Toughness and Delaunay Triangulations

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Abstract

We show that nondegenerate Delaunay triangulations satisfy a combinatorial property called 1-toughness. A graph with set of sites S is 1-tough if for any set $P \subseteq S$, $c(S-P) \leq |S|$, where $c(S-P)$ is the number of components of the subgraph induced by the complement of P and $|P|$ is the number of sites in P. We also show that, under the same conditions, the number of interior components of $S-P$ is at most $|P|-2$. These appear to be the first nontrivial properties of a purely combinatorial nature to be established for Delaunay triangulations. We give examples to show that these bounds can be attained, and we state and prove several corollaries. In particular, we show that maximal planar graphs inscribable in a sphere are 1-tough.

1 Introduction

The connection between Delaunay triangulations and Hamiltonian graphs has been a question of some interest. In his thesis, M. I. Shamos posed a variant of the question by asking whether every Delaunay triangulation contained a traveling salesman cycle for its sites [Sham78]. The answer to this question was shown to be negative in [Dill86a].

More recently, the question of when Delaunay triangulations are Hamiltonian has arisen in the contexts of pattern recognition and shape representation. Consider the problem of constructing a "reasonable" simple curve through a given planar set of points. One approach that has been suggested is to construct the Delaunay triangulation of the points, and then to construct a cycle through this triangulation either by "growing" a single triangle [ORou84] or by "sculpting" the convex hull [Bois84]. Clearly, these algorithms will be successful only if the Delaunay triangulation has a Hamiltonian cycle.

It has been shown [Dill86b] that not all nondegenerate Delaunay triangulations are Hamiltonian. (A degenerate example is in [Kant83]). However, O'Rourke and Boissonat both report that their algorithms appear to work in practice. In fact, Boissonat has run a number of simulations with randomly generated point sets containing up to 2000 points, and all his examples have yielded Hamiltonian Delaunay triangulations [Bois86]. Thus there is evidence that Delaunay triangulations are Hamiltonian with high probability.

In this paper, we establish two results that may partially explain this phenomenon. These results appear to be the first nontrivial properties of a purely combinatorial nature to be established for Delaunay triangulations. In particular, we show that nondegenerate Delaunay triangulations enjoy a property called *1-toughness* [Chva73]. A graph is 1-tough if, for any k, removing k points splits the graph into at most k components. It is easy to show that any Hamiltonian graph is 1-tough, although the converse is not true. The connection between Hamiltonicity and toughness is discussed in [Chva73] and [Berm78].

An interesting consequence of the 1-toughness of Delaunay triangulation is the following fact. Suppose, in a distributed computing system, each node is connected to its Voronoï neighbors. Then if k of the nodes fail at once, there are at most k groups of remaining nodes that are completely isolated from one another. It is not hard to show that, for a general (i.e., non-Delaunay) triangulation, there can be as many as $2k-2$ such groups.

Related to the 1-toughness result is another theorem, also proved in Section 3, which says that removing k sites can split a nondegenerate Delaunay triangulation into at most $k-2$ components that do not contain a point of the boundary of the triangulation. In Section 4, we show, using the the transformations of [Brow79] and [EdSe85], that maximal planar graphs inscribable in a sphere or in the paraboloid $z = x^2 + y^2$ are 1-tough. These results are relevant to a problem that is currently open, namely determining whether all graphs so inscribable are Hamiltonian. We also show that the two theorems proved in Section 3 cannot be improved.

2 Mathematical preliminaries

Except as noted, we use the same terminology as in [Hara69]. We use the notation $|S|$ to indicate the cardinality of a set S. If S and P are sets, $S - P$ indicates the set of elements of S that are not in P. If G is a graph, S its set of sites, and $P \subseteq S$, we denote by P^* the subgraph of G *induced* by P; this subgraph consists of P and all edges of G that join two points of P. By a component of P, we mean a component of P^*.

A *plane graph* is a graph that is embedded in the plane. Such a graph divides the plane into regions, called *faces*. Exactly one of these faces is unbounded; it is called the *exterior face*, while the remaining faces are called *interior faces*. The set of sites and edges incident with the exterior face is called the *boundary* of the graph. An edge of a plane graph is called an *interior* edge if it is incident on both sides to interior faces, a *boundary* edge if it is incident to an interior face and the exterior face, and an *exterior* edge if it is incident on both sides to the exterior edge. If G is a plane graph with S the set of sites, and $P \subseteq S$, a component of $S - P$ is an *interior component* if it contains no point of S that is on the boundary of G.

A *triangulation* is a plane graph in which every edge is a line segment, every interior face is bounded by a triangle, and the boundary of the graph is a convex polygon. An *elementary triangle* is a triangle that bounds an interior face.

The Delaunay triangulation is a structure that is well known to computational geometers; see [PrSh85] for the relevant definitions. A Delaunay triangulation is *nondegenerate* if it is impossible to find four or more generating sites that lie on a common circle that contains no additional generating sites in its interior, and if no three consecutive points on its convex hull are collinear. It follows from the "general lemma" in [Dela34] and some elementary geometry that a triangulation is a nondegenerate Delaunay triangulation if and only if, for each pair of elementary triangles ABC and ABD that share a common edge AB, $ABC + ABD < 180$. (Note that here, and throughout this paper, we use the notation ABC to represent either a triangle or the measure of an angle in degrees; since it is always clear from the context which we mean, this notational overloading should not cause confusion.)

3 Toughness conditions for Delaunay triangulations

Two important combinatorial properties of Delaunay triangulations are captured in the following theorems.

Theorem 3–1. Let T be a nondegenerate Delaunay triangulation of a set of sites S, and let $P \subseteq S$. Then $S - P$ has at most $|P| - 2$ interior components (where an interior component of $S - P$ is a component that does not contain a point on the convex hull of S).

Theorem 3–2. Let T be a Delaunay triangulation of a set of sites S, and let $P \subseteq S$. Then $S - P$ has at most $|P|$ components. In other words, T is 1-tough.

Due to space limitations, we prove these two theorems under the additional assumptions that P^* is connected and that P^* does not contain any exterior edges. Proving the theorems in full generality is then a simple matter. We also omit proofs of several lemmas. Full details appear in [Dill87].

Since P^* is a planar graph, it divides the plane into faces, all but one of which are interior faces. We partition the interior faces into two types. We call faces with no points of $S - P$ in their interior ***type 1 faces***, while faces with points of $S - P$ in their interior are called ***type 2 faces***. Clearly, each component of $S - P$ is contained in a type 2 face. The following lemma tells us that, in a triangulation, distinct components of $S - P$ are contained in distinct faces of P^*.

Lemma 3–3. Let T be a triangulation with sites S, and let P be a connected subset of S. Each type 2 face of P^* contains exactly 1 component of $S - P$. Furthermore, any type 1 face of P^* is bounded by a triangle.

In order to bound the number of type 2 faces of P^*, we associate with each type 1 and type 2 face of P^* certain ***distinguished angles***, defined as follows. For each type 1 face (which, by Lemma 3–3, must be a triangle), we take the three internal angles of the triangle. We call these ***type 1 angles***. For each type 2 face, we take all angles of the form AXB, where AB is an edge of the face boundary and X is the point of $S - P$ inside the face such that triangle AXB is in T. We call these ***type 2 angles***. The type 1 and type 2 angles are illustrated in Figure 1. Two important properties of these angles are contained in the following lemmas.

Lemma 3–4. Each interior edge of P^* is opposite two distinguished angles, and each boundary edge is opposite one distinguished angle.

Lemma 3–5. The sum of the type 2 angles associated with a given type 2 face of P^* is at least 360.

Proof of Theorem 3–1. Let T be a Delaunay triangulation of a set of sites S. In accordance with remarks made earlier in this section, assume that P is a connected subset of S with $|P| = k$ and that P has no exterior edges. By Lemma 3–3, it is sufficient to show that P^* can have at most $k - 2$ type 2 faces.

The proof is based on a counting argument. We establish minimum values for the total measure of the distinguished angles, based on the number of type 2 faces of P^*. We then eliminate pairs of angles that are opposite a common edge, using the Delaunay condition that such pairs must sum to less than 180 degrees. Since the remaining angles are internal angles of triangles, they must all be less than 180 degrees. This permits us to derive a bound on the number of type 2 faces we had to begin with.

Let t be the number of type 1 faces, and let a_j be the number of type 2 faces bounded by j edges. Note that each type 1 face contributes 3 angles of total measure 180, and each type 2 face bounded by j edges contributes j angles of total measure at least 360, by Lemma 3–5. Thus if we let d_n denote the total number of distinguished angles and d_v denote the sum of their measures, we have

$$d_n = 3t + \sum_j ja_j, \text{ and} \qquad (1)$$

$$d_v \geq 180(t + 2\sum_j a_j). \qquad (2)$$

Let f be the number of interior faces of P^*, and observe that $f = t + \sum_j a_j$. Let e be the number of edges of P^*, let e_i be the number of interior edges of P^*, and recall that k is the number of points in P. By Lemma 3–4, each interior edge of P^* is opposite two distinguished angles and that each boundary edge of P^* is opposite one distinguished angle. Since there are no exterior edges, $e + e_i = d_n$. By Euler's formula,

$$e = k + (t + \sum_j a_j) - 1.$$

Combining these last two equations with (1), we have

$$\begin{aligned} e_i &= 3t + \sum_j ja_j - \{k + (t + \sum_j a_j) - 1\} \\ &= 2t + \sum_j (j-1)a_j - k + 1. \qquad (3) \end{aligned}$$

Let r_n be the number of distinguished angles remaining after we reduce the total number of distinguished angles by removing all pairs that are opposite a common internal edge, and let r_v be the total measure of this reduced collection. Then since each internal edge is opposite two distinguished angles, we have, by (1) and (3),

$$\begin{aligned} r_n &= (3t + \sum_j ja_j) \\ &\quad -2\{2t + \sum_j (j-1)a_j - k + 1\} \\ &= 2k - 2 - t + \sum_j (2-j)a_j. \qquad (4) \end{aligned}$$

Since T is a Delaunay triangulation, the sum of the two distinguished angles opposite each internal edge is less than 180. Thus, by (2) and (3),

$$\begin{aligned} r_v &\geq 180(t + 2\sum_j a_j) \\ &\quad -180\{2t + \sum_j (j-1)a_j - k + 1\} \\ &= 180(k - 1 - t + \sum_j (3-j)a_j). \qquad (5) \end{aligned}$$

Since each interior angle in a triangulation must be less than 180, we must have $r_v < 180r_n$. So by (4) and (5), we have

$$k - 1 - t + \sum_j (3-j)a_j < 2k - 2 - t + \sum_j (2-j)a_j.$$

Simplifying this inequality yields $\sum_j a_j < k - 1$. Since $\sum_j a_j$ is the number of type 2 faces, this proves Theorem 3–1. ■

In order to prove Theorem 3–2, we introduce a new type of distinguished angle, to supplement the type 1 and type 2 angles defined in the proof of Theorem 3–1. We define a *type 3 angle* to be an angle of the form p_1qp_2, where p_1 and p_2 are points of P and q is a point of an exterior component of $S-P$ (See Figure 2). The following lemma is somewhat analogous to Lemma 3–5.

Lemma 3–6. Let T be a triangulation, S the set of sites, and $P \subseteq S$. Let c_e be the number of exterior components of $S - P$. Then the sum of the measures of all type 3 angles of $S - P$ is at least $180(c_e - 2)$.

Proof. For each exterior component Q of $S - P$, define the *p-boundary* of Q to be the path through P such that each edge of the path is the base of a triangle whose apex is in Q and is to the left of the edge. Define the *q-boundary* of Q to be the path from the first point of the p-boundary to the last point of the p-boundary such that every point (except the first and last points) is in Q and such that each edge of the path is the base of a triangle whose apex is in P and is to the right of the edge. The p-boundary and q-boundary are illustrated in Figure 1.

Our first goal is to establish a bound for the total measure of the type 3 angles of Q in terms of the total measure of the angles along the q-boundary. Indeed, assume that the q-boundary is the path $q_0q_1 \ldots q_sq_{s+1}$ (where q_0 and q_{s+1} are in P, and all other points are in Q). Then if α is the sum of the measures of the type 3 angles of Q, we claim that

$$\alpha \geq \sum_{j=1}^{s} q_{j-1}q_jq_{j+1} - 180(s-1) \qquad (6)$$

This follows from observing that if we start with the q-boundary and throw out all triangles that are inside the q-boundary, outside the p-boundary, and do not contain any type 3 angles, we will have thrown out at most $s - 1$ triangles. (The eliminated triangles are shaded in Figure

1).

The next step in the proof involves constructing a polygon, which we call R, obtained by taking the convex hull of S (with the vertices enumerated in counterclockwise order) and "cutting across it" with q-boundaries. That is, if $s_1, \ldots, s_n, s_1$ is an enumeration of the vertices of the convex hull of S, then for each pair s_u and s_v of vertices that form the opposite ends of a q-boundary, replace $s_{u+1} \ldots, s_{v-1}$ with the points of $S-P$ on that q-boundary. The polygon R has two kinds of vertices — points of P that are on the convex hull of S, and points of $S-P$ that lie on q-boundaries. Suppose that there are x_i points of $S-P$ along the q-boundary of component number i, and let $x = \sum_{i=1}^{c_e} x_i$. Let β_i be the sum of the internal angles of R at points of the q-boundary of component number i, and let $\beta = \sum_{i=1}^{c_e} \beta_i$ (i.e., β is the sum of the measures of all internal angles of R at points of $S-P$.) Since all vertices of R that are not points of $S-P$ are convex vertices (i.e., they are less than 180), it follows that

$$\beta > 180(x-2) \qquad (7)$$

Let α_i be the sum of the measures of all type 3 angles at vertices in component number i. By summing both sides of (6) over all exterior components of $S-P$, we obtain

$$\sum_{i=1}^{c_e} \alpha_i \geq \sum_{i=1}^{c_e} \{\beta_i - 180(x_i - 1)\} = \beta - 180(x - c_e).$$

Hence, by (7), we have

$$\sum_{i=1}^{c_e} \alpha_i > 180(x-2) - 180(x-c_e) = 180(c_e - 2),$$

which was to be proved. ∎

Proof of Theorem 3–2. Let T be a Delaunay triangulation of a set of sites S. Let P be a connected subset of S with $|P| = k$ such that P^* has no exterior edges. We must show that $S-P$ can have at most k components.

The proof is quite similar to the proof of Theorem 3–1. Let a_j be the number of type 2 faces of P^* that have j edges on the boundary, t the number of type 1 faces of P^*, and c_e the number of exterior components of $S-P$. By Lemma 3–4, the number of components of $S-P$ is given by $c_e + \sum_j a_j$.

Let d_n be the total number of distinguished angles (i.e., all angles of types 1, 2, and 3), and let d_v be their total measure. Also, let x be the total number of edges in all p-boundaries of exterior components of $S-P$. Then since there are no exterior edges of P^*, there is a 1-1 correspondence between edges of p-boundaries and type 3 angles, so we have

$$d_n = 3t + \sum_j j a_j + x. \qquad (8)$$

By (2) and Lemma 3–6,

$$d_v \geq 180(t + 2\sum_j a_j + c_e - 2). \qquad (9)$$

Let e_s be the number of edges that separate two distinguished angles from one another. There are two types of such edges — those that separate type 1 and/or type 2 angles from one another, and those that separate type 3 edges from type 1 or type 2 edges. Since P^* has no exterior edges, there are x edges of the second type, so it follows from (3) that

$$e_s = 2t + \sum_j (j-1)a_j - k + 1 + x. \qquad (10)$$

As in the proof of Theorem 3–1, we reduce the supply of distinguished angles by removing pairs of angles that are opposite a common edge. If we let r_n be the number of angles in this reduced set and let r_v be their total measure, then (8), (9), and (10) imply

$$r_n = 2k - 2 - t + \sum_j (2-j)a_j - x, \qquad (11)$$

and

$$r_v \geq 180\{k - t + \sum_j (3-j)a_j + c_e - 3 - x\}. \qquad (12)$$

Arguing as in the proof of Theorem 3-1, (11) and (12) imply

$$k - t + \sum_j (3-j)a_j + c_e - 3 - x < 2k - 2 - t + \sum_j (2-j)a_j - x,$$

which simplifies to

$$\sum_j a_j + c_e < k + 1.$$

Since $\sum_j a_j + c_e$ is the number of components of $S - P$, this proves Theorem 3–2. ■

4 Extensions and Remarks

We say that a graph G is *inscribable* in a surface if it is isomorphic to the convex hull of a set of points on the surface. A *maximal planar graph* is a planar graph that can be embedded in the plane in such a way that all faces (including the exterior face) are bounded by triangles.

Theorem 4–1. Any maximal planar graph inscribable in the paraboloid $z = x^2 + y^2$ is 1-tough.

Proof. Let S be the set of vertices of a maximal planar graph G inscribed in this paraboloid. Let S' be the set of points obtained by projecting the points S vertically downward onto the plane. By results in [EdSe85], the edges of G are exactly those edges obtained by projecting edges of the Delaunay triangulation of S' and the edges of the dual of the "farthest point" Voronoï diagram of S' back up onto the paraboloid. It also follows from these results that the Delaunay triangulation of S' is nondegenerate, since all faces of G (and, in particular, all those on the "underside") are triangles. Thus the theorem follows from Theorem 3–2 and the simple observation that a 1-tough graph remains 1-tough if additional edges (but no new sites) are added. ■

Theorem 4–2. Any maximal planar graph inscribable in a sphere is 1-tough.

Proof. Identical to Theorem 4–1, using the inversion transformation of [Brow79]. ■

Figure 3 shows a triangulation that fails to satisfy the conclusion of Theorem 3–1, because removing A, B, C, and D splits it into 3 internal components. Since the triangulation is 1-tough (in fact, it is Hamiltonian), this shows Theorem 3–1 is not implied by Theorem 3–2. Conversely, the example in Figure 4 which is not 1-tough, satisfies the conclusion of Theorem 3–1. These two examples, taken together, show that Theorems 3–1 and 3–2 are indeed independent of one another.

The triangulation in Figure 5 shows that neither Theorem 3–1 nor Theorem 3–2 can be improved. It is easy to verify that the figure is in fact a Delaunay triangulation. Removing the 3 vertices A, B, and C separates it into 3 components, one of which is interior. This shows that the bounds on the number of components proved in Section 3 can be attained.

The connection between Hamiltonicity and 1-toughness is not fully understood. Perhaps the results of this paper will motivate further research in this area. Such research might lead to answers to several questions that are currently open, such as whether it is indeed true that "most" Delaunay triangulations are Hamiltonian (in a probabilistic sense) and how difficult it is to determine whether a given Delaunay triangulation is Hamiltonian (i.e., does there exists a polynomial-time algorithm?).

The proofs of the theorems in Section 3 rely heavily on the condition that, in a Delaunay triangulation, opposite angles sum to less than 180 degrees. This condition is only true for the Euclidean metric. Thus it is an interesting question whether the theorems hold for other metrics as well.

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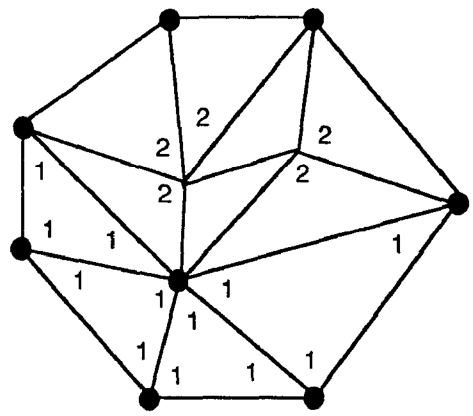

Figure 1. Type 1 and type 2 angles. The thick circles represent the nodes of P.

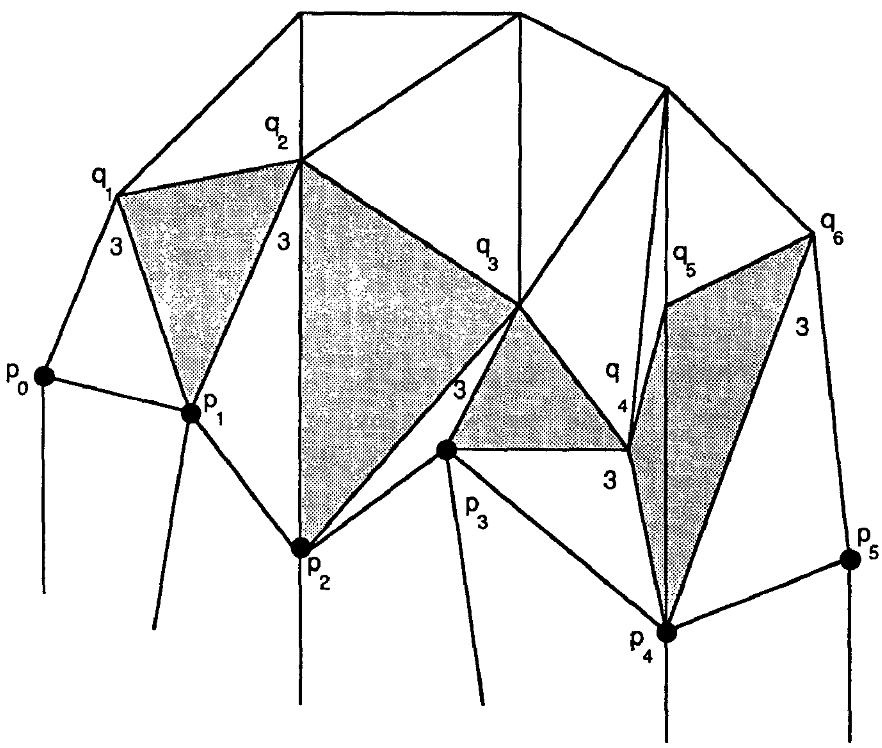

Figure 2. The p-boundaries and q-boundaries for a single exterior component Q. (Note that only a portion of the triangulation is shown.) Here r = 5 and s = 6. The shaded triangles are the triangles that are eliminated in the derivation of equation (6). The type 3 angles are as indicated.

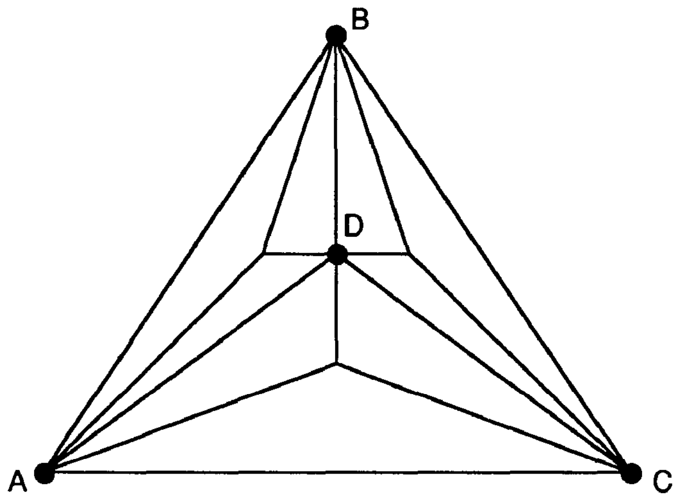

Figure 3. A 1-tough triangulation that fails to satisfy the conclusion of Theorem 3-1.

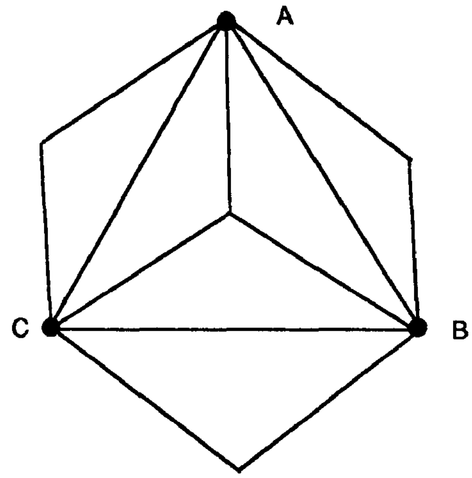

Figure 4. A triangulation that is not 1-tough. This example shows that Theorem 3-1 does not imply Theorem 3-2.

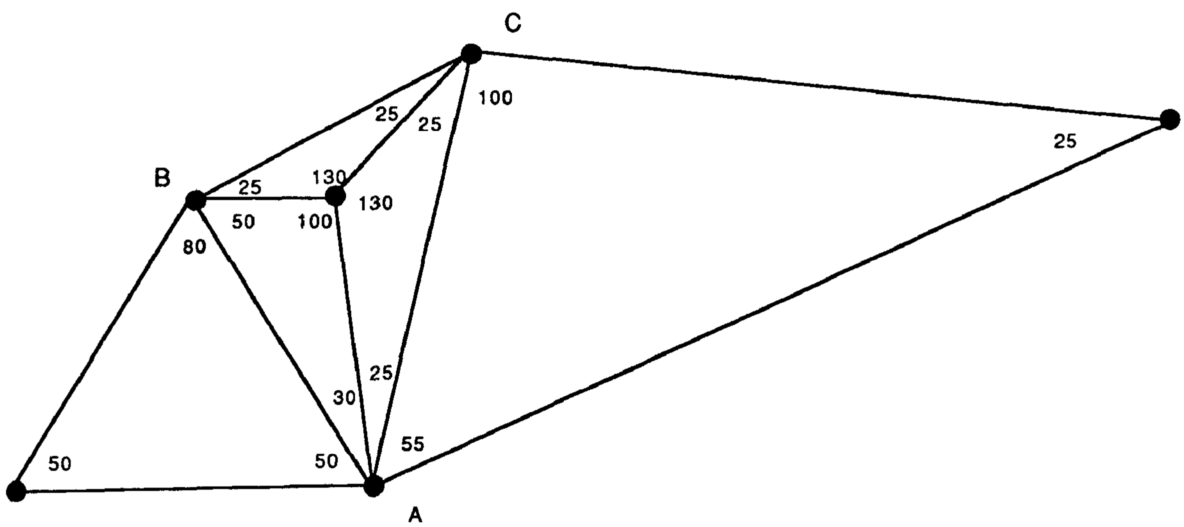

Figure 5. A Delaunay triangulation illustrating that Theorem 3-1 and Theorem 3-2 are both sharp. The angle measurements are in degrees.