i1 := i; ai1 := ai; rhi1 := rhi
    end i
  end j;
  comment equate (m+1)th difference to zero to determine h;
  h := −a[mplus1]/rh[mplus1];
  comment with h known, combine the function and deviation
    differences;
  for i := 0 step 1 until mplus1 do
    a[i] := a[i] + rh[i] × h;
  comment compute polynomial coefficients;
  for j := m − 1 step −1 until 0 do
  begin
    xj := rx[j]; i := j; ai := a[i];
    for i1 := j + 1 step 1 until m do
    begin
      ai1 := a[i1];
      a[i] := ai − xj × ai1;
      ai := ai1; i := i1
    end i1
  end j;
  comment if the reference deviation is not increasing mono-
    tonically then exit;
  hmax := abs(h);
  if hmax ≤ prevh then
  begin a[mplus1] := −hmax; go to fit end;
  comment find the index, imax, and value, himax, of the largest
    absolute error for all sample points;
  a[mplus1] := prevh := hmax; imax := r[0]; himax:= h;
  j := 0; rj := r[j];
  for i := 1 step 1 until n do
    if i ≠ rj then
  begin
    xi := x[i]; hi := a[m];
    for k := m − 1 step −1 until 0 do
    hi := hi × xi + a[k];
    hi := hi − y[i]; abshi := abs(hi);
    if abshi > hmax then
    begin hmax := abshi; himax := hi; imax := i end
  end
  else
  if j < mplus1 then
  begin j := j + 1; rj := r[j] end;
  comment if the maximum error occurs at a nonreference
    point, exchange this point with the nearest reference point
    having an error of the same sign and repeat;
  if imax ≠ r[0] then
  begin
    for i := 0 step 1 until mplus1 do
    if imax < r[i] then go to swap;
    i := mplus1;
swap: nexthi := if i − i ÷ 2 × 2 = 0 then h else −h;
    if himax × nexthi ≥ 0 then r[i] := imax
    else
    if imax < r[0] then
    begin
      j1 := mplus1;
      for j := m step −1 until 0 do
      begin r[j1] := r[j]; j1 := j end;
      r[0] := imax
    end
    else
    if imax > r[mplus1] then
    begin
      j := 0;
      for j1 := 1 step 1 until mplus1 do
      begin r[j] := r[j1]; j := j1 end;
      r[mplus1] := imax
    end
    else r[i−1] := imax;
    go to start
  end;
fit:
end chebfit
```

CERTIFICATION OF ALGORITHM 91 [E2] CHEBYSHEV CURVE-FIT [Albert Newhouse *Comm. ACM 5* (May 1962), 281; *6* (April 1963), 167; *7* (May 1964), 296]

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In addition to the corrections noted by R. P. Hale [op. cit., April 1963] and P. Naur [op. cit., May 1964], the following changes are necessary:

1. The first statement should be $k := entier((m-1)/(n+1))$
2. A semi-colon should precede label *L*1.

With these changes the procedure ran successfully using Elliott 503 Algol.

Although this procedure is an implementation of a finite algorithm, roundoff errors may give rise to cyclic changes of the reference set causing the procedure to fail to terminate.

Algorithm 318 [J. Boothroyd, Chebyshev Curve-Fit(Revised), *Comm.ACM 10* (Dec. 1967), 801] avoids this cycling difficulty, uses less than half the auxiliary array space of Algorithm 91 and, on test, appears to be at least four times as fast.

Deutsch and Lampson—Cont'd from p. 799

be consciously aware of them. Many small details throughout the system have been arranged to permit smooth and rapid operation.

Acknowledgment. The basic framework of QED was designed and the program written by Mr. Deutsch. Many people have contributed suggestions for its improvement; we are especially indebted to M. W. Pirtle and R. Morris in this respect.

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