



EARLY NUMERICAL ANALYSIS IN THE UNITED KINGDOM

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1. Introduction

Rumour has it that the term "Numerical Analysis" was coined sometime in the late nineteen forties by the numerical statistician J.H. Curtiss at the National Bureau of Standards in Washington, D.C., the NBS being effectively the American National Physical Laboratory, the English version of which I shall mention a bit later. That sort of date makes numerical analysis a rather new subject, and in fact I lived through quite a lot of the early history in the UK. But in some respects the subject has a quite long history, and these I shall mention briefly.

Throughout history individuals have wanted numerical solutions for simple problems like the volume of a rectangular solid with given sides to very complicated matters like the determination of the position in space at a particular time of a vehicle launched from a specified point on earth. Such individuals are not really numerical analysts, and I think of them as engineers or scientists. But then there are others, perhaps of a more mathematical bent, who decide that they can help the scientists in general rather than in particularly numerical contexts. These are the people I do think of as numerical analysts, and indeed in the early days, before numerical analysis became a topic in a mathematics degree, at least in the U.K. assistance to the scientists was a very important motivation for their work. One of the earliest such operations, which continued for many years, was the construction and publication of mathematical tables.

2. Table making

When all arithmetic was done by pencil and paper, multiplication and division, at least, were tedious and time-consuming operations. To ease this some early mathematical tables were produced which gave the results of multiplying any number say up to four figures by any other such number. Allied tables of reciprocals helped with a corresponding division operation to a certain level of accuracy which self-respecting

tables would discuss in a suitable introduction. The invention of logarithms more or less eliminated the need for multiplication and division, and many tables of logarithms were produced by numerical analysts, differing mainly in the selected arguments and the number of figures given in the tables.

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Other functions of integers were found to be useful, and one of the very first books of such tables was first published in 1814 by Barlow, and recast in two more modern editions in 1930 and 1941 by the famous English table-maker L.J. Comrie. The last edition gave n^2 , n^3 , $n^{\frac{1}{2}}$, $(10n)^{\frac{1}{2}}$, $n^{\frac{1}{3}}$ and n^{-1} for $n = 1(1)12500$, with extra attention for reasonably small n for n^4 , $n!$ and $n^{-\frac{1}{2}}$, integer powers up to n^{10} for $n = 1(1)100$ and up to n^{20} for $n = 1(1)10$, binomial coefficients for $n = 1(1)12$, and a list of useful constants. The non-exact numbers have 7,8 or 9 significant figures, with some facilities for interpolation and a relevant description thereof in the introduction.

Barlow's 1814 preface makes interesting reading, the following being part of it, with address The Royal Military Academy, Woolwich (July 1, 1814).

"In presenting the following Mathematical Tables to the attention of the public, the far greater part of which are the result of laborious calculation, little need be said to prove that I have not had in view the accomplishment of any pecuniary object, as the time employed in the computation, the expense of publication, and the limited number of purchases which from the nature of the subject is to be apprehended, preclude any idea of adequate remuneration. And as little is to be expected of mathematical reputation, nothing more being requisite for the execution of such an undertaking than a moderate skill in computation and a persevering industry and attention; which are not precisely the qualifications a mathematician is most anxious to be thought to possess.

"In fact the only motive which prompted me to engage in this unprofitable task was the utility I conceived might result from my labour; and if I have succeeded in facilitating any of the more abstruse arithmetical calculations, and thereby rendered mathematical investigations more pleasant and easy, I have obtained the principle object I had in view."

As applied mathematicians developed their skills their computational problems became increasingly complex, and more advanced mathematical tables were needed and indeed produced. The first group included the trigonometric functions and their inverses, the corresponding treatment of hyperbolic functions, together with the increasing and decreasing exponentials and the logarithmic functions. Common logarithms of these elementary functions were also frequently tabulated for obvious purposes.

The next group included the so-called higher functions of mathematical physics, commonly occurring for example in certain methods of solving partial differential equations. For this and other purposes they included the functions of Bessel, Legendre, etc, gamma and allied functions, Weber parabolic cylinder functions, exponential and logarithmic integrals, elliptic functions, elliptic integrals and many others.

More and more numerical analysis was now needed because the calculation of relevant tabular values was no longer trivial. Moreover, close preliminary attention was needed to the question of what auxiliary function or functions should be tabulated, particularly for the simplification of interpolation in difficult regions. As a simple example consider the tabulation of the exponential integral

$$-Ei(-x) = \int_x^\infty t^{-1} e^{-t} dt. \quad (2.1)$$

For small x we have the series expansion

$$-\text{Ei}(-x) = -\gamma - \ln x + \sum_{n=1}^{\infty} (-1)^{n-1} (x^n / n \cdot n!), \quad (2.2)$$

but the singularity at $x=0$ makes it desirable to tabulate the function $-\text{Ei}(-x) + \ln x$, which is not singular and interpolates nicely. For large x there is the asymptotic expansion

$$-\text{Ei}(-x) \sim \frac{e^{-x}}{x} \left(1 - \frac{1!}{x} + \frac{2!}{x^2} - \dots \right) = \frac{e^{-x}}{x} S(x). \quad (2.3)$$

Here S can be tabulated nicely and conveniently with argument $z=x^{-1}$, and the required quantity is easily recovered.

Of course the ascending series may not be economic for good accuracy for medium-sized x , and the asymptotic series may not give the required accuracy for too small an x , and there may be a middle range in which other methods are desirable if not completely necessary. For example, for the function

$$f(x) = \int_0^{\infty} (u+x)^{-1} e^{-u^2} du \quad (2.4)$$

there are two series corresponding respectively to (2.2) and (2.3), but in a middle range of x it is more convenient to integrate by numerical methods the ordinary differential equation

$$f' + 2xf = \pi^{\frac{1}{2}} - x^{-1}. \quad (2.5)$$

Other frequent computations involved recurrence relations. For example the Bessel function $J_r(x)$, for fixed argument x and variable order r , satisfies the recurrence relation

$$J_{r+1}(x) = \frac{2r}{x} J_r(x) - J_{r-1}(x), \quad (2.6)$$

and to all intents and purposes this can be used to compute successive $J_r(x)$ for integer r , starting say with known values of $J_0(x)$ and $J_1(x)$. The other obvious task was the direct evaluation of definite integrals, and all these various operations had to be performed on desk calculating machines, sometimes with very high accuracy and always as economically as possible.

The final important topic in table-making was the systematic use of finite-difference formulae for checking computed values by inspecting differences, for sub-tabulating them as mechanically as possible to obtain other tabular values very easily, and then for providing accurate and reasonably economic methods for interpolation in the published tables. The sub-tabulation, which is systematic interpolation at a constant fraction of the original interval, usually one-fifth or one-tenth thereof, was performed quite mechanically by machines like the Hollerith punched card machine or the National Accounting Machine.

The interpolation by the user was based on finite-difference formulae typified by the Everett formula

$$f_p = (1-p)f_0 + pf_1 + E_2\delta^2 f_0 + F_2\delta^2 f_1 + E_4\delta^4 f_0 + F_4\delta^4 f_1 + \dots, \quad (2.7)$$

where the δ^2 and δ^4 are central-difference symbols and the E and F functions are simple polynomials in p , the fraction of the (constant) distance between tabular points. Comrie found that the fourth difference could be "thrown back" into the second difference, with the explicit part of (2.7) replaced by

$$f_p = (1-p)f_0 + pf_1 + E_2\delta_m^2 f_0 + F_2\delta_m^2 f_1, \quad \delta_m^2 f = \delta^2 f - 0.184 \delta^4 f, \quad (2.8)$$

and that (2.8) is only very slightly less accurate than (2.7) and clearly much more convenient.

The construction of the more advanced mathematical tables and relevant publishing continued until the nineteen sixties. The main table-making activities were organised by the British Association Mathematics Tables Committee, starting about 1930, and then from 1948 onwards by the Royal Society.

Other early publications included work by Sheppard (1906) on the accuracy of finite-difference interpolation, Bickley (1939, 1941) on formulae for numerical integration and differentiation, Comrie (1931) on "throwback interpolation" and (1936) on mechanical operations with the National Accounting Machine, and Bickley and Miller (1936) and Airey (1937) on the summation of slowly-convergent series. Fletcher, Miller and Rosenhead (1946) published the comprehensive Index of Mathematical Tables. Miller (1949) wrote about table-making in general and on his solution of ordinary differential equations in particular. This he performed with the Taylor-series method, not too difficult when, as often occurred, the relevant differential equations were linear, but Miller thought nothing of using up to twelfth derivatives with a large interval of tabulation.

Miller was probably the dominant member of the relevant British Association and Royal Society committees, and much of his work appeared for the first time in the introductions to the various tables which were written singly or jointly by members of the committees and included quite important numerical analysis. Prominent in this respect is the introduction to B.A. Vol. 10 (1952), which includes the famous Miller algorithm in connexion with the recurrence relation (2.6). Miller quickly realised that the forward recurrence produced increasing inaccuracy as r increased beyond x . He solved this problem by backward recurrence with a replacement of (2.6) given by

$$\bar{J}_{r-1}(x) = \frac{2r}{x} \bar{J}_r(x) - \bar{J}_{r+1}(x), \quad \bar{J}_N(x)=0, \quad \bar{J}_{N-1}(x)=1, \quad (2.9)$$

and then by scaling the computed $\bar{J}_r(x)$ to give for example

$$J_r(x) = k \bar{J}_r(x), \quad k = J_0(x)/\bar{J}_0(x). \quad (2.10)$$

For sufficiently large N this gives very good results, accuracy increasing as r decreases.

Perhaps the final useful publication was the booklet "Interpolation and Allied Tables", developed at H.M. Nautical Almanac Office. It first appeared in 1936 when Comrie was Superintendent. It was reissued at frequent intervals and with amendments and additions until the last appearance in 1956, when D.H. Sadler was superintendent. The original booklet contains finite-difference formulae of all kinds, and the 1942 edition also gave a method for solving ordinary differential equations which actually used central differences with estimation and subsequent correction, in the spirit of more modern predictor-corrector methods. A companion booklet "Subtabulation", published in 1958, gives a comprehensive version of the relevant methods developed over many years in H.M. Nautical Almanac Office.

3. Other early numerical analysis

Apart from table making and the much earlier contributions by Gauss, Newton, Runge and Kutta and Bashforth and Adams, a few other workers, particularly astronomers and theoretical scientists, suggested

numerical methods for both ordinary and partial differential equations and a few other topics. But by 1939, the start of the second world war, there was little in the way of numerical literature and numerical analysis was hardly a mathematical topic. Published in the UK, there were only a few books with a numerical content, such as Brunt (The combination of observations, 1923), Whittaker and Robinson (The calculus of observations, 1924), Steffenson (Interpolation, 1927), Scarborough (Numerical mathematical analysis, 1930), Milne-Thomson (The calculus of finite differences, 1933) and Levy and Baggott (Numerical studies in differential equations, 1934).

Scattered in the journal literature of this period were papers for example by Aitken (1926,1937) on Bernoulli's method for solving algebraic equations and his own δ^2 method for accelerating the convergence of such iterations, Hartree and Womersley (1937) on mathematical and mechanical (differential analyser) methods for the solution of parabolic partial differential equations, and Richardson and Gaunt (1926) on "the deferred approach to the limit" for accelerating the convergence of finer-net approximations to the numerical solution of ordinary differential equations.

The last mentioned method is still in common use, and Richardson, a major figure in this field, also wrote important papers on the solution of partial differential equations. Perhaps the most famous of these is Richardson (1910), and Richardson (1925) gives a short summary of this and other work. The 1910 paper discussed finite-difference methods for what he called "jury" problems given by

$$\nabla^2 \phi = 0, (\nabla^2 + k^2)\phi = 0, \nabla^4 \phi = 0, (\nabla^4 - k^4)\phi = 0 \quad (3.1)$$

with suitable boundary conditions. He postulated a "deferred approach to the limit" rule when central differences are used, not only for the function in all cases but also for the eigenvalues. He solved the finite-difference equations by direct methods if their number was small enough, and otherwise he used an iterative method, now known as Richardson's method.

The following example appears in the 1925 paper. The equations

$$Ax = b, \quad A = \begin{bmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 0 & 1 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} -3 \\ -7 \\ 0 \\ 0 \end{bmatrix} \quad (3.2)$$

obviously relate to a particular member of the first of (3.1), with boundary values on a unit square and with interval $h=\frac{1}{3}$ in both directions. He uses the iteration

$$x^{(r+1)} = x^{(r)} + a_r^{-1} (Ax^{(r)} - b), \quad (3.3)$$

showing in this example that if $x^{(1)} = (1, 2, 0.3, 0.2)^T$, then with $a_1=4$, $a_2=2$, $a_3=6$, the computed $x^{(4)}$ is the exact solution of (3.2). This, of course, follows from the fact that the eigenvalues of A are -4 , -4 , -2 and -6 , but Richardson was aware that the eigenvalues are not usually available. He observed that the largest and smallest can be obtained with rough accuracy, that a single $a_r > \frac{1}{2} |\lambda_{\max}|$ in (3.3) will produce ultimate convergence, but that "it saves time to spread out the values of the a_r over the range

covered by the eigenvalues."

This was a remarkable piece of work, on which Golub and Varga and others did more research in the nineteen-fifties with the name "semi-iterative method". The paper has many other interesting sections which suggest other things about finite differences, what to do for example near boundaries which are not rectangular, and the importance of a non-dimensional treatment of the problem prior to computation. He also considered the parabolic problem

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}, \quad (3.4)$$

with appropriate boundary conditions, but his suggested

$$\phi_{r,s+1} = \phi_{r,s-1} + 2 \frac{\Delta t}{(\Delta x)^2} (\phi_{r+1,s} - 2\phi_{r,s} + \phi_{r-1,s}) \quad (3.5)$$

is now known to be unstable.

In passing it is interesting to note that it was this pre-war numerical analysis which was mainly examined in postgraduate courses in the subject, courses which did not start seriously until the early nineteen-fifties. They were usually organised by the Computing Laboratory rather than by the Mathematics Department, and one of the very first was the Cambridge Diploma in Numerical Analysis and Automatic Computing. At the start this had one theoretical paper and one practical paper on numerical analysis and one paper on the hardware and software of the new stored-program computers. The two numerical papers of the first examination in 1954 reveal that the material so far discussed is very well represented. This is perhaps not surprising since Miller was the dominant force in this part of the Diploma, but later diplomas had very little more variety. Even by 1959 the theoretical paper at Cambridge had three questions on interpolation, two on quadrature, one on the Taylor-series method for a particular (non-linear) ordinary differential equation, one on Richardson's method for elliptic equations, one on Aitken and other iteration topics, and one on three methods for the eigenvalues of symmetric matrices of small and large order with comparison of desk machines and automatic computation.

4. War-time groups

(i) Relaxation at Oxford

In 1939 I had just started my D.Phil research at Oxford with R.V. Southwell, who had told my tutor that he needed a mathematician to work on extensions of his "relaxation method". In the early thirties he had invented what was originally called the method of "systematic relaxation of constraints" for solving problems of loaded frameworks, and in the decade 1932 - 42 he had a regular group of research students at Oxford working on these problems and somewhat similar problems in the finite-difference solution of elliptic partial differential equations. From 1939 onwards arrangements in the second world war caused applied and even pure mathematicians to work on military problems with whatever techniques they had available, but the Oxford group was one of the first of these and, in particular, the name "relaxation", if not the original method, has carried over to quite modern techniques for relevant problems.

Two papers by Southwell in 1935 described the method for frameworks. Basically this used iteration

to solve the linear equations

$$Ax = b, \quad (4.1)$$

x being the vector of displacements and b of the forces at the joints of the framework. The matrix A was sparse and generally diagonally dominant. Southwell considered in an engineering sense not only the problem but also its method of solution. He postulated a system of "constraints" at the joints which could bear the forces without allowing any displacements. Then, usually selecting the joint with the currently largest force, he permitted a displacement at this point by "relaxing the constraint", wholly or partially at this stage so that at this joint the framework was now bearing all or at least some of its force. This also changed the forces at other joints, in an easily calculable manner, and by systematically "relaxing the constraints" (the word "systematically" originally meaning "in descending order of magnitude of forces still borne by the constraints") he expected on engineering principles that the process would converge. In other words the residual forces still borne by the constraints, components of the residual vector

$$r^{(n)} = Ax^{(n)} - b \quad (4.2)$$

at stage n of the iteration, would systematically be reduced to zero or to very small quantities as n increases. In fact Southwell contemplated the acceptance of any solution for which the residual forces were less than some "engineering fraction" of the original forces, since the latter are quite unlikely to be known very accurately.

Now if at joint s the residual force r_s is reduced temporarily to zero by a change in the displacement x_s at that joint, then this is one step of Gauss-Seidel iteration, and indeed for some problems this method had already been used by other workers. But Southwell concentrated on the residuals, which were actually recorded at every joint, and he and his research students used a variety of methods to reduce them sensibly to zero. The following simple examples illustrate some of these methods.

First we solve a one-dimensional problem with equations

$$f_{r+1} - 2f_r + f_{r-1} = b_r, \quad f_0=100, \quad f_5=-1000, \quad b_1=20, \quad b_2=80, \quad b_3=-40, \quad b_4=600, \quad (4.3)$$

the selected values of b_r and f_0 and f_5 being effectively arbitrary numbers. Suppose that we start with the guess $f_1 = f_2 = f_3 = f_4 = 0$, so that the first relevant picture is that of Figure 1, in which the current f values are to the left and the current residuals to the right of the "nodal lines". The first residuals are just the $-b_r$ at $r=2$ and 3 , and $-b_1$ and $-b_4$ plus the respective contributions from the specified boundary values at the two nodes next to the boundaries.

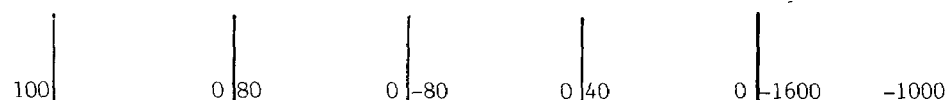


Figure 1

In the relaxation process we use the same diagram throughout, recording additions to the displacements

C	100	B	100	B	100	C	100
B	100	A	100	A	100	B	100
B	100	A	100	A	100	B	100
C	100	B	100	B	100	C	100

Figure 4

We now use the word "point" instead of "joint" since in Southwell's language the framework had become a tensioned net, and "net point" became the accepted terminology. Other useful words were "overrelax", deliberately to change the sign of the relevant residual(s) when adjacent points have residuals of the same sign and have a "wash-back" effect, and "underrelax" in regions in which the signs of residuals alternate. These words are still used in modern methods but with rather different applications. We also note with respect to (4.6) (and indeed also with respect to (4.3)) that the algebraic sum of residuals is unchanged unless a displacement is made at one or more points next to the boundary, so that residuals should be "swept" from the centre of the region towards the boundaries rather than in the reverse direction. A useful first step is to use a complete block operation which reduces the algebraic sum of residuals virtually to zero.

There is much symmetry in Figure 4, and indeed there are only three independent values, respectively at points marked A, B and C. Table 1 gives a list of operations and the resulting residuals, a displacement at A meaning the same displacement at all A points in Figure 4, and similarly for B and C.

Table 1

Operation	Displacement changes			Current residuals		
	A	B	C	A	B	C
(i)	100	100	100	100	0	-100
(ii)	70			-40	70	-100
(iii)		20		0	10	-60
(iv)			-16	0	-6	4
(v)	-3	-3		0	0	-2

Operation (i) reduces the sum of residuals to zero, and the remaining operations would be understood quite easily by any competent and experienced operator. Notice that simple numbers are used throughout, with no useless early attempts to make any residuals exactly zero. The fact that at the end of Table 1 there are only zero or negative residuals tells us immediately that all the values are too large, that of C perhaps especially. But a complete extra block of -1 would leave residuals of 0, 1 and 0 for A, B and C, so that every value would then be slightly too small. Table 1 gives A=167, B=117, C=84, the exact values being $166\frac{2}{3}$, $116\frac{2}{3}$, $83\frac{1}{3}$. This table, of course, would nowhere be recorded, and all the operations would be performed on a single sheet of paper, with perhaps only one-eighth of Figure 4, by an experienced operator who takes the symmetries in his or her stride. Figure 5 shows all that is needed.

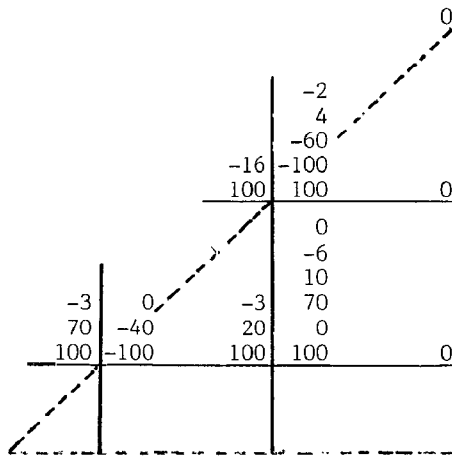


Figure 5

We learnt a lot about the "condition" of various problems, measured by the size of the displacements needed to liquidate sets of residuals. The condition of course worsens as the interval is reduced, but more to the point is the fact that the biharmonic equations are much more ill-conditioned than the Laplace equations. As the condition worsens the need for significant overrelaxation increases. A good starting approximation, of course, helped considerably with the convergence, and in an engineering background some workers could envisage pictorially and really quite accurately the nature of the correct solution. We simplified the use of a finer mesh, first by interpolating quite accurately or, where possible, using the differential equation to get a good start at the finer net points, and then by a process which now has the name "multi-grid". Here any oscillation in residual signs was removed by a few simple point relaxations, and by taking averages over small relative regions of the resulting one-signed residuals these could effectively be transferred back to the original net, and liquidated there with easier relaxation. The results were then transferred back to the finer mesh, and one further interpolation and a relatively trivial amount of fine net relaxation produced the required results quite quickly. Unlike modern multi-grid methods we never used more than one coarser mesh for this purpose.

The latter technique did not obtain written publicity, but most of the useful devices appear in the book by Allen (1954). This, together with Southwell's last books (1946, 1956) also give a full account of the problems solved by relaxation, some non-linear, some involving eigenvalues, some with boundaries of initially unknown position, some in three dimensions and some with parabolic and hyperbolic systems. The eigenvalue techniques were rather interesting. Normally a guess at the eigenfunction gave a starting estimate of the eigenvalue with the use of Rayleigh's principle, and some relaxation was then performed. This cannot proceed too far because the equations do not have a solution at this stage, and a favourite trick was to try to arrange for displacements which made the residual at each point reasonably proportional to its displacement. The computation of a new eigenvalue estimate then gives better results and a good start for further operations. When this was very difficult a method called intensification was used, which turns out to be just the method of inverse iteration for

$$(A - \lambda I)x = 0, \quad (4.6)$$

given by

$$Ax^{(r+1)} = x^{(r)}. \quad (4.7)$$

Occasionally the operator $A - kI$ might be used in (4.7), not so much to increase the rate of convergence as to simplify the relaxation solution of the linear equations.

One final comment on the relaxation method is essential. The success of the method (and it was successful even with the meagre computing equipment then available), depended significantly on the ability of the human eye and brain very quickly to pick out the largest of a sequence of numbers or a cluster of such numbers, to recognise patterns of numbers and to forecast the overall effects of relaxation operations. In fact it was rather like a game of chess, and I return briefly to this point a little later.

(ii) Admiralty Computing Service

In 1943 I joined the new Admiralty Computing Service at Bath, probably the first group with the words "Computing Service" in its title. It was headed by D.H. Sadler, who was Comrie's successor as Superintendent of the Nautical Almanac Office, and it had as consultants Miller, Erdelyi and John Todd. Its workers also included E.T. Goodwin, F.W.J. Olver and H.H. Robertson, whose names are well known in the literature of numerical analysis. We solved a fair number of problems for the Admiralty, we learnt a lot about the numerical methods of Miller and Sadler, and I extended my knowledge of and capabilities with the relaxation method. Some problems were written up as reports for "Department of Scientific Research and Experiment - Admiralty Computing Service", mainly in 1945, and listed in the references are two of the problems which have particular interest for me.

The first is the evaluation of the two-variable function

$$f(x,y) = \int_0^\infty e^{-k} (J_0(kx) \cosh(ky) - 1) \operatorname{cosech}(k) dk \quad (4.8)$$

at the points $x = 0(0.1)5.0$, $y = 0(0.1)1.0$. This is how the problem was presented, but we discovered that $f(x,y)$ satisfies the elliptic equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{1}{x} \frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial y^2} = 0, \quad (4.9)$$

and that boundary values can be calculated with some interesting numerical analysis giving quite rapid techniques. I then solved the problem by relaxation methods, and the first point of interest is that this is the first publication of my use of the "difference correction" for correcting a first approximate solution on the same finite-difference mesh. (The year in which these computations were performed was either 1943 or 1944). The second interesting point is that in this very early problem (4.9) was known originally, but a mathematician deduced (4.8) quite cleverly but without knowing that a direct treatment of (4.9) is here computationally preferable. Just how much early mathematics is valuable in numerical work has always been a matter of some speculation and dispute!

The other interesting problem was the solution by Goodwin and myself of a Volterra integral equation of the first kind, with a mixture of Laplace transforms, Taylor's series, and direct numerical solution of a corresponding second-order equation in which the trapezoidal rule had attached to it several correc-

ting expressions from Gregory's quadrature formula. Here Sadler was a source of great strength, with a wealth of finite-difference knowledge of the kind contained in 'Interpolation and Allied Tables'. He published very little himself, but he was always able and willing (if not determined) to make suggestions about methods which were almost always exceedingly useful, and he had a genius for spotting errors in our computation. He insisted that all this should be done on good paper, in ink, and he delighted to peer frequently over our shoulders and triumphantly note an error before we had wasted too much sequential time!

5. Mathematics Division, National Physical Laboratory

The Admiralty Computing Service was quite successful, and its success was one of the main reasons for the setting up in 1945 of the Mathematics Division, a new division of the National Physical Laboratory. Goodwin, Fox, Olver and Robertson went in 1945 from Bath to the NPL at Teddington, J.H. Wilkinson joined in the following year, and roughly at that time we also recruited Clenshaw and Gill and Hayes and a number of others with perhaps less well-known names. The famous Turing came to contemplate building his version of the new idea of stored-program computers, and we had quite a number of junior workers on desk machines and punched-card machines. We also acquired from Germany a large differential analyser. Our duties were to help other divisions of NPL with their 'mathematical and computational' problems, to do the same for other stations of the current version of the Department of Scientific and Industrial Research, and indeed for many other government or government-type laboratories, and above all to engage in research in the theory and practice of numerical computation.

About this time there were other small groups in other government and government-type laboratories, and at several universities, particularly Cambridge and Manchester, who were also working on computer construction and use. Comrie had formed the London Scientific Computing Service, which Miller joined, but as far as the mathematics of numerical analysis was concerned the NPL group was by far the largest and the most experienced. There is no doubt that what you would call the more modern numerical analysis in the UK started with this group which, indeed, was dominant in our numerical work for at least 30 years.

The history of the NPL work has two parts, the smaller for a decade or so until the mid-fifties, in a period in which the new computer was not generally available, and the larger after the appearance of the lusty Pilot ACE computer in a form from which useful computations could be obtained. My history virtually ends with the first of these parts, in which much useful research was still performed. I mention in what follows a few of the topics and resulting publications.

Table-making continued, and indeed NPL started its own series of mathematical tables, a project for which Fox (1956) wrote a lengthy Vol. 1 which extended much of the Chebyshev theory of Lanczos and Miller for interpolation and other relevant formulae. Work on ordinary differential equations produced papers by Fox (1947, 1949), Fox and Goodwin (1949), Gill (1951, in which the effect of the new computer was already foreshadowed), and Clenshaw and Olver (1951). Clenshaw (1954, 1955, 1957) started important work on Chebyshev methods for ordinary differential equations, and Olver, after a comprehensive paper on computing the zeros of polynomials (1952), collaborated with Clenshaw (1955) on the use of economized polynomials in

mathematical tables. Fox and Goodwin (1953) continued their ACS work on integral equations with a comprehensive account of finite-difference methods for both Volterra and Fredholm equations, and Goodwin and Staton (1948) and Goodwin (1949) added to earlier work on methods for evaluating particular integrals. There was some curve-fitting by Hayes and Vickers (1951), and a little linear algebra by Fox, Huskey and Wilkinson (1948), Goodwin (1950), Fox and Hayes (1951), Fox (1950a, 1954), but the main papers for the stimulation of future work in this area came from Turing (1948) and Wilkinson (1954a, b). We did little on partial differential equations except papers on further relaxation by Fox (1947, 1950b), including the difference-correction method. Some independent workers, however, contributed significantly in this field, including of course Crank (1956) and Crank and Nicolson (1947), which produced one of the very useful stable methods for parabolic equations; and Motz (1946) and Woods (1953) did useful work on singularities in elliptic problems. Singularities in some integral equations were also treated by Young (1954).

The NPL group joined together to produce the book 'Modern Computing Methods' (1957, second edition 1961), which includes an extensive bibliography. This was one of the first quite modern books on numerical analysis, somewhat more up-to-date at that time than the very readable 'Numerical Analysis' by Hartree (1952). My book (1957) on "The numerical solution of two-point boundary problems in ordinary differential equations" put into print the work started some fifteen years earlier on relaxation methods and the "difference-correction" method. This, again, must be one of the earliest books on this topic.

And that is really the end of the "Early Numerical Analysis" story. In the middle nineteen-fifties and onwards there was a flood of books and papers on numerical analysis of all kinds and from many places, largely stimulated by the development of the stored-program computer. The NPL contribution to this feast was supplied very largely by J.H. Wilkinson. His third relevant paper (1955) was merely the first of a series which for the next thirty years transformed both the theory and the practice of virtually all problems in numerical linear algebra.

But that is another story. I end the current story by making a few comments on the effect of the new computing machine on our earlier work. First, in 1958 at a meeting of the Royal Society Mathematical Tables Committee, the chairman M.V. Wilkes raised the question of its role in the new computer world. This led to considerable and lengthy debate, but the extent of table-making decreased quite rapidly and the committee virtually ceased to exist around 1965. Second, the old relaxation methods were never used in the same spirit with the new computers. For the latter did not match the human eye and brain in picking out relatively quickly the largest of a sequence of numbers, or recognise useful patterns, and the new relaxation method developed by David Young and others worked in a virtually completely systematic way. This, of course, led to some useful and very interesting mathematical theories, but the modern method bears only slight relation to the original relaxation concept.

My "difference-correction" method for differential and integral equations of all kinds was also treated afresh by V. Pereyra and others, and they also made some changes, though perhaps not quite so violent as those of the relaxation story. For example, for the two-point boundary problem

$$y'' + f(x)y' + g(x)y = k(x), \quad y(a) = \alpha, \quad y(b) = \beta, \quad (5.1)$$

I replaced the differential equation by the recurrence relation

$$(1 - \frac{1}{2}hf_r)y_{r-1} - (2-h^2g_r)y_r + (1 + \frac{1}{2}hf_r)y_{r+1} = c(y_r), \quad y(a) = \alpha, \quad y(b) = \beta, \quad (5.2)$$

where $c(y_r)$ is the difference-correction at mesh point x_r which I expressed in terms of central differences, here involving third, fourth and higher-order differences. I proposed to solve (5.2) iteratively in the form

$$(1 - \frac{1}{2}hf_r)y_{r-1}^{(n+1)} - (2-h^2g_r)y_r^{(n+1)} + (1 + \frac{1}{2}hf_r)y_{r+1}^{(n+1)} = c(y_r^{(n)}), \quad c(y_r^{(0)}) = 0, \quad (5.3)$$

a device very similar to the modern use of "iterative refinement" for simultaneous linear algebraic equations. I inspected the differences of $y_r^{(1)}$ to discover what orders of differences at this interval made contributions to $c(y_r^{(1)})$ for the required accuracy, whether from this point of view the interval length was satisfactory and, really quite accurately, how the interval should be changed for this purpose. All further calculations were performed at this "satisfactory" interval, starting with a new $y_r^{(1)}$ and continuing with the iterative sequence. Using only the differences at every stage which were expected to contribute to $c(y_r)$ I performed the iteration as many times as needed to reach consistency in the computed results. Some external values had to be computed and even "corrected" to produce the central differences near boundary points.

Pereyra, however, showed that whereas $y_r^{(1)}$ has global error $O(h^2)$, $y_r^{(2)}$ has global error $O(h^4)$ if $c(y_r^{(1)})$ uses only third and fourth differences, and $y_r^{(3)}$ has global error $O(h^6)$ if $c(y_r^{(2)})$ is computed using only up to sixth differences. Normally the number of differences to be used finally would be decided before the computation started, and if consistency had not been reached at this stage the process would be repeated at a smaller interval. I am not clear what the present position is, but in the early routines external values were not computed and forward or backward differences were used for at least some y_r in $c(y_r)$. Again, this new theory is very important, but the method has undoubtedly changed, at least to some extent including the fact that, as with initial value problems, (5.1) is now likely to be treated as simultaneous first-order equations with the trapezoidal rule.

Finally, the new computers were so powerful that they quickly put an effective end to the use of analogue equipment like the differential analyser for the solution of partial differential equations, and other various pieces of equipment for various problems in which the data and answers were measured by physical quantities like length, voltage, current and so on. Another analogue device was the construction of alignment nomograms which up to this time had been a regular feature of problem solving of certain kinds and had developed quite a literature.

The basic idea of the alignment nomogram can be demonstrated by a very simple example, the solution of the quadratic equation

$$a^2 + pa + q = 0, \quad (5.4)$$

which has three variables a , p and q . The nomogram depends on our ability to express (5.4) in the deter-

minantal form

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0, \quad (5.5)$$

which represents the condition that in two dimensions the points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) are collinear. For (5.4) there are various such possibilities, of which one is given by

$$\begin{vmatrix} p^{-1} & 0 & 1 \\ 0 & q^{-1} & 1 \\ -a^{-1} & -a^{-2} & 1 \end{vmatrix} = 0. \quad (5.6)$$

The corresponding nomogram has the scale $x = p^{-1}$ on $y=0$, $y = q^{-1}$ on $x=0$, and an a -scale on the parabolic curve $x = -a^{-1}$, $y = -a^{-2}$. A line joining a "p-point" to a "q-point" intersects the "a-curve" at two, one or no points, these points giving the real roots of the quadratic equation.

We constructed a fair number of nomograms in the early nineteen-fifties, one of them I discovered from forgotten notes relating to the equation

$$\left(\frac{R^2}{AQ^2}\right)^{\frac{1}{3}} = K_T^{\frac{1}{3}} e^{-1 \cdot 3R/V} - K_O^{\frac{1}{3}}, \quad (5.7)$$

where we wanted R from specified A , K_T , V , and Q and K_O which were given functions of a fifth parameter β_S . With five parameters s , t , u , v , w we need to be able to produce a determinant like

$$\begin{vmatrix} f(s,t) & g(s,t) & 1 \\ p(u,v) & q(u,v) & 1 \\ \lambda(s) & \mu(s) & 1 \end{vmatrix} = 0, \quad (5.8)$$

to provide an alignment nomogram. I failed to do this for (5.7), but introduced two other parameters

$$\alpha = A^{\frac{1}{3}} Q^{\frac{2}{3}} K_O^{\frac{1}{3}} = A^{\frac{1}{3}} F(\beta_S), \quad \beta = K_O^{\frac{1}{3}} K_T^{-\frac{1}{3}}, \quad (5.9)$$

leading to the determinants

$$\begin{vmatrix} \alpha & 0 & 1 \\ -A^{\frac{1}{3}} & 1 & 1 \\ 0 & f(1+f)^{-1} & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & \beta & 1 \\ 1 & -K_O^{\frac{1}{3}} & 1 \\ (1+K_T)^{-\frac{1}{3}} & 0 & 1 \end{vmatrix} = 0, \quad (5.10)$$

and the ability to find α and β from given A , K_T and β_S . Equation (5.7) can then be expressed in the form

$$R^{\frac{2}{3}} \alpha^{-1} = e^{-1 \cdot 3R/V} \beta^{-1} - 1, \quad (5.11)$$

with the relevant determinantal equation

$$\begin{vmatrix} \alpha & 0 & 1 \\ 0 & \beta & 1 \\ -R^{\frac{2}{3}} & e^{-1 \cdot 3R/V} & 1 \end{vmatrix} = 0 \quad (5.12)$$

which permits the determination of R from given α , β , and V.

The accuracy obtainable depends upon various things including the determination of appropriate scaling factors for the variables, and the literature gave this some close attention. At NPL my colleague J.G.L. Michel was our "analogue" expert, both with the differential analyser and with nomography, and he joined the other authors in producing the fourth edition of a very good book on the subject by Alcock et al (1950).

Since that time I have heard no more about nomography, but of course it is quite proper that old methods should be reviewed, readapted and if necessary discarded when new equipment becomes available, and this is one of the important ways in which numerical analysis continues to make good progress in its initial task of helping the scientists in their work.

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