

Numerical Solution of a Thin Plate Heat Transfer Problem

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The numerical solution of a system of linear equations resulting from a discrete approximation to a thin plate heat transfer problem is considered. The slow convergence of point iterative methods is analyzed and shown to be caused by one of the boundary conditions. The difficulty may be removed by a standard line iterative technique.

KEY WORDS AND PHRASES: heat transfer problem, Poisson equation, boundary value problem, thin domain, successive overrelaxation (SOR), block SOR

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1. Introduction

We consider a two-dimensional heat transfer problem in a long, thin rectangular plate which is itself generating heat. Three of the boundaries of the plate are insulated while heat is being transferred across the fourth boundary by forced convection.

The problem is to establish the steady state temperature distribution throughout the plate. An attempt to solve the system of difference equations corresponding to this problem (Section 3) by point successive overrelaxation (SOR) resulted in exceedingly slow convergence. In Section 4 an informal analysis is given to show that the slow convergence is caused by the convective boundary conditions. In Section 5 we describe how the problem was finally solved using a line iterative technique which treats simultaneously all finite difference equations and boundary conditions associated with a vertical line of the mesh.

2. Statement of the Problem

The problem is mathematically formulated as the following system, where Q is the heat generation rate in the

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plate, k is the thermal conductivity of the plate, T_{s1} and T_{s2} are constant "sink" temperatures, h_1 and h_2 are the film coefficients, l_x and l_y are the length and width of the plate, and $\partial T/\partial n$ is the outward normal derivative of T along the appropriate boundary.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{Q}{k}, \quad 0 < x < l_x, \quad 0 < y < l_y, \quad (1)$$

$$\left. \begin{aligned} \frac{\partial T}{\partial n} &= 0, & x &= 0, & 0 < y < l_y, \\ & & x &= l_x, & 0 < y < l_y, \\ & & y &= 0, & 0 < x < l_x, \end{aligned} \right\} \quad (2)$$

$$\left. \begin{aligned} k \frac{\partial T}{\partial n} &= h_1(T_{s1} - T), & y &= l_y, & 0 < x < \frac{l_x}{3}, \\ k \frac{\partial T}{\partial n} &= h_2(T_{s2} - T), & y &= l_y, & \frac{l_x}{3} < x < l_x. \end{aligned} \right\} \quad (3)$$

In the numerical example considered, the orders of magnitude of the constants appearing in eqs. (1) to (3) are $Q \sim 10^6$, $k \sim 10$, $l_x \sim 10^{-1}$, $l_y \sim 10^{-4}$, $h_1 \sim 10$, and $T_{s1} \sim 10^2$.

3. The Finite Difference Equations

The difference equations are obtained by subdividing the plate into rectangular regions and approximating the derivatives in eqs. (1) to (3) by differences of temperatures. The nodal temperatures are labeled $T_{i,j}$ ($i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$). The spacings between points are $h_x = l_x/(m-1)$ and $h_y = l_y/(n-1)$ along the x and y axes.

With standard approximations, the difference equations for eqs. (1) to (3) can be written in the form:

$$2(h_x^2 + h_y^2)T_{i,j} = h_x^2(T_{i-1,j} + T_{i+1,j}) + h_y^2(T_{i,j-1} + T_{i,j+1}) + \frac{Q}{k}h_x^2h_y^2, \quad (4)$$

$$\begin{aligned} i &= 2, \dots, m-1; \quad j = 2, \dots, n-1, \\ T_{i,1} &= T_{i,2}, & i &= 2, \dots, m-1, \\ T_{i,n} &= T_{i,n-1}, & i &= 2, \dots, m-1, \end{aligned} \quad (5)$$

$$\begin{aligned} T_{1,j} &= T_{2,j}, & j &= 2, \dots, n-1, \\ T_{m,j} &= \lambda_\nu T_{m-1,j} + (1 - \lambda_\nu)T_{s\nu}, & j &= 2, \dots, n-1; \quad \nu = 1, 2, \end{aligned} \quad (6)$$

where

$$\lambda_\nu = (1 + h_y h_\nu/k)^{-1}.$$

As the referee has noted, eqs. (5) and (6) can be reformulated so that the truncation error is reduced [1, 2]. This reformulation does not affect the main result of this paper, although it does complicate the analysis somewhat.

Standard methods [3] establish the convergence of the Gauss-Seidel and SOR iteration schemes for eqs. (4) to (6). In practice, however, the convergence of these iteration methods was very slow, whatever overrelaxation factor was used. An analysis of the entire problem was performed which showed why this slow convergence could be expected. Rather than present the complete analysis, we shall instead analyze a modified form of the original problem which shares its salient features.

4. A One-dimensional Analogue

The modified problem is a one-dimensional analogue of the original problem along a vertical line. It is motivated by the fact that the vertical temperature profiles all have about the same shape.

The problem is

$$\begin{aligned} \frac{d^2 T}{dy^2} &= -\frac{Q}{k}, & 0 < y < l_y, \\ \frac{dT}{dy} \Big|_{y=0} &= 0, & k \frac{dT}{dy} \Big|_{y=l_y} = h(T_s - T). \end{aligned} \quad (7)$$

Its solution is

$$T(y) = -\frac{Q}{2k}(y^2 - l_y^2) + \frac{Q}{h}l_y + T_s.$$

The difference equations for (7) then become

$$2T_i = T_{i-1} + T_{i+1} + h_y^2 Q/k, \quad i = 2, \dots, m-1, \quad (8)$$

$$T_1 = T_2, \quad T_m = \lambda T_{m-1} + (1 - \lambda)T_s,$$

where

$$\lambda = (1 + h_y h/k)^{-1}.$$

Note that when $h_y h/k$ is small, as will happen when the plate is sufficiently thin, λ is near unity.

System (8) can be written as a matrix equation

$$AT = b, \quad (9)$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & & -\lambda & 1 \end{bmatrix}, \quad 2b = \left(0, \frac{h_y^2 Q}{k}, \dots, \frac{h_y^2 Q}{k}, 2(1 - \lambda)T_s\right)^T,$$

and

$$T = (T_1, \dots, T_m)^T.$$

The point Jacobi matrix for eq. (9) is

$$B = \begin{bmatrix} 0 & 1 & 0 & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \\ & & & \lambda & 0 \end{bmatrix}.$$

Let $\rho(B)$ be the spectral radius of B . Then [3]

$$\lambda < \rho(B) < 1.$$

The optimum overrelaxation factor, ω_b , and the asymptotic convergence rate, R_∞ , for the SOR iteration method are given by

$$\omega_b = \frac{2}{1 + \sqrt{1 - \rho^2(B)}} \quad \text{and} \quad R_\infty = -\ln(\omega_b - 1).$$

In the limit the number of iterations to reduce the error in the solution by a factor of e will be about $1/R_\infty$. For example, with $m = 11$, we have $\lambda \sim .99999$, $\omega_b \geq 1.991$, $R_\infty \leq .009$ and $1/R_\infty \geq 110$. Therefore, we would expect slow convergence of the SOR method.

The main point to note is that the rate of convergence depends on the parameter λ which arises from the convective boundary condition. If λ is near unity, as it must be when the plate is thin or the mesh is fine, then the convergence must be slow. All this suggests that if some way could be found to remove λ from the original problem, then the convergence would be faster. In the next section we do this by resorting to a line iterative method.

5. Method of Solution

The slow convergence was due to the presence of the convective boundary condition. We can satisfactorily overcome this difficulty by using a line iterative technique which simultaneously treats all finite difference equations and boundary conditions for each vertical line of the mesh. This is done as follows.

Let

$$T_j = (T_{1,j}, \dots, T_{m,j})^T.$$

Then eqs. (4) to (6) can be written in the block matrix form

$$\begin{bmatrix} D_v & -P & 0 \\ -P & D_v & -P \\ & \ddots & \ddots \\ & -P & D_v & -P \\ 0 & & & -P & D_v \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ \vdots \\ T_{n-2} \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} C_2 \\ C_3 \\ \vdots \\ C_{n-2} \\ C_{n-1} \end{bmatrix},$$

where

$$D_v = \begin{bmatrix} h_x^2 & -h_x^2 & & 0 \\ -h_x^2 & q & -h_x^2 & \\ & \ddots & \ddots & \\ & -h_x^2 & q & -h_x^2 \\ 0 & & -h_x^2 & h_x^2/\lambda_y \end{bmatrix}, \quad q = 2(h_x^2 + h_y^2),$$

and

$$P = \begin{bmatrix} 0 & & 0 \\ & h_y^2 & \\ & & h_y^2 \\ 0 & & & 0 \end{bmatrix}.$$

Observe that the boundary conditions on the left-hand and right-hand sides of the plate have not been included.

This block matrix equation generates a natural block Gauss-Seidel iteration scheme as follows.

$$\begin{aligned} D_r T_2^{(k+1)} &= P T_3^{(k)} + C_2 \\ D_r T_3^{(k+1)} &= P T_2^{(k+1)} + P T_4^{(k)} + C_3 \\ &\vdots \\ D_r T_{n-2}^{(k+1)} &= P T_{n-3}^{(k+1)} + P T_{n-1}^{(k)} + C_{n-2} \\ D_r T_{n-1}^{(k+1)} &= P T_{n-2}^{(k+1)} + C_{n-1}. \end{aligned}$$

The missing boundary conditions may be adjusted after each application of these equations.

The proposed iterative scheme requires that the system of equations involving the D_r be solved. Since the D_r are positive definite [3], the solution of each of the above equations by Gaussian elimination without pivoting is stable [4]. Since the D_r are tri-diagonal, the number of operations involved in their solution will be within the range of practicability. A further savings of computations can be effected by taking advantage of the fact that the D_r are identical except for their last diagonal element.

A Comparison of the Correlational Behavior of Random Number Generators for the IBM 360

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Hutchinson states that the "new" (prime modulo) multiplicative congruential pseudorandom generator, attributed to D.H. Lehmer, has passed the usual statistical tests for random number generators. It is here empirically shown that generators of this type can produce sequences whose autocorrelation functions up to lag 50 exhibit evidence of nonrandomness for many multiplicative constants. An alternative generator proposed by Tausworthe, which uses irreducible polynomials over the field of characteristic two, is shown to be free from this defect.

The applicability of these two generators to the IBM 360 is then discussed. Since computer word size can affect a generator's statistical behavior, the older mixed and simple congruential generators, although extensively tested on computers having 36 or more bits per word, may not be optimum generators for the IBM 360.

KEY WORDS AND PHRASES: random numbers, pseudorandom number generators, autocorrelation function, serial correlation, digital shift-register generators, linear recurrence modulo two, irreducible polynomials, primitive trinomials modulo two, congruential generators, prime numbers, statistical tests for randomness, IBM 360, 32-bit versus 36-bit word size

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The procedure outlined above was tried on the problem with $m = 11$. The starting values used were those predicted by the one-dimensional model. After approximately 250 iterations the procedure became stationary. This fast convergence was expected, since we incorporated the convective boundary conditions into the equations involving the D_r , which were solved directly. No attempt was made to overrelax the method. The problem was also solved analytically using Fourier series. The values of this solution agreed very well with the values of our numerical solution.

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REFERENCES

1. FOX, L. *Numerical Solution of Ordinary and Partial Differential Equations*. Pergamon, New York, 1962, p. 264.
2. SMITH, G. D. *Numerical Solution of Partial Differential Equations*. Oxford U. Press, New York, 1965, pp. 134-136.
3. VARGA, R. S. *Matrix Iterative Analysis*. Prentice-Hall, Englewood Cliffs, N.J., 1962.
4. WILKINSON, J. H. Error analysis of direct methods of matrix inversion. *J. ACM* 8, 3 (July 1961), 281-330.

1. Correlations from the Lehmer Generator

Hutchinson [1], and earlier, Holz and Clark [2, 3], Greenberger [4], and D. H. Lehmer [5], have suggested a method, attributed to Lehmer, for obtaining uniform pseudorandom numbers by means of the generators:

$$Y_{i+1} = AY_i \pmod{p}, \quad (1)$$

where p is the largest prime less than some 2^s .

In this note we investigate certain empirical and theoretical properties of autocorrelation functions or "correlograms" calculated from sequences of numbers generated by several different pseudorandom generators.

The autocorrelation function is a measure widely used in studying stochastic processes, especially in applied fields such as brain research, geophysics, and oceanography [6]. If we let X_i , $i = 1, 2, \dots$ be a sequence from a zero mean stochastic process, then we may define the autocorrelation function of a sample of length N , from this sequence, as

$$R(t) = \frac{1}{N} \sum_{i=1}^N X_i X_{i+t}. \quad (2)$$

(Such a sequence can be generated by letting $X_i = Y_i - (p+1)/2$.) What we in fact employ in this note is the *normalized* autocorrelation function, defined by

$$R_{xx}(t) = R(t)/R(0), \quad (3)$$

where the symbol $R_{xx}(t)$ is used to distinguish the *auto*-correlation function from the *cross*-correlation function $R_{xy}(t)$.

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