Digital Synthesis of Correlated Stationary Noise*

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In this note we propose a method of generating stationary noise with a prescribed auto-covariance function by digital methods. The need for such a technique often arises in testing the performance of data processing and engineering systems, where inputs corrupted with correlated noise (of a known form) are required.

The technique is quite simple and produces strict-sense stationary noise which agrees approximately with $R(\tau)$, the prescribed auto-covariance function (acf), over an interval $[-T_0, T_0]$. The method consists of approximating the spectral density by a periodic process with spectral lines, and then synthesizing the periodic noise with random phases and appropriate amplitudes.

In order to simplify discussion of the statistical properties of the noise generated, the technique is first presented in terms of exact harmonic analysis. In practice, discrete harmonic analysis as presented in the third section is used.

The Technique in Theory

Let $R(\tau)$ be the given acf and $[-T_0, T_0]$ the interval in τ for which noise with acf $R^*(\tau) = R(\tau)$ is to be generated. Using the fact that $R(-\tau) = R(\tau)$ and $|R(\tau)| \leqq R(0) < \infty$, we write

$$R^*(\tau) = A_0 + 2\sum_{n=1}^{\infty} A_n \cos nw_0\tau \qquad w_0 = \frac{\pi}{T_0} \quad (1)$$

so that $R^*(\tau) = R(\tau)$ for $|\tau| \leqq T_0$. The coefficients A_n are obtained from the formula

$$A_n = \frac{1}{T_0}\int_0^{T_0} R^*(\tau) \cos nw_0\tau \, d\tau \quad (2)$$

The A_n are related to the spectral density $S^*(w)$ of $R^*(\tau)$, which is the Fourier transform of $R^*(\tau)$, by

$$S^*(w) = \sum_{n=-\infty}^{+\infty} A_n\delta(w - nw_0) \quad (3)$$

where $\delta(w)$ is the unit impulse at $w = 0$. Thus the spectrum of $R^*(\tau)$ consists of lines of height A_n at the frequencies nw_0. The coefficients A_n are non-negative since an acf is realizable by a random time series if and only if if the Fourier transform is non-negative.

Let $\phi_1, \phi_2, \cdots$ be a sequence of independently and identically distributed random variables with a uniform distribution over the interval $[0, 2\pi]$. Set $B_0 = (2A_0)^{\frac{1}{2}}$, $B_n = 2(A_n)^{\frac{1}{2}}$ for $n \geqq 1$ and define

$$x(t) = \sum_{n=0}^{\infty} B_n \cos(nw_0t + \phi_n). \quad (4)$$

It can easily be verified that $E[x(t)] = 0$ and $E[x(t)x(t+\tau)] = R(\tau)$ for $|\tau| \leqq T_0$.

The Technique in Practice

In practice we have only a sampled $R(\tau)$, say $R_k = R(k\Delta)$, $k = 0, 1, \cdots, N$, $N\Delta = T_0$; and hence only $N+1$ coefficients A_n may be determined. The integration in Eq. (2) must also be replaced by numerical integration. If the trapezoidal rule is used, the following formulas are obtained:

$$A_n = \frac{2}{N}\left[\frac{R_0}{2} + \sum_{k=1}^{N-1} R_k \cos\frac{kn\pi}{N} + \frac{R_N}{2}\cos n\pi\right] \quad (5a)$$

for $0 \leqq n \leqq N-1$, and

$$A_N = \frac{1}{N}\left[\frac{R_0}{2} + \sum_{k=1}^{N-1} (-1)^k R_k + (-1)^N \frac{R_N}{2}\right] \quad (5b)$$

which yields

$$R^{**}(\tau) = A_0 + 2\sum_{n=1}^{N} A_n \cos nw_0\tau$$

so that $R^{**}(k\Delta) = R(k\Delta)$.

Because of roundoff and truncation errors introduced by the numerical integration, it is possible for some of the A_n coefficients to be negative. Thus we put

$$B_0 = (2A_0)^{\frac{1}{2}}, \quad B_n = \begin{cases} 2(A_n)^{\frac{1}{2}} & \text{if } A_n \geqq 0, \\ 0 & \text{if } A_n < 0. \end{cases}$$

We choose a sample of size $N+1$ of random phases $\phi_0, \phi_1, \cdots, \phi_N$ and define

$$x(t) = \sum_{n=0}^{N} B_n \cos(nw_0t + \phi_n). \quad (6)$$

Thus $x(t)$ is a sample function from a stationary zero-mean random process with acf equal to $R^{**}(\tau)$.

Periodicity of Sample Functions

The construction process given is equivalent to sampling the spectrum $S(w)$ of $R(\tau)$ at the frequencies $w_0, 2\omega_0, \cdots, N\omega_0$, where $w_0 = \pi/T_0$. The sample-function $x(t)$ is periodic with period $2T_0$, and care must be exercised not to sample $x(t)$ over an interval longer than $2T_0$. Suppose we sample $x(t)$ at intervals of length δ, and let $M\delta$ be the desired length of time. Then $M\delta$ must be equal to or less than $2N\Delta$.

If a time period longer than $2T_0$ is desired, then either (a) T_0 must be extended by sampling $R(\tau)$ longer, or (b) R_k may be set equal to zero for $k > N$, so that $R_k = R(k\Delta)$ for $0 \leqq k \leqq N$, and $R_k = 0$ for $N < k \leqq N_1$, where where $2N_1\Delta > M\delta$. It should be noticed that if the latter approach is used, the A_n remain the same for $0 \leqq n \leqq N$, and non-trivial A_n are added for $n > N$.

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Asymptotic Normality

For each t, $x(t)$ in Eq. (6) is composed of a linear combination of identically and independently distributed random variables. It might be suspected that as $N \rightarrow \infty$ the distribution of $x(t)$ approaches a normal distribution with mean-zero and variance equal to $R(0)$. Care must be exercised, however, since the A_n coefficients are functions of N, as evidence by Eqs. (5). In other words, each case must be analyzed individually, as $R(\tau) = \cos \tau$ provides an example of a case in which the central limit theorem does not hold.

The authors have tested a few cases, using a chi-squared goodness-of-fit test, and obtained results which indicate that when the central limit theorem applies, convergence is quite rapid. For example, let

$$R(\tau) = 1 - |\tau| \qquad |\tau| \leqq 1$$

and let

$$R_k = 0.02k \qquad 0 \leqq k \leqq 50.$$

A very small chi-squared value with 9 degrees of freedom indicated normality at the 99 percent level of confidence.

On the Computation of Rational Approximations to Continuous Functions

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Introduction

For computer programs which require as nearly as possible approximations of uniform accuracy, the Tchebychev approximations which minimize the maximum error are better than others although perhaps, as Kogbetliantz [1] points out, not significantly better for some purposes. The main reason for not using them in the past has been the difficulty connected with obtaining them, but this situation has changed rapidly with the improvement of algorithms for their determination. Since the publication of the review by Stiefel [2] in 1959, the Remes' algorithm in particular has been developed into a stable efficient method for finding "best" polynomial approximations. Of importance in this connection has been the improvement by Maehly [3] of the initial development of Murnaghan and Wrench [4].

As mentioned by Kogbetliantz [q.v.], the use of rational approximations converted into continued fraction form offers a potentially great improvement over polynomials in the evaluation of functions by computer sub-routines if the operations of multiplication and division require approximately the same time and these are long in comparison with addition and subtraction. In general, we can expect to produce a stated accuracy using less computer time or with approximately equal times get greater accuracy. The purpose of this article is to present computational details of the extension of Remes' algorithm for finding polynomial approximations to the determination of "best" rational approximations, and to present some results obtained by using it. For the functions sin x and cos x, comparison with rational approximations given by Spielberg [8] shows noticeable improvement, while coefficients for the approximation of $\Gamma(1 + x)$ have not to our knowledge been tabulated before. An approximation for $|x|$ is included because it illustrates the behavior in the case of a function which is continuous but not everywhere differentiable. It is hoped that the mathematical aspect of the extension, which is not yet complete, will be published later.

Remes' Second Algorithm

It will be helpful in describing the algorithm for minimax rational approximation to make reference to the algorithm for polynomial approximation since many of the computing details are exactly the same and hence directly transferable. Defining $\Delta(x) = f(x) - P_n(x)$, $a \leqq x \leqq b$, it is known that corresponding to the best polynomial approximation of degree n to $f(x)$, there exists a set of points $\{x_i\}$ and a minimum deviation E such that

$$\Delta(x_i) = (-1)^i E \qquad (i = 0, 1, \cdots, n+1), \quad (1)$$

$$|\Delta(x_i)| \geqq |\Delta(x)|, \; a \leqq x \leqq b, \quad (i = 0, 1, \cdots, n+1). \quad (2)$$

The determination of the set of points $\{x_i\}$, the minimum deviation E, and the polynomial $P_n(x)$ of best approximation is achieved by replacing the system of equations (1) and (2) by the following iterative scheme, in which the superscript k applies to the kth iteration:

$$\Delta^k(x_i^k) = (-1)^i E^k \qquad (i = 0, 1, \cdots, n+1), \quad (3)$$

$$\left.\begin{aligned} &\Delta^k(x_i^{k+1}) = \text{extreme} \qquad (i = 0, 1, \cdots, n+1) \\ &\text{and for at least one } i, \; |\Delta^k(x_i^{k+1})| = \max_{a \leqq x \leqq b} |\Delta^k(x)|. \end{aligned}\right\} \quad (4)$$

The solution is started by choosing an arbitrary set of points $\{x_i^0\}$ and using the resulting linear system (3) to determine E^0 and $P_n^0(x)$. The maxima and minima (including the greatest extreme) of $\Delta^0(x)$ become by (4) the set of points $\{x_i^1\}$, and a cycle of the computation is complete.

Novodvorskii and Pinsker [7] show that using any arbitrary set of initial points $\{x_i^0\}$ such that $E^0 \neq 0$, this process is convergent. In addition, Veidinger [5] shows that if $f(x)$ is differentiable the rate of convergence is