# Notes on computing peaks in k-levels and parametric spanning trees 

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#### Abstract

We give an algorithm to compute all the local peaks in the $k$-level of an arrangement of $n$ lines in $O(n \log n)+\tilde{O}\left((k n)^{2 / 3}\right)$ time. We can also find $\tau$ largest peaks in $O\left(n \log ^{2} n\right)+\tilde{O}\left((\tau n)^{2 / 3}\right)$ time. Moreover, we consider the longest edge in a parametric minimum spanning tree (in other words, a bottleneck edge for connectivity), and give an algorithm to compute the parameter value (within a given interval) maximizing/minimizing the length of the longest edge in MST. The time complexity is $\tilde{O}\left(n^{8 / 7} k^{1 / 7}+\right.$ $\left.n k^{1 / 3}\right)$.


## 1 Introduction

The $k$-level of an arrangement of lines is one of popular geometric objects in computational geometry [12, 29]. The $k$-level is the union of $k$-th lowest (closed) line-segments of the arrangement, and it can be considered as the trajectory of the $k$-th smallest element in a set of $n$ data each of which depends on a parameter $x$ linearly. Thus, the $k$-level is a special case of the locus of the largest element of the minimum base of a parametric matroid $17,11,13,19,21]$ with one parameter; in precise, the $k$-level is the locus of the maximum element in the minimum base of a parametric uniform matroid of rank $k$. From a different aspect, the $k$-level is a dual concept of the $k$-set 12], and the complexity $g_{k}(n)$ of the $k$-level of an arrangement of $n$ lines is asymptotically same as the number of different $k$-sets in a set of $n$ points in a plane.

Lovász 23] first gave a nontrivial $O\left(n^{3 / 2}\right)$ upper bound for $g_{\lfloor n / 2\rfloor}(n)$, and also introduced Straus's $\Omega(n \log n)$ lower bound. The current best upper and lower bounds for $g_{k}(n)$ are $O\left(k^{1 / 3} n\right)$ [11] and $2^{\Omega(\sqrt{\log k})} n$ [28], respectively. The upper bound holds for any parametric matroid (with a linear parameter) of rank

[^0]$k$ in $n$ elements 13. Moreover, Cole et al. gave an $O\left(g_{k}(n) \log ^{2} n\right) 8$ time algorithm to compute the $k$-level; this time complexity has been improved due to improvement of algoirthms for the dynamic convex hull computation employed as a key subroutine [5, 6]. The current best time complexity (randomized expected time) is $O\left(g_{k}(n) \alpha\left(g_{k}(n)\right) \log n\right)$ time given by Har-Peled 20, where $\alpha()$ is the inverse Ackermann function. Thus, the time complexity is very close to $O\left(g_{k}(n) \log n\right)$.

However, we often need a compact "outline" of a trajectory of a parametric problem by using a small number of characteristic points on it. Such an outline, generally speaking, will be useful as a compact data to control parametric problems, and possibly utilized in designing kinetic data structures [4, 3]. Local peaks in the trajectory are considered to be natural characteristic points. A key observation to investigate the $k$-level is that it has at most $2 k-1$ local peaks (at most $k$ maximal peaks and at most $k-1$ minimal peaks) [2, 13]. One interesting problem is to compute all the local peaks efficiently. This enables us to give a decomposition of the $k$-level into monotone chains, and hence create an outline with a size $O(k)$ of the $k$-level.

We give an algorithm to compute all the local peaks in the $k$-level of an arrangement of $n$ lines in $O(n \log n)+\tilde{O}\left((k n)^{2 / 3}\right)$ time, where $\tilde{O}$ is the big-O notation ignoring polylogarithmic factors. The current estimate for the polylogarithmic factor of the second term is less than $\log ^{5} n$; however we do not give it explicitly in this paper, since it is probably loose and will confuse readers. The time complexity is better than $O\left(g_{k}(n) \log n\right)$ for some restricted range of $k$ even if the current lower bound of $g_{k}(n)$ by Tóth 28 is tight. If we substitute the current $O\left(k^{1 / 3} n\right)$ upper bound to $g_{k}(n)$, the time complexity is better than $O\left(g_{k}(n) \log n\right)$ if $k=O\left(n / \log ^{c} n\right)$, where $c$ is a suitable constant.

Another interesting question is how fast we can compute $\tau$ largest maximal peaks for $\tau \leq k$. If $\tau=1$, Roos and Widmayer [26] gave a neat method to compute the maximum point in the $k$-level in $O\left(n \log n+(n-k) \log ^{2}(n-k)\right)$ time by using an efficient slope selection algorithm. We can compute $\tau$ largest peaks in $O\left(n \log ^{2} n\right)+\tilde{O}\left((\tau n)^{2 / 3}\right)$ time by combining Roos and Widmayer's technique and the above mentioned method for computing all the peaks.

Finally, we investigate whether we can analogously treat some parametric matroids: Compute peaks in the trajectory of the largest element in the minimum weight base of a parametric matroid. In particular, the graphic matroid is of wide interest [13, 17, 19]: Consider a weighted undirected connected graph $G(x)$ with $k$ nodes and $n$ edges, such that each edge has a parametric weight that is linear in a parameter $x$. Here, $k$ and $n$ become the rank and size of the graphic matroid, respectively. Let $T(x)$ be the minimum weight spanning tree of $G(x)$ and consider the longest edge $e(x)$ in $T(x)$. Note that the minimum weight spanning tree becomes a spanning tree that minimizes the length of the longest edge. We call the edge $e(x)$ the spanning bottleneck edge( SBE ), and write $S B E(x)$ and $w_{S B E}(x)$ for $e(x)$ and its weight, respectively. The naming comes from the fact that $w_{S B E}(x)$ is the minimum value of $w$ such that the subgraph of $G(x)$ consisting of edges with weights less than or equal to $w$ is connected.

The following problems are important in sensitivity analysis: (1). Compute the maximum value and the minimum value of $w_{S B E}(x)$ for $x \in I$, where $I$ is a given interval. (2). Compute all peaks of the trajectory $y=w_{S B E}(x)$.

For example, imagine a system represented by the graph $G$ where a link represented by an edge with a weight larger than a (controllable) threshold value becomes unreliable, and the edge weight depends on a parameter $x$ linearly within an interval $I$. For a given subinterval $J \subset I$, we want to know the threshold value of the edge weights so that the graph remains connected for every $x \in J$. This can be reduced to the problem (1). Moreover, if we have computed all peaks in $I$ as a preprocessing (problem (2)), we can efficiently query for the threshold value, provided that we have an efficient method (shown in Section 4) to compute $w_{S B E}(x)$ at endpoints of $J$.

Both of problems (1) and (2) can be solved by computing the whole picture of the transitions of the minimum spanning trees, and the time complexity of the computation is $O(k n \log n)$ by using an algoirthm given by Fernandez-Baca et al. 15. Note that $O\left(h_{k}(n) n^{2 / 3} \log ^{3 / 4}\right)$ time output-sensitive algorithm of Agarwal et al. 3 ] is better for some range of $k$, where $h_{k}(n)$ is the number of of transitions of the minimum spanning tree. An $O\left(k^{1 / 3} n\right)$ upper bound 11 and an $\Omega(n \alpha(k))$ lower bound 13] are known for $h_{k}(n)$.

Roos and Widmayer's method can be directly applied to the first problem. By using dynamic maintenance algorithms [14] of a minimum spanning tree, the time complexity becomes $O(\sqrt{k} n \log n)$. Combined with range searching techniques, we improve the time complexity to $\tilde{O}\left(n^{8 / 7} k^{1 / 7}+n k^{1 / 3}\right)$. We give some discussion on the second problem, although theoretical improvement on the $O(k n \log n)$ time method remains open.

## 2 Preliminaries

### 2.1 Roos and Widmayer's algorithm

Given a set $\mathcal{H}$ of $n$ lines in the $x-y$ plane, let $\mathcal{L}_{k}$ be the $k$-level of the arrangement of $\mathcal{H}$. Let $\boldsymbol{p}$ be a point on $\mathcal{L}_{k}$ that has the maximum $y$-value $y_{\max }$. Without loss of generality, we assume that such a point is unique. For any given value $\alpha$, one can decide whether $y_{\max } \geq \alpha$ or not in $O(n \log n)$ time: We sweep on the line $h$ : $y=\alpha$ from the leftmost intersection point to the right to compute the levels of all intersection points on $h$ with lines in the arrangement. If all intersection points are above the $k$-level, $\alpha>y_{\max }$; otherwise, $\alpha \leq y_{\max }$. By using this decision method, a binary search algorithm works to compute $\boldsymbol{p}$, and a weakly polynomial time algorithm with a time complexity $O(n \log n \log \Gamma)$ can be obtained, if each coefficient of the lines is a quotient number of integers with $\log \Gamma$ bits. Roos and Widmayer 26] applied an efficient slope selection method [7] to transform the binary search algorithm into strongly polynomial, and gave an $O\left(n \log ^{2} n\right)$ time algorithm. They further improved the time complexity to $O\left(n \log n+k \log ^{2} k\right)$ for computing the minimum and $O\left(n \log n+(n-k) \log ^{2}(n-k)\right)$ for computing the maximum.

### 2.2 Range query and Matoušek's point set decomposition

We use well-known (although sophisticated) simplex range query data structures [1]: We construct a data structure for a set $S$ of $n$ points in a plane such that given a query halfplane $H$, we can compute the number of points in $S$ located in $H$ efficiently. If we spend $O(m)$ time for constructing the data space for $n \log n \leq m \leq n^{2}$, the query time is $\widetilde{O}\left(n / m^{1 / 2}\right)$. The query can be done in polylogarithmic time by using $O\left(n / m^{1 / 2}\right)$ processors. The data structure uses $\tilde{O}(m)$ space, although we do not discuss space complexity in this paper. Moreover, we can query the number of points in the intersection of two (or three) halfplanes in the same query time if we ignore a polylogarithmic factor. We can also do reporting query by spending additional $O(N)$ time if the region contains $N$ points.

Given a set $\mathcal{H}$ of $n$ lines in a plane, we consider the set $\mathcal{D}(\mathcal{H})$ of their dual points: The dual point of a line $y=a x-b$ is $(a, b)$. We construct a range searching data structure for $\mathcal{D}(\mathcal{H})$. Given a point $p=\left(x_{0}, y_{0}\right)$, the set of dual points of lines below $p$ is the set of points in $\mathcal{D}(\mathcal{H})$ located below the line $Y=x_{0} X-y_{0}$, where $X$ and $Y$ correspond to coordinates of the dual plane. Thus, we can compute the level of the point $p$ in the arrangement of $n$ lines by using half-plane range searching. Moreover, we have the highest line below $p$ in the same query time. Also, we can query the number of lines which lie below both of a pair of query points.

A main building block for the range query is the point set decomposition structure of Matoušek [24], which we also need to utilize directly (we only describe its two-dimensional version):

Theorem 2.1 (Matoušek) Given a set $S$ of $n$ points in the plane, for any given $r<n$, we can subdivide $S$ into $r$ disjoint subsets $S_{i}(i=1,2, \ldots, r)$ such that $\left|S_{i}\right| \leq 2 n / r$ satisfying the following condition: Each $S_{i}$ is enclosed in a triangle $\sigma_{i}$, and any line in the plane cuts at most cr $r^{1 / 2}$ triangles among $\sigma_{1}, \sigma_{2}, \ldots, \sigma_{r}$ where $c$ is a constant independent of $n$ and $r$. Such a decomposition can be constructed in $O(n \log n)$ time.

## 3 Computing all peaks in $k$-level

We assume $k \leq n / 2$ for simplicity from now on; if $k>n / 2$, replace $k$ by $n-k$ and exchange maximal and minimal in the statements. A key observation for the $k$-level is that it is a subset of a union of $k$ concave chains such that all concave vertices of the $k$-level are vertices of these concave chains [2]; thus, a $k$-level has at most $k$ maximal peaks and $k-1$ minimal peaks. We want to compute all the local peaks in a given interval $I$ of the $x$-coordinate value.

We apply a version of parametric search paradigm 25, 27. However, before applying the parametric search, we start with a simpler " $k$-branching binary search" method. Without loss of generality, we assume that no line in the arrangement is horizontal nor vertical.

We prepare two key-subroutines: one-shot query and peak counting: Let $\boldsymbol{p}\left(x_{0}\right)$ be the point on the $k$-level at the $x$-coordinate value $x_{0}$. Let $\ell_{k}^{-}\left(x_{0}\right)$ (resp. $\ell_{k}^{+}\left(x_{0}\right)$ ) be the line in the $k$-level at the $x$-coordinate value $x_{0}-\epsilon$ (resp. $x_{0}+\epsilon$ ) for infinitesimally small $\epsilon>0$. If $x_{0}$ is not an $x$-coordinate value of a vertex on the $k$-level, $\ell_{k}^{-}\left(x_{0}\right)=\ell_{k}^{+}\left(x_{0}\right)$. The above operation to compute the point (together with lines containing the point) on $\mathcal{L}_{k}$ at a given $x$-coordinate value is called one-shot query. One-shot query is an analogue of ray shooting $\mathbb{1}]$, and thus the following lemma is basically well-known. The complexity $q(n, m)$ given in the lemma is called one-shot query-time for the $k$-level:

Lemma 3.1 If we preprocess the lines in $\mathcal{H}$ with $O(m)$ time for $n \log n<m<$ $n^{2}$, given an $x$-coordinate value $x_{0}$, we can compute $\boldsymbol{p}\left(x_{0}\right), \ell_{k}^{-}\left(x_{0}\right)$ and $\ell_{k}^{+}\left(x_{0}\right)$ in $q(n, m)=\tilde{O}\left(n / m^{1 / 2}\right)$ time, and also in polylogarithmic time by using $O\left(n / m^{1 / 2}\right)$ processors.

Proof By using the method given in the preliminary section, we can compute the level of any given point $\left(x_{0}, y_{0}\right)$ in polylogarithmic time by using $O\left(n / m^{1 / 2}\right)$ processors. We now apply parametric searching to have the sequential time bound to compute the point $\boldsymbol{p}\left(x_{0}\right)$. A parametric searching algorithm is usually stated as a sequential algorithm; however, it is naturally a parallel algorithm if we use a parallel decision algorithm and also a parallel sorting algorithm. We remark that we can do it easier without using the parametric searching if we examine the range searching method in precise; however, we omit it in this paper.

The peak-counting is a routine to compute the number of peaks of the $k$-level in a given interval $J=\left[x_{0}, x_{1}\right]$ of $x$-coordinate values efficiently. The following elementary lemma is essential:

Lemma 3.2 Let $f\left(x_{0}\right)$ and $f\left(x_{1}\right)$ are numbers of positive-slope lines below or on the $k$-level at $x_{0}$ and $x_{1}$, respectively. Then, the number of maximal peaks of $\mathcal{L}_{k}$ in the interval $J$ is $f\left(x_{0}\right)-f\left(x_{1}\right)$.

Proof At-most- $k$-level (the part of the arrangement below $k+1$-level) is a union of $k$ concave chains such that all concave peaks in the chains appear in the $k$-level [2]. If a concave chain among them has a peak in $J$, the slope of the chain must be changed from positive to negative. Thus, the number of maximal peaks within $J$ is the difference between the numbers of positive slope lines at two endpoints.

We remark that $f\left(x_{0}\right)-f\left(x_{1}\right)$ equals the number of positive slope lines intersecting the segment between $\boldsymbol{p}\left(x_{0}\right)$ and $\boldsymbol{p}\left(x_{1}\right)$ if the segment has a nonpositive slope.

Lemma 3.3 For a given interval $J$ of $x$-coordinate value, the number $\kappa(J)$ of peaks of $\mathcal{L}_{k}$ in $J$ can be computed in $O(q(n, m))$ time if we preprocess the lines with $O(m)$ time. Also, the number of maximal peaks can be computed in $O(q(n, m))$ time.

Proof If we construct the dual of range search data structure for the set of lines with positive slopes, the number of positive slope lines below a given point $\left(x_{0}, y_{0}\right)$ can be computed in $O(q(n, m))$ time. Hence, $f\left(x_{0}\right)$ and $f\left(x_{1}\right)$ can be computed in $O(q(n, m))$ time, and the number of maximal peaks can be computed by using Lemma 3. The number of minimal peaks is easily computed from the number of maximal peaks and slopes of the $k$-level at endpoints.

Now, we can apply a binary search paradigm to design a weakly-polynomial time algorithm. First, for the input interval $I$, we compute the number of peaks $\kappa=\kappa(I)$ within the interval. The time complexity for this initialization is negligible, and obviously $\kappa \leq 2 k-1$. Next, we construct a data structure for the one-shot query in $O(m)$ time, where the choice of $m$ will be explained later. Suppose that coefficients of the equations of lines are quotient numbers of $\log \Gamma$ bit integers. We apply $\kappa$-branching binary search to find all peaks; At each stage of the binary search, we have at most $\kappa$ subintervals which has at least one local peak of $\mathcal{L}_{k}$ (such a subinterval is called an active interval), and we recursively search in active subintervals. Thus, after examining $\kappa \log \Gamma$ candidates of $x$-coordinate values, we can find all of the peaks.

Proposition 3.4 All the local peaks of $\mathcal{I}_{k}$ in the interval I can be computed in $O(\kappa q(n, m) \log \Gamma)$ time.

To make the complexity into strongly polynomial, we apply parametric search by using the parallel algorithm for the one-shot query given in Lemma 3 as its guide algorithm. We run the guide algorithm without inputting the parameter value (in our case, an $x$-coordinate value), and decide the $x$-coordinate values of the peaks by using the sequential one-shot query and the counting algorithm of Lemma 3 as decision algorithms. The counting algorithm needs two values of the parameter, which corresponds to endpoints of each of intervals obtained by splitting active intervals in the parametric searching process. Usually, parametric searching method solves optimization problems on monotone or convex functions. Here, $k$-level is neither monotone nor convex, but it consists of $\kappa$ monotone fragments. Thus, while running the guide algorithm, there are at most $\kappa$ different critical parameter values to determine all the comparisons in the current parallel step that are necessary to proceed into the next parallel step. In precise, the number of different choices is the number of active intervals obtained breaking $I$ by the critical parameter values found so far in the guide algorithm. We make a clone of the guide algorithm for each active interval. If the current interval is split into $f$ active subintervals, $f-1$ new clones are created. Naturally, we create at most $\kappa$ clones in our process. There is only one critical parameter value to determine the comparisons in a "usual" parametric search, and such a value can be found if we run the decision algorithm $O(\log N)$ times if the guide algorithm is a parallel algorithm on $N$ processors. In our case, we run the decision algorithm $O(\kappa \log N)$ times at each level. Thus, we obtain the following theorem:

Theorem 3.5 All the peaks on $\mathcal{L}_{k}$ within an interval $I$ can be computed in $O(n \log n)+\tilde{O}\left((\kappa n)^{2 / 3}\right)$ time if $I$ has $\kappa$ local peaks.

Proof The parametric searching method gives $\tilde{O}(\kappa q(n, m))$ time complexity apart from the $O(m)$ preprocessing time. We balance $\kappa q(n, m)=\tilde{O}\left(\kappa n / m^{1 / 2}\right)$ and $m$ to have $m=(\kappa n)^{2 / 3}$. If $(\kappa n)^{2 / 3}<n$, we instead use $m=n \log n$. This gives the time complexity.

Since $\kappa \leq 2 k$, we have the following:
Corollary 3.6 All the peaks on $\mathcal{L}_{k}$ can be computed in $O(n \log n)+\tilde{O}\left((k n)^{2 / 3}\right)$ time.

### 3.1 Computing selected peaks

When $\kappa$ is large, it may be too expensive to compute all the local peaks. Suppose that we want to compute $\tau$ largest maximal peaks in the input interval $I$ for $\tau \ll \kappa$ more efficiently than computing all the peaks. This can be done by combining Roos and Widmayer's algorithm and the algorithm given above.

We first run a binary search process with respect to $y$-coordinate values similarly to Roos and Widmayer's algorithm. At the intersection of the arrangement with a horizontal line $y=y_{0}$, we compute intervals on the line which are below the $k$-level (we call them semi-active intervals) in $O(n \log n)$ time. We next compute the sum $s\left(y_{0}\right)$ of numbers of local peaks in the semi-active intervals. We could apply our counting method of Lemma 3 for each semi-active intervals to compute the sum of maximal peaks in the intervals by using $O(\tau q(n, m))$ time. More simply, we can compute it in $O(n \log n)$ time by counting the number of intersecting positive slope lines with the horizontal line during the sweep. In precise, we also need information of the arrangement at endpoints of the input interval $I$ if one (or both) of them is below $k$-level (we omit details).

If $s\left(y_{0}\right)$ is not between $\tau$ and $2 \tau$, we continue binary search on $y_{0}$ : If $s\left(y_{0}\right)>$ $2 \tau$, we increase $y_{0}$ while if $s\left(y_{0}\right)<\tau$, we decrease $y_{0}$. Thus, we can eventually find a value $y_{0}$ such that $\tau \leq s\left(y_{0}\right) \leq 2 \tau$. We have spent $O\left(n \log ^{2} n\right)$ time so far. Now, we search all peaks in the union of active intervals by using the method given in the previous section. The following theorem is easy to see:

Theorem 3.7 We can compute $\tau$ largest maximal peaks of $\mathcal{L}_{k}$ in an interval $I$ in $O\left(n \log ^{2} n\right)+\tilde{O}\left((\tau n)^{2 / 3}\right)$ time. We can also compute $\tau$ largest local peaks (including both maximal and minimal peaks) in the same time complexity.

Note that if we only use Roos and Widmayer's algorithm in a naive fashion to find $\tau$ largest peaks, it would cost $O\left(\tau n \log ^{2} n\right)$ time. Analogously, we can compute the $\tau$-smallest minimal peaks.

Theorem 3.8 We can compute $\tau$ smallest minimal peaks of $\mathcal{L}_{k}$ in an interval I in $O\left(n \log ^{2} n\right)+\tilde{O}\left((\tau n)^{2 / 3}\right)$ time. We can also compute $\tau$ smallest local peaks (including both maximal and minimal peaks) in the same time complexity.

For this $\tau$ smallest peak finding problem, we can modify Roos and Widmayer's method [26] to compute them in $O\left(n \log ^{2} n+k \tau \log ^{2} k\right)$ time. Although $n+(\tau n)^{2 / 3} \leq 2(n+k \tau)$ always holds, the time complexity is better than the
one in Theorem 3.1 by a polylogarithmic factor if $n^{1 / 2} \log ^{-c} n<k<n^{1 / 2} \log ^{c} n$ for some constant $c$. Moreover, the algorithm does not use range search data structure, and hence much simpler and uses smaller data space.

Proposition 3.9 The $\tau$ smallest local peaks in an interval $I$ can be found in $O\left(n \log ^{2} n+k \tau \log ^{2} k\right)$ time.

Proof First we search for a horizontal line $h$ such that it intersects $k$-level, and the number of peaks below it is not less than $\tau$. Such a line $h$ can be found in $O\left(n \log ^{2} n\right)$ time. Next, let $v$ and $w$ be the leftmost intersection and the rightmost intersection with the $k$-level on $h$, and let $J$ be the interval between them. If the $k$-level at an endpoint of the input interval $I$ is below $h$, we connect the point on $k$-level at the $x$-value of the endpoint to $J$ with a segment to form a chain $C$ with at most three segments. Let $\mathcal{H}_{0}$ be the set of lines in the arrangement intersecting with the chain $C$. The cardinality of $\mathcal{H}_{0}$ is at most $2 k$ because of Lemma 3. Finally, we find all peaks below $h$ by using the $\tau$-branching binary search. Here, by using the windowing method of 26], we should only take care of lines in $\mathcal{H}_{0}$ together with lines below endpoints of the chain $C$. There are at most $4 k$ such lines. Hence, this second step can be done in $O\left(k \tau \log ^{2} k\right)$ time.

## 4 Bottleneck edge length in a parametric spanning tree

Next, we consider the parametric spanning tree problem. Consider a connected graph $G=(V, E)$ with $k$ nodes and $n$ edges. Because of the connectivity, $k-1 \leq n \leq k(k+1) / 2$. For each edge $e \in E$, we associate a weight function $w_{e}(x)$, which is linear on a parameter $x$. We assume that the arrangement generated by lines $y=w_{e}(x): e \in E$ is simple, i.e., no three lines intersect at a point. We can remove this assumption by giving a symbolic perturbation. $G$ is denoted by $G(x)$ if it is considered as a weighted graph with parametric weights. For a given value $x$, we consider the minimum spanning tree $T(x)$ of $G(x)$.

It is known that the number of transitions of the structure of the minimum spanning tree $T(x)$ is $O\left(k^{1 / 3} n\right)$, and all the transitions can be computed in $O(k n \log n)$ time 15]. Moreover, the average edge weight in the minimum spanning tree is a concave function in $x$, and the value of $x$ maximizing the average edge weight of $T(x)$ can be computed in $O(n \log n)$ time 15].

As parametric matroid problems, the average edge weight is a counterpart of the average of $y$-values of $k$ lines below (or on) the $k$-level. A natural counterpart of the $k$-level itself in the minimum spanning tree is the longest (i.e. maximum weight) edge in the minimum spanning tree. The edge is also called the spanning bottleneck edge at $x(S B E(x)$ in short $)$, and its weight is denoted by $w_{S B E}(x)$. It is easily observed that $W_{S B E}(x)$ is the minimum value of $w$ such that the subgraph of $G(x)$ constructed from the set of edges whose weights are less than or equal to $w$ is connected.

It is natural and important problem in sensitivity analysis 18 to trace the trajectory $y=w_{S B E}(x)$ of the weight of $S B E(x)$. Analogously to the $k$-level, there are at most $k$ maximal peaks in the trajectory $y=w_{S B E}(x)$. We want to compute peaks in the trajectory.

### 4.1 One-shot query for the longest edge in MST

We first consider efficient query for $S B E\left(x_{0}\right)$ at any given value of $x_{0}$ of the parameter. This query is called one-shot query for the $S B E$. A naive method is the following: First construct $T\left(x_{0}\right)$ in $O(n)$ time, and select its longest edge. Instead, we use the Matoušek's set partition. In the dual space, the dual points of $n$ weight functions of the edges are partitioned into $r$ subsets of size $O(n / r)$. Each subset is contained in a triangle, and $O\left(r^{1 / 2}\right)$ triangles are cut by any query line.

Accordingly, we partition the set of $n$ edges into $r$ subsets each has $O(n / r)$ edges. For each subset, we compute a spanning forest (irrelevant to edge weights) and store the connected components except singletons. Thus, each component has a forest with $O(\min \{k, n / r\})$ edges. This computation can be done in $O(n)$ additional time.

If we are given a parameter value $x_{0}$, we sort $O(r)$ vertices of the triangles with respect to the inner product of them with the vector $\left(x_{0}, 1\right)$. We do binary search on this sorting list. We guess a vertex $v$, and consider a line $\ell: Y=$ $x_{0} X+c$ which goes through $v$. We recognize the triangles which are below $\ell$; thus, the edges in the subsets associated with the triangles has weights less than $c$. We construct a spanning forest $F$ of the union of forests in these subsets: since they have $O(r k)$ edges, this can be done in $O(r k)$ time. If the forest $F$ is a spanning tree, we decide that $v$ is too large in the sorting list, and continue the binary search.

Otherwise, we consider the subsets associated with the triangles cut by $\ell$. They contain $O\left(n / r^{1 / 2}\right)$ points in total. We sort them with respect to the weights, and greedily insert them into $F$ until we have a spanning tree. If we do not have a spanning tree, we decide $v$ is too small, and continue the binary search. If we have a spanning tree, we decide $v$ is a candidate, but it may be too large, and continue to search for the lowest vertex $v$ satisfying the above condition, and return the longest edge in the tree for that $v$. Note that the spanning tree is not a minimum spanning tree in general; however, we correctly recognize the longest edge in a minimum spanning tree.

This process needs $O\left(r k+n / r^{1 / 2}\right)$ time, and we do this process $O(\log r)$ times during the binary search. Thus, if we set $r=(n / k)^{2 / 3}$, the time complexity is $O\left(n^{2 / 3} k^{1 / 3} \log (n / k)\right)$, which is slightly better than $O(n)$ if $k=o(n)$. By applying a hierarchical subdivision, we can further improve it: We fist start $r=r_{1}$, and decompose the subset of size $O\left(r_{1} n\right)$ into $r_{2}$ smaller subsets, where $r_{2}=r_{1}^{1 / 2}$, and we further continue the refinement for $r_{i}=r_{i-1}^{1 / 2}$ until $r_{i}$ becomes below a constant. The query time becomes
$k\left(r_{1}+r_{1}^{1 / 2} r_{2}+\ldots+\left(r_{1} r_{2} \ldots r_{i-1}\right)^{1 / 2} r_{i}\right)+n /\left(r_{1} r_{2} \ldots r_{i}\right)^{1 / 2}$. Setting $r_{1}=$
$(n / k)^{1 / 2}$, this enables $\tilde{O}\left((n k)^{1 / 2}\right)$ time computation for $S B E\left(x_{0}\right)$. Similarly to the case of halfspace range searching, we can combine hierarchical cutting [9] of the arrangement to have a preprocess-query trade-off (we omit details in this version). Indeed, we have the following proposition:

Proposition 4.1 If we spend $\tilde{O}(m)$ preprocessing time for $n \leq m$, we can do the one-shot query for SBE in
$\tilde{O}\left(n /(m / k)^{1 / 2}+k\right)$ time.
Moreover, we will later use the following two-shot reporting query for a spanning forest, which reports a spanning forest consisting of edges whose weight functions are below both of given two query points $\left(x_{0}, y_{0}\right)$ and $\left(x_{1}, y_{1}\right)$. This can be done similarly to one-shot query (this is a counterpart of the simplex range searching if the one-shot query is a counterpart of the halfplane range searching).

Proposition 4.2 If we spend $\tilde{O}(m)$ preprocessing time for $n \leq m$, we can do the two-shot reporting query in
$\tilde{O}\left(n /(m / k)^{1 / 2}+k\right)$ time.

### 4.2 Computing the maximum peak

Let us consider the problem of computing the maximum peak in $I$. First, we straightforwardly apply Roos-Widmayer's algorithm. For a given $y$-value $y_{0}$, we want to decide whether $\operatorname{Max}_{x \in I} w_{S B E}(x) \leq y_{0}$ or not. We dynamically update the spanning forest associated with edges with weight below $y_{0}$ from $x=x_{0}$ to $x=x_{1}$ if $I=\left[x_{0}, x_{1}\right]$. If we find a value $x \in I$ such that the spanning forest becomes a spanning tree, we know $\operatorname{Max}_{x \in I} w_{S B E}(x) \leq y_{0}$. It costs $O\left(k^{1 / 2}\right)$ time to update a minimum spanning forest for insertion and deletion of edges. Suppose that we sweep on the line $y=y_{0}$ updating the minimum spanning forest. The line $y=y_{0}$ has at most $n$ intersections with lines associated with weight functions, and hence the method needs $O\left(n k^{1 / 2}\right)$ time for the decision. Thus, the maximum peak can be found in $O\left(n k^{1 / 2} \log n\right)$ time.

We try to improve the above time complexity. We subdivide the line $y=$ $y_{0}$ into $\lceil n / s\rceil$ intervals such that each interval contains at most $s$ intersection points. For each interval $I_{i}=\left[x_{i}, x_{i+1}\right]$, we perform the two-shot reporting query at $\left(x_{i}, y_{0}\right)$ and $\left(x_{i+1}, y_{0}\right)$. The reported forest $F$ is constructed from edges whose weight is less than $y_{0}$ both at $x=x_{i}$ and $x=x_{i+1}$. If the forest has more than $s+1$ connected components, it is impossible that $w_{S B E}(t) \leq y_{0}$ for a $t \in I_{i}$. Otherwise, we dynamically maintain the spanning tree, where we contract nodes into at most $s+1$ super nodes each of which associate with a connected component of the forest $F$. Our graph has only $s$ edges, and hence the update can be done $O(\sqrt{s})$ time per intersection.

Hence, total time complexity becomes
$\tilde{O}\left(n \sqrt{s}+(n / s)\left[n /(m / k)^{1 / 2}+k\right]+m\right)$.
If we optimize this, we have $\tilde{O}\left(n^{8 / 7} k^{1 / 7}+n k^{1 / 3}\right)$. This is an improvement over $O\left(n k^{1 / 2}\right)$, since $n \leq k(k+1) / 2$.

The minimum of $w_{S B E}(x)$ can be analogously computed. Hence, we have the following theorem:

Theorem 4.3 For a given interval I of the parameter value $x$, we can compute both the maximum and the minimum of $w_{S B E}(x)$ for $x \in I$ in $\tilde{O}\left(n^{8 / 7} k^{1 / 7}+\right.$ $n k^{1 / 3}$ ) time.

We can generalize the above theorem for the truncated matroid of the graphic matroid to obtain the following proposition (we omit the proof):

Proposition 4.4 For a constant $c$, let $w_{S B E-c}(x)$ be the minimum value of $w$ such that the set of edges in $G(x)$ with weights less than or equal to $w$ has at most c connected components. We can compute both the maximum and the minimum of $w_{S B E-c}(x)$ in $\tilde{O}\left(n^{8 / 7} k^{1 / 7}+n k^{1 / 3}\right)$ time.

### 4.3 Computing all local peaks for $S B E$

It is desired to apply the method of computing all peaks in the $k$-level to $S B E$ in a parametric graph. It is known that there are at most $k$ maximal peaks in $y=w_{S B E}(x)$. Unfortunately, we have no good method to know the number of peaks within an interval $I=\left[x_{0}, x_{1}\right]$ exactly, since we do not have a property that is a counterpart of Lemma 3. The only known method for the authors is to compute the minimum spanning trees $T\left(x_{0}\right)$ and $T\left(x_{1}\right)$ explicitly, and compute the difference $d(I)$ between the number of edges whose weight functions are positive slopes. For any disjoint set of intervals, the sum of $d(I)$ over the intervals is at most $k$, and $d(I)$ gives an upper bound of maximal peaks of $y=w_{S B E}(x)$ within $I$. However, it is often an overestimate, since $d(I)$ gives the number of maximal peaks of $k$ trajectories of weights of all edges (not only maximum one) in the parametric minimum spanning tree, where we only include peaks where an edge in the spanning tree is replaced by an edge outside the spanning tree.

By using the above observation, we have an $\tilde{O}(k f(n, k))$ time algorithm if we have an algorithm to compute $T\left(x_{0}\right)$ for a given $x_{0}$ in $O(f(n, k))$ time. Unfortunately, we only have $f(n, k)=O(n)$, which leads to an $\tilde{O}(k n)$ time algorithm, which is inferior to a known algorithm to compute all transitions of the parametric minimum spanning tree. We can compute $\tau$ largest peaks in transitions of edge weights in MST in $\tilde{O}(\tau n)$ time, if we include all peaks of all edges in MST; however, the number of peaks appeared at the transitions of the longest edge among them may be much smaller than $\tau$.

Although the above method is not attractive for the minimum spanning tree, the method is applicable to any parametric matroid, and hence it is useful if we do not have a dynamic algorithm to maintain a minimum weight base. The current $O(k n \log n)$ time algorithm to compute the transitions of parametric minimum spanning tree needs $O\left(k^{2 / 3}\right)$ time method (indeed, it can be done in $O\left(k^{1 / 2}\right)$ time) to update a minimum spanning tree. Thus, the analogue of the algorithm needs $\tilde{O}\left(k^{1 / 3} n q+k n\right)$ time to compute all the transitions of a parametric matroid of rank $k$, where $q$ is the time complexity to update its minimum weight base.

## 5 Concluding remarks

The number of maximal peaks in a $k$-level is known to be at most ${ }_{k} C_{d}$ (number of combinations choosing $d$ elements from $k$ elements) if we have $d$ dimensions [10]. Hence, this is much smaller than the complexity of whole arrangement, especially if $k$ is much smaller than $n$. However, to the author's knowledge, the problem of computing peaks in the $k$-level for a higher dimensional arrangement is open. Although the algorithm of Roos and Widmayer can be applied to 3dimensional case, it needs $\tilde{O}\left(n^{2}\right)$ time to compute the largest peak (i.e. global maximum) if we naively implement it. One necessary constituent is to develop a counterpart of Lemma 3: Given an arrangement of $n$ hyperplanes in the threedimensional space, preprocess it, and for any given three points $A, B$, and $C$ in the plane $z=0$, decide whether the triangle $A B C$ contains (a projection) of a peak in the $k$-level or not efficiently. For the purpose, we probably need a counterpart of Lemma 3: Give a criterion of the existence of a peak from the information of the set of hyperplanes below $k$-level at each of $A, B$, and $C$. In two-dimensional space, the lines are classified into positive slope lines and negative slope lines, while this kind of natural discrete classification of planes in the space does not exist. Moreover, it is difficult to find a counterpart of concave chain decomposition for three dimensional $k$-level 22]. These lacks make the problem difficult, although the authors think it is quite attractive.

Another interesting problem is an an extension of parametric $S B E$ problem to the $c$-edge-connectedness for $c \geq 2$. Here, we hope we can develop efficient solutions by combining geomertic methods and graph theoretical methods 14 , 16.

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