A Finite Sequentially Compact Process for the Adjoints of Matrices Over Arbitrary Integral Domains

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Let $\xi = (a_{ij})$ be a square matrix of order n whose entries are elements of an integral domain D. Denote ξ by $\xi^{(0)}$ and a_{ij} by $a_{ij}^{(0)}$. A sequential process for $n \times n$ matrices

$$\xi^{(k)} = (a_{ij}^{(k)}), \quad 1 \leqq k \leqq \text{n}, \quad (1)$$

will be described which is of such nature that:

(a) each $a_{ij}^{(k)}$ is an element of D;
(b) with one exception the elements $a_{ij}^{(k)}$ are formed from the elements $a_{ij}^{(k-1)}$ and indeed in such manner that a specific $a_{pq}^{(k)}$ can replace $a_{pq}^{(k-1)}$ in the array leaving available therein all elements needed for calculating the remaining unfound $a_{ij}^{(k)}$;
(c) except for a permutation of rows and columns $\xi^{(n)}$ is the adjoint of $\xi^{(0)}$;
(d) if the number of independent rows is $m < n$ then a selection of m independent rows can be made;
(e) the sequential process can be varied rather arbitrarily (for example so as to attempt to minimize in the case, say, of real or Gaussian integers, the size of the new entries formed) and the required permutation of rows and columns mentioned in (c) can be built as part of the process.

The technique used can be quickly described as being essentially Gauss-Jordan elimination, modified to include the principle of exact elimination except that at each stage a common factor (one occurring systematically) is removed; the whole process being arranged in a "compact" manner suggestive of Crout's method. Perhaps the technique for building an adjoint (rather than an inverse) is new.

At each step k of the recursive process one chooses any nonzero element $a_{i_{k+1}j_{k+1}}^{(k)}$ $(0 \leqq k \leqq n - 1)$ under the further restrictions that the sequence of sensed pairs

$$S = ((i_1, j_1), (i_2, j_2), \cdots, (i_n, j_n)) \quad (2)$$

shall be recorded and that the sequences $(i_1, i_2, \cdots, i_n)$ and $(j_1, j_2, \cdots, j_n)$ individually shall have no duplicate terms. If all entries available for selection are zero at a given stage then a maximum number of independent rows will have been found. If this occurs at step $n-1$, completing the process still gives the adjoint. The element $a_{i_{k+1}j_{k+1}}^{(k)}$ so chosen must be carried over to step $k+2$ and is the exception mentioned in (b) above as well as the systematic factor mentioned in the second paragraph.

The sequential matrix $\xi^{(k)}$, $1 \leqq k \leqq n$, is further defined as follows, first requiring for notational convenience that $a_{i_0j_0}^{(-1)} = 1$. Let

$$a_{i_kj_k}^{(k)} = a_{i_{k-1}j_{k-1}}^{(k-2)}, \quad (3)$$

thus $a_{i_1j_1}^{(1)}$ is always one. Proceed by defining

$$\begin{aligned} a_{i_kj}^{(k)} &= a_{i_kj}^{(k-1)} \quad \text{if} \quad j \neq j_k; \\ a_{ij_k}^{(k)} &= -a_{ij_k}^{(k-1)} \quad \text{if} \quad i \neq i_k. \end{aligned} \quad (4)$$

Finally, for $j \neq j_k$ and $i \neq i_k$, let

$$a_{ij}^{(k)} = [a_{ij}^{(k-1)}a_{i_kj_k}^{(k-1)} - a_{ij_k}^{(k-1)}a_{i_kj}^{(k-1)}]/a_{i_kj_k}^{(k)}. \quad (5)$$

Ostensibly (5) requires a field. It will be shown subsequently that the denominator is a factor of the numerator. For convenience in proof, the elements of D will be temporarily viewed as rational forms over the quotient field of D, and the original entries of $\xi^{(0)}$ will be considered as n^2 indeterminates.

Proof is accomplished by use of related matrices. Let δ_{ij} denote the Kronecker delta, and let

$$\begin{aligned} \theta^{(k)} &= \frac{1}{a_{i_{k-1}j_{k-1}}^{(k-2)}} (b_{ij}^{(k)}), \\ b_{ij}^{(k)} &= \delta_{ij}a_{i_kj_k}^{(k-1)} \quad \text{if} \quad j \neq i_k, \\ b_{ii_k}^{(k)} &= a_{ij_k}^{(k)}. \end{aligned} \quad (6)$$

Define

$$\eta^{(0)} = \xi^{(0)}, \quad \eta^{(k)} = \theta^{(k)}\eta^{(k-1)}, \quad 1 \leqq k \leqq n. \quad (7)$$

For $1 \leqq l \leqq n$, let $\Delta_{ij}^{(l)} = 1$ if $i = i_l$ and $j = j_l$, and let $\Delta_{ij}^{(l)} = 0$ otherwise. Then using (3), (4), (5), (6) and (7) it follows in routine manner that

$$\begin{aligned} \eta^{(k)} &= (d_{ij}^{(k)}); \\ d_{ij}^{(k)} &= \Delta_{ij}^{(l)}a_{i_kj_k}^{(k-1)} \quad \text{if} \quad j = j_l, \quad 1 \leqq l \leqq k; \\ d_{ij}^{(k)} &= a_{ij}^{(k)} \quad \text{otherwise.} \end{aligned} \quad (8)$$

Define

$$\psi^{(1)} = \theta^{(1)}, \quad \psi^{(k)} = \theta^{(k)}\psi^{(k-1)}; \quad 1 \leqq k \leqq n. \quad (9)$$

It follows, also inductively, that

$$\begin{aligned} \psi^{(k)} &= (e_{ij}^{(k)}); \\ e_{ij}^{(k)} &= a_{ij_l}^{(k)} \quad \text{for} \quad j = i_l, \quad 1 \leqq l \leqq k; \\ e_{ij}^{(k)} &= \delta_{ij}a_{i_kj_k}^{(k-1)} \quad \text{otherwise.} \end{aligned} \quad (10)$$

Now let ρ denote the $n \times n$ permutation matrix which sends row i_k into row j_k, $1 \leqq k \leqq n$. Note that the determinant of ρ is 1 or -1 and is readily calculated from (2) without use of determinants. Also, pre- and postmultiplication by ρ can be effected from (2) without actual use of matrix multiplication. By (7) and (9)

$$\psi^{(k)} = \theta^{(k)}\theta^{(k-1)} \cdots \theta^{(1)}; \quad \psi^{(k)}\xi^{(0)} = \eta^{(k)}. \quad (11)$$

If $I^{(k)} = (f_{ij}^{(k)})$ where $f_{ij}^{(k)} = 1$ if $i = j = i_l$, $1 \leqq l \leqq k$ and is otherwise zero, then

$$\psi^{(k)}\xi^{(0)}\rho I^{(k)} = \eta^{(k)}\rho I^{(k)}.$$

The right side has entries of zero unless $i = j = i_l$,

$1 \leqq l \leqq k$. If $i = j = i_l$, the entry is $a^{(k-1)}_{i_kj_k}$. Thus the above becomes

$$\psi^{(k)}\xi^{(0)}\rho I^{(k)} = a^{(k-1)}_{i_kj_k} I^{(k)}. \tag{12}$$

Multiplying both sides by $I^{(k)}$ leaves the right side unaltered. However, observe that $I^{(k)}\theta^{(k)}$ has nonzero entries only in rows i_p, $1 \leqq p \leqq k$, and columns i_q, $1 \leqq q \leqq k$. Observe also that if such a matrix have as a right multiplier a matrix of the type $\theta^{(l)}$, $1 \leqq l \leqq k$, the result again has nonzero entries in only these rows and columns. Thus by (11), $I^{(k)}\psi^{(k)}$ has this property. Since $\xi^{(0)}\rho I^{(k)}$ can have nonzero entries only in columns i_q, $1 \leqq q \leqq k$, the matrix $\xi^{(0)}$ contributes only in the form of elements of the columns numbered j_q, $1 \leqq q \leqq k$. Moreover, since the only nonzero columns of the matrix $I^{(k)}\psi^{(k)}$ are those numbered i_q, $1 \leqq q \leqq k$, the matrix $\xi^{(0)}$ contributes only elements from these rows.

A study of (3), (4), and (5) shows inductively that all of the elements of $I^{(k)}\psi^{(k)}$ are expressible in terms of these same elements from $\xi^{(0)}$. Thus the left member of (12) is dependent only on the indeterminates $a^{(0)}_{i_pj_q}$, $1 \leqq p \leqq k$, $1 \leqq q \leqq k$.

Proceed by letting $\xi^{(l)}_k$ be the submatrix of order k obtained from $\xi^{(l)}$, $1 \leqq l \leqq k$, by deleting the rows and columns other than rows i_q and columns j_q, $1 \leqq q \leqq k$. Let ρ_k be constructed from ρ by deleting rows other than those numbered j_q and columns other than those numbered i_q, $1 \leqq q \leqq k$. Let $\psi^{(l)}_k$ and $\theta^{(l)}_k$ be constructed from $\psi^{(l)}$ and $\theta^{(l)}$ respectively by deleting all rows and columns except rows i_q and columns i_q, $1 \leqq q \leqq k$. The expression (12) then yields

$$\psi^{(k)}_k \xi^{(0)}_k \rho_k = a^{(k-1)}_{i_kj_k} I_k, \tag{13}$$

where I_k is the identity matrix of order k. It is also true that

$$\psi^{(l)}_k = \theta^{(l)}_k \theta^{(l-1)}_k \cdots \theta^{(1)}_k, \tag{14a}$$

$$\det \theta^{(l)}_k = \left[\frac{a^{(l-1)}_{i_lj_l}}{a^{(l-2)}_{i_{l-1}j_{l-1}}}\right]^{k-1}, \tag{14b}$$

provided $1 \leqq l \leqq k$.

Combining (14a) with (13) yields

$$\theta^{(k)}_k \theta^{(k-1)}_k \cdots \theta^{(1)}_k \xi^{(0)}_k \rho_k = a^{(k-1)}_{i_kj_k} I_k .$$

This in turn leads by virtue of (14b) to

$$a^{(k-1)}_{i_kj_k} = \det [\xi^{(0)}_k \rho_k], \tag{15}$$

and since $\xi^{(k)}_k \rho_k = \psi^{(k)}_k$, to

$$\xi^{(k)} \rho_k \xi^{(0)}_k \rho_k = \det [\xi^{(0)}_k \rho_k] I_k,$$

or

$$\operatorname{adj} \xi^{(0)}_k = [\det \rho_k] \rho_k \xi^{(k)}_k \rho_k . \tag{16}$$

In particular, since $\rho_n = \rho$, etc.,

$$\operatorname{adj} \xi^{(0)} = [\det \rho] \rho \xi^{(n)} \rho. \tag{17}$$

Turn now to the proof that the entries of $\xi^{(k)}$ are elements of the integral domain D. In routine manner the following formal relations are established using (3), (4) and (5). If $j \neq j_{k-1}$ and $j \neq j_k$ then

$$\begin{aligned} a^{(k)}_{i_{k-1}j} \cdot a^{(k-2)}_{i_{k-1}j_{k-1}} &= \begin{vmatrix} a^{(k-1)}_{i_{k-1}j} & a^{(k-1)}_{i_{k-1}j_k} \\ a^{(k-1)}_{i_kj} & a^{(k-1)}_{i_kj_k} \end{vmatrix} \\ &= \frac{a^{(k-2)}_{i_{k-1}j_{k-1}}}{a^{(k-3)}_{i_{k-2}j_{k-2}}} \begin{vmatrix} a^{(k-2)}_{i_{k-1}j} & a^{(k-2)}_{i_{k-1}j_k} \\ a^{(k-2)}_{i_kj} & a^{(k-2)}_{i_kj_k} \end{vmatrix} . \end{aligned} \tag{18}$$

If $i \neq i_{k-1}$ and $i \neq i_k$ then

$$\begin{aligned} a^{(k)}_{ij_{k-1}} \cdot a^{(k-2)}_{i_{k-1}j_{k-1}} &= \begin{vmatrix} a^{(k-1)}_{ij_{k-1}} & a^{(k-1)}_{ij_k} \\ a^{(k-1)}_{i_kj_{k-1}} & a^{(k-1)}_{i_kj_k} \end{vmatrix} \\ &= \frac{a^{(k-2)}_{i_{k-1}j_{k-1}}}{a^{(k-3)}_{i_{k-2}j_{k-2}}} \begin{vmatrix} a^{(k-2)}_{i_kj_{k-1}} & a^{(k-2)}_{i_kj_k} \\ a^{(k-2)}_{ij_{k-1}} & a^{(k-2)}_{ij_k} \end{vmatrix} . \end{aligned} \tag{19}$$

Also, if $i \neq i_k$, $i \neq i_{k-1}$, $j \neq j_k$ and $j \neq j_{k-1}$, then

$$\begin{aligned} a^{(k)}_{ij} \cdot a^{(k-2)}_{i_{k-1}j_{k-1}} &= \begin{vmatrix} a^{(k-1)}_{ij} & a^{(k-1)}_{ij_k} \\ a^{(k-1)}_{i_kj} & a^{(k-1)}_{i_kj_k} \end{vmatrix} \\ &= \frac{a^{(k-2)}_{i_{k-1}j_{k-1}}}{[a^{(k-3)}_{i_{k-2}j_{k-2}}]^2} \begin{vmatrix} a^{(k-2)}_{ij} & a^{(k-2)}_{ij_{k-1}} & a^{(k-2)}_{ij_k} \\ a^{(k-2)}_{i_{k-1}j} & a^{(k-2)}_{i_{k-1}j_{k-1}} & a^{(k-2)}_{i_{k-1}j_k} \\ a^{(k-2)}_{i_kj} & a^{(k-2)}_{i_kj_{k-1}} & a^{(k-2)}_{i_kj_k} \end{vmatrix} . \end{aligned} \tag{20}$$

Proof is by induction, and the postulate is essentially that $\xi^{(p)}$, $\xi^{(p-1)}$ have entries which are polynomials in the indeterminates $a^{(0)}_{ij}$, and that $a^{(p-1)}_{i_pj_p}$ and $a^{(p-2)}_{i_{p-1}j_{p-1}}$ are relatively prime. It is readily determined that $\xi^{(1)}$ and $\xi^{(0)}$ are polynomials in the $a^{(0)}_{ij}$, $1 \leqq j \leqq n$, $1 \leqq i \leqq n$. Moreover, $a^{(0)}_{i_1j_1}$ and $a^{(-1)}_{i_0j_0}$ are relatively prime. Assume true for $p = k - 1$. By (3) and (4) it is seen that only the nature of $a^{(k)}_{ij}$, $j \neq j_k$ and $i \neq i_k$, need be considered insofar as polynomial character is concerned. Since $a^{(k-2)}_{i_{k-1}j_{k-1}}$ and $a^{(k-3)}_{i_{k-2}j_{k-2}}$ are relatively prime by hypothesis, analysis of the last two members of (18) shows that $a^{(k-3)}_{i_{k-2}j_{k-2}}$ must be a factor of the determinant shown. That being so, analysis of the first and third members of (18) shows that $a^{(k)}_{ij_{k-1}}$ is a polynomial in the original indeterminates. In the case of the expressions in (19) and (20) the reasoning is similar. It remains to show that $a^{(k-2)}_{i_{k-1}j_{k-1}}$ and $a^{(k-1)}_{i_kj_k}$ have no common factors.

To accomplish this, consider the submatrix $\xi^{(0)}_k$ which can be viewed as evolved from $\xi^{(0)}_{k-1}$ by bordering with $2k-1$ new indeterminates. Since these matrices have as determinantal values (except for sign) the polynomials $a^{(k-1)}_{i_kj_k}$ and $a^{(k-2)}_{i_{k-1}j_{k-1}}$, in light of the presence of the new indeterminates they can have no common factor.

A program for the IBM 709 has been written which finds the adjoint of 50×50 square matrices, using integer arithmetic. This program can handle matrices such that no calculated elements of $\xi^{(k)}$, $1 \leqq k \leqq n$, can exceed $2^{280} - 1$.

Clearly the components of $\xi^{(k)}$ are minors of ξ of varying orders. A preliminary study indicates that this feature may be used to find the characteristic function of ξ. A second line of present investigation is the creation of a similar compact scheme diagonalizing quadratic forms over integral domains.