using successive approximation. The $f_N(\beta)$ is obtained by substituting this value of u_N in equation (13), for which we also need

$$\int_{u_N}^{\beta} [g(x)]^2 \, dx = S(\beta, u_N), \text{ say.}$$

One could alternatively determine u_N with greater computer time consumption, by successive approximation after a discrete search, from equation (13) directly. The integral in this equation reduces to

$$S(\beta, u_N) - \frac{2}{\beta - u_N} \Big\{ [\beta g(u_N) - u_N g(\beta)] I(\beta, u_N) + [g(\beta) - g(u_N)] J(\beta, u_N) + \frac{(\beta - u_N)}{6} [g(\beta) + g(u_N)]^2 \Big\}. \quad (17)$$

It should finally be noted that in order to obtain $f_N(\beta)$, the ranges over which $f_1, f_2, \cdots, f_{N-1}$ must be calculated are all $[\alpha, \beta]$.

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A Set of Matrices for Testing Computer Programs†

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1. Introduction

In this article, a set of matrices is defined, and the complete solution of each matrix is given. The matrices have arbitrary dimension, and a large number of parameters. By adjusting these parameters, a matrix of arbitrary dimension can be constructed which has two eigenvectors that are nearly coincident, which is deficient (has fewer than the maximum possible number of eigenvectors) or is as nearly singular as desired. Thus a set of matrices is available which forms a convenient set of check matrices for testing computer programs which purport to solve[1] or invert a matrix. The matrices of the set enjoy certain properties of regularity to which the term symmetry might well be applied; however, the matrices are not necessarily symmetric in the ordinary sense.

For selected values of the parameters, the matrices appear as variance-covariance matrices in statistics. In particular cases, the eigenvalues (but not the eigenvectors) and determinants have been found [1]. The methods of this note are shorter, more elementary, and more effective; from them, immediate insight into the matrix structure is obtained.

2. Basic Case

Let f_n be the $1 \times n$ column-vector whose entries are all 1's; f_n^* is the corresponding $n \times 1$ row-vector. For arbitrary n, k, write $J_{nk} = f_n f_k^*$; thus J_{nk} is the $k \times n$ matrix whose entries are all 1's. The matrix J_{nn} has the following properties: f_n is an eigenvector (with eigenvalue n); every vector v orthogonal to f_n $[f_n^* v = 0]$ is an eigenvector, with eigenvalue 0. This space of vectors v is spanned by

$$\{g_i\}, \quad g_i = f_n - n e_n^{\,i}, \qquad i = 1, 2, \cdots, n,$$

and has ex.gr. the first $n - 1$ of these for a basis. Here $e_n^{\,i}$ is the $1 \times n$ vector $[0, \cdots, 0, 1_i, 0, \cdots, 0]^*$. This proves the following theorem.

THEOREM 1. *By a change of basis, the matrix* $aI_n + bJ_{nn}$ *can be transformed to diag* $[a, \cdots, a, a + bn]$.

Note that the set of $n \times n$ matrices $aI_n + bJ_{nn}$ forms a semi-group, and the invertible ones form a group. The inverse of $aI_n + bJ_{nn}$ is $a^{-1}I_n - ba^{-1}(a + bn)^{-1}J_{nn}$.

3. General Case

The general matrix that can be solved by the above methods is the partitioned matrix:

$$A = \begin{bmatrix} a_1 I_{n_1} + b_{11} J_{n_1 n_1}, & b_{12} J_{n_1 n_2}, & \cdots, & b_{1t} J_{n_1 n_t} \\ b_{21} J_{n_2 n_1}, & a_2 I_{n_2} + b_{22} J_{n_2 n_2}, & \cdots, & b_{2t} J_{n_2 n_t} \\ \vdots & \vdots & & \vdots \\ b_{t1} J_{n_t n_1}, & \cdots, & & a_t I_{n_t n_t} + b_{tt} J_{n_t n_t} \end{bmatrix}$$

By choosing a suitable basis, the matrix A can be transformed into

$$B = \operatorname{diag}[A_1, A_2 \quad \cdots, A_t, A_{t+1}],$$

where for $i = 1, 2, \cdots, t$, A_i is the $n_i - 1 \times n_i - 1$ scalar

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[1] I.e., find the eigenvalues and corresponding invariant subspaces.

matrix $\mathbf{a}_i$, and A_{t+1} is the $t \times t$ matrix defined by

$$A_{t+1} = \begin{bmatrix} a_1 + b_{11}n_1, & b_{12}n_2, & \cdots, & b_{1t}n_t \\ b_{21}n_1, & a_2 + b_{22}n_2, & \cdots, & b_{2t}n_t \\ \vdots & \vdots & & \vdots \\ b_{t1}n_1, & \cdots, & & a_t + b_{tt}n_t \end{bmatrix}.$$

Thus the matrix A can be solved by solving a $t \times t$ matrix of constants. To establish the assertion just made, it is only necessary to exhibit the "suitable basis". A trivial lemma suffices.

LEMMA 1. *Let the subspaces $S_1, S_2, \cdots, S_t, S_{t+1}$ of $\Re_{n_1+\cdots+n_t}$ be defined as follows:*

S_{t+1} has the basis

$$\{f_i', \quad i = 1, 2, \cdots, t\}, f_i' = [0, 0, \cdots 0, f_{n_i}, 0, \cdots, 0]^*,$$

S_r has the basis

$$\{g_{ir} \equiv [0, 0, \cdots 0, f_{n_r} - n_r e^i_{n_r}, 0, \cdots, 0]^*,$$
$$i = 1, 2, \cdots, n_r - 1\}, \quad r = 1, 2, \cdots, t.$$

These $t + 1$ spaces are mutually orthogonal; they span the entire $(n_1 + n_2 + \cdots + n_t)$-dimensional space $\Re_{n_1+\cdots+n_t}$.

The action of A on the vectors of these various bases is easily checked; in this way it is verified that if a transformation of $\mathbf{R}$ has matrix A with respect to the basis vectors $\{e^i_n\}$, the matrix of the transformation with respect to the new basis V: $\{g_{ir}, f_i'\}$ is B: $V^{-1}AV = B$.

4. Examples

The results of Section 3 are powerful enough to yield many useful examples.

The matrix

$$C = \begin{bmatrix} aI_n + bJ_{nn} & cJ_{nk} \\ dJ_{kn} & hI_k + lJ_{kk} \end{bmatrix}$$

has determinant

$$a^{n-1}h^{k-1}[(a + bn)(h + lk) - cndk],$$

and inverse

$$C^{-1} = \begin{bmatrix} a^{-1}I_n + b'J_{nn} & c'J_{nk} \\ d'J_{kn} & h^{-1}I_k + l'J_{kk} \end{bmatrix},$$

$$b' = \frac{h + lk - a^{-1}\Delta}{\Delta n}, \quad c' = -\frac{c}{\Delta}$$

$$d' = -\frac{d}{\Delta}, \qquad l' = \frac{a + bn - h^{-1}\Delta}{\Delta k}$$

$$\Delta = (a + bn)(h + lk) - cdnk.$$

This inverse is conveniently discovered by writing the matrix

$$\begin{bmatrix} a + bn & ck \\ dn & h + lk \end{bmatrix}^{-1}$$

in the form

$$\begin{bmatrix} a^{-1} + b'n & c'k \\ d'n & h^{-1} + l'n \end{bmatrix}.$$

If $c = 0$, the matrix C is deficient only if the conditions

$$a + bn = h + lk, \quad d \neq 0$$

are both satisfied. In this case, the relations

$$[A - (a + bn)]^2 f_{n+k} = 0,$$
$$V = [A - (a + bn)]f_{n+k} \neq 0$$

hold.

Determinants, roots, and vectors of the matrices in [1] (in all of which $t \leqq 3$ and the a_i are equal) are quickly computed by use of the results of Section 3. Note that the individual basis vectors g_{ir} of S_r $(i = 1, \cdots, n_r - 1, r = 1, \cdots, t)$ are always eigenvectors of A. The rest of the structure of A is found from the structure of B (i.e. of S_{t+1}) in the following way. If the $t \times t$ matrix S_{t+1} has Jordan canonical form J, with respect to the basis vectors $V_j = \sum \alpha_{ji} e_t{}^i$, the vectors of $\Re$ needed to reduce A to canonical form are $w_j = \sum \alpha_{ji} f_i'$.

A matrix C for which the exact inverse can be expressed on a machine with 7 significant decimal digits is the following $(t = 2)$:

$$a = b = c = h = l = 1, \quad d = 1.259999, \quad n = 20,$$
$$k = 5, \quad b' = 3000. - .05, \quad c' = -10000,$$
$$d' = -12599.99, \; l' = 42000. - .2$$

5. Generalization

The essential point in the definition of A is that the matrices J_{nk} have rank 1. The theory is unaltered if f_n is replaced by any $1 \times n$ nonzero vector.

Dr. A. S. Householder has kindly communicated the fact that the matrix A can be written as a product $(I - zw^*)D$, where z, w are $1 \times n$ vectors, and D is a diagonal matrix.

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