Bounded Concurrent Timestamp Systems Using Vector Clocks^{*}

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Abstract

Shared registers are basic objects used as communication mediums in asynchronous concurrent computation. A concurrent timestamp system is a higher typed communication object, and has been shown to be a powerful tool to solve many concurrency control problems. It has turned out to be possible to construct such higher typed objects from primitive lower typed ones. The next step is to find efficient constructions. We propose a very efficient wait-free construction of bounded concurrent timestamp systems from 1-writer multireader registers. This finalizes, corrects, and extends, a preliminary bounded multiwriter construction proposed by the second author in 1986. That work partially initiated the current interest in wait-free concurrent objects, and introduced a notion of discrete vector clocks in distributed algorithms.

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1 Introduction

Consider a system of asynchronous processes that communicate among themselves by executing read and write operations on a set of shared variables (also known as shared *registers*) only. The system has no global clock or any synchronization primitives. Every shared variable is associated with a process (called *owner*) which writes it and the other processes may read it. An execution of a write (read) operation on a shared variable will be referred to as a *Write* (*Read*) on that variable. A Write on a shared variable puts a value from a pre determined finite domain into the variable, and a Read reports a value from the domain. A process that writes (reads) a variable is called a *writer* (*reader*) of the variable.

Wait-free shared variable: We want to construct shared variables in which the following two properties hold. (1) Operation executions are not necessarily atomic, that is, they are not indivisible, and (2) every operation finishes its execution within a bounded number of its own steps, irrespective of the presence of other operation executions and their relative speeds. That is, operation executions are *wait-free*. These two properties give rise to a classification of shared variables, depending on their output characteristics. Lamport [29] distinguishes three categories for 1-writer shared variables, using a precedence relation on operation executions defined as follows: for operation executions A and B, A precedes B, denoted $A \longrightarrow B$, if A finishes before B starts; A and B overlap if neither A precedes B nor B precedes A. In 1-writer variables, all the Writes are totally ordered by " \longrightarrow ". The three categories of 1-writer shared variables defined by Lamport are the following.

- 1. A *safe* variable is one in which a Read not overlapping any Write returns the most recently written value. A Read that overlaps a Write may return any value from the domain of the variable.
- 2. A *regular* variable is a safe variable in which a Read that overlaps one or more Writes returns either the value of the most recent Write preceding the Read or of one of the overlapping Writes.

3. An *atomic* variable is a regular variable in which the Reads and Writes behave as if they occur in some total order which is an extension of the precedence relation.

A shared variable is $boolean^1$ or *multivalued* depending upon whether it can hold only two or more than two values.

Multiwriter shared variable: A multiwriter shared variable is one that can be written and read (concurrently) by many processes. Lamport [29] constructed a shared variable that could be written by one process and read by one other process, but he did not consider constructions of shared variables with more than one writer or reader. Vitányi and Awerbuch [42] were the first to construct an atomic multiwriter shared variable from 1-writer variables. They propose two constructions: one from 1-writer multireader shared variables using bounded control information that turned out to be incorrect [43] (just regular and not atomic as claimed), and the other from 1-writer 1-reader variables using unbounded control information. The latter construction is correct. It is made bounded in [31], yielding one of the most optimal implementations that are currently known. (In this paper we correct and extend the first construction to obtain an efficient version of the more general notion of bounded concurrent timestamp system as defined below.) Related work is [1, 5, 6, 16, 17, 19, 23, 24, 29, 30, 31, 34, 35, 36, 37, 38, 40]. In particular, it is now possible to construct bounded multiwriter atomic variables from 1-writer 1-reader safe bits. See [31], and the last section of this paper, for a brief history of the subject.

Timestamp system: In a multiwriter shared variable it is only required that every process keeps track of which process wrote last. There arises the general question whether every process can keep track of the order of the last Writes by all processes. This idea was formalized by Israeli and Li [21]. They introduced and analyzed the notion of *timestamp system* as an abstraction of such a higher typed communication medium. In a timestamp system every process owns an *object*, an abstraction of a set of shared variables. One of the requirements of the system is to determine the temporal order in which the objects are written. For this purpose, each object is given a *label* (also refer to as *timestamp*) which indicates the latest (relative) time when it has been written by its owner process. The processes assign labels to their respective objects in such a way that the labels reflect the real-time order in which they are written to. These systems must support two operations, namely *labeling* and *scan*. A

¹Boolean variables are referred to as *bits*.

labeling operation execution (Labeling, in short) assigns a new label to an object, and a scan operation execution (Scan, in short) enables a process to determine the ordering in which all the objects are written, that is, it returns a set of labeled-objects ordered temporally. We are concerned with those systems where operations can be executed *concurrently*, in an overlapped fashion. Moreover, operation executions must be *wait-free*, that is, each operation execution will take a bounded number of its own steps (the number of accesses to the shared space), irrespective of the presence of other operation executions and their relative speeds.

Wait-free constructions of concurrent timestamp systems (CTSs, in short) have been shown to be a powerful tool for solving concurrency control problems such as *fcfs*-mutual exclusion [8, 25], multiwriter multireader shared variables [42], probabilistic consensus [2, 7], *fcfs l*exclusion [12] by synthesizing a "wait-free clock" to sequence the actions in a concurrent system.

Here, we are interested in constructing concurrent timestamp systems using 1-writer shared variables. It is not difficult to construct a timestamp system if the shared space is unbounded (there is no limit on the size of some shared variables). The problem gets much harder for bounded (shared space) systems. A *bounded timestamp system* is a timestamp system with a finite set of bounded size labels. In the rest of the paper, unless stated otherwise, by a timestamp system we mean a wait-free bounded concurrent timestamp system.

Israeli and Li [21] constructed a bit-optimal bounded timestamp system for sequential operation executions. The *concurrent* case of bounded timestamp system is harder and the first generally accepted solution is due to Dolev and Shavit [9]. Their construction is of the type as in [21] and uses shared variables of size O(n), where n is the number of processes in the system. Each Labeling requires O(n) steps, and each Scan $O(n^2 \log n)$ steps. In their construction, no Scan writes any shared variables: It is a 'pure' reading operation execution. (But, by the theorem of Lamport [29, page 91], all such constructions become de facto impure if we break them down to the lowest level of system building.) Following Dolev and Shavit, several researchers have come up with other constructions. Israeli and Pinhasov [22] use shared variables of size $O(n^2)$; Labeling and Scan require O(n) steps. Gawlik, Lynch and Shavit [14] use shared variables of size $O(n^2)$; Labeling and Scan access $O(n \log n)$ shared variables. In [10], Dwork and Waarts introduce a powerful communication abstraction called "traceable use abstraction" to recycle values of shared variables. They demonstrate the use-fulness of the abstraction by constructing a CTS, borrowing the basic ideas and techniques

from [42] for recycling private values. Their construction requires shared variables of size $O(n \log n)$; Labeling and Scan require O(n) steps. Later, they along with Herlihy and Plotkin [11] propose a construction using shared variables of size O(n); Labeling and Scan access O(n) shared variables. Unlike the Israeli-Li and Dolev-Shavit constructions, Scans in other proposed constructions are not pure; they write a lot of shared space.

Our result and related work: Among the constructions mentioned above, the one of Dwork and Waarts [10] is relatively simple and efficient as well². They introduce "traceable use abstraction" to bound the size of labels. Like in [42], each label is a vector of n private values, one for each of n processes. Using a strategy similar to, and extending, [42], the abstraction helps each process to keep track of its private values that are in use in the system. At any point in time, a process can use only a bounded number of private values of another process. Exploiting that feature, the abstraction helps in bounding the set of private values needed. The labels are read by executing a *traceable-read* function, and written by executing a traceable-write procedure. When the traceable-read function is executed to read a label, the executing process explicitly informs all other processes which of their private values it is going to use. A process can find which of its private values are in use by other processes even if the values propagate through these processes in tandem one after another. To determine which of its private values are currently not in use, a process executes a *garbage collection* routine. This routine helps processes to safely recycle their respective private values that are not in use. These three routines are at the heart of implementing the traceable use abstraction. Dwork and Waarts [10] have shown how these routines are used in constructing a bounded concurrent timestamp system. The most intricate among these routines is the garbage collection, whose time complexity is $O(n^2)$ that could be, though nonstandard, uniformly amortized over $O(n^2)$ labeling operation executions. To achieve this, each process needs to maintain a private, separate, pool of $22n^2$ private values. The costliest part of their construction is the use of multireader 'order' variables. The construction uses, for each process, $\Theta(n)$ sets of 22*n*-many 1-writer *n*-reader atomic variables of size $\Theta(n \log n)$ bits each. Let us roughly estimate their space complexity at the fundamental level, i.e., at the level of 1-writer 1-reader safe bits. (To implement a 1-writer *n*-reader atomic variable of size *m* bits, the constructions in [29, 40]together require 3mn 1-writer 1-reader safe bits, 2n 1-writer 1-reader atomic bits and one 1writer *n*-reader atomic bit. Each 1-writer 1-reader atomic bit can be implemented from O(1)

²We find it is the easiest one to understand; also see comments in [44] by Yakovlev.

1-writer 1-reader safe bits [16, 29, 39, 41]. A 1-writer *n*-reader atomic bit can be implemented from $O(n^2)$ safe bits [16]. Thus, we require a total of $3mn + O(n^2)$ 1-writer 1-reader safe bits to implement a 1-writer *n*-reader atomic variables of size *m* bits.) Thus, there is a need of at least $\Omega(n^4 \log n)$ bits at the fundamental level just for the order variables in each process. Consequently, we need at least $\Omega(n^5 \log n)$ 1-writer 1-reader safe bits for all order variables of all processes. In addition, there are other shared variables for the processes.

The bounded multiwriter shared variable construction of Vitányi and Awerbuch [42], while falling short of the claimed atomicity [43], has brought into prominence many techniques that were used later in wait-free computing. An example is the idea of a label as a vector of nindividual clocks.³ (In [42], vector entries are called 'tickets'.) Even better, it turns out that the corrected version presented here suffices to implement the higher communication object type of bounded CTS. The current paper is the final version of the pioneering preliminary [42], and its correction [18]. Dwork and Waarts [10], without stating this explicitly, used the idea of (bounded) vector clocks and other techniques introduced in [42], and hence their solution bears a close resemblance to the construction proposed here (and, in fact, to other constructions [35, 36] based on [42]). On the other hand, our construction uses some ideas from their traceable use abstraction. We observe that in CTSs the propagation of private values is restricted to only one level of indirection, and not to arbitrary levels. Consequently, the propagation of private values can be tracked down by their respective owner processes with relative ease. And, the one level indirect propagation of private values by other processes need not be informed to the original owner of these private values. Thus, one doesn't need the complete power of the traceable use abstraction for constructing a CTS. In our construction, we use less powerful traceable-read and traceable-write. But, we prefer to use the same function/procedure names of [10] just keep conformity with the literature. We do not require a garbage collection routine, thereby simplifying the proposed CTS construction and its correctness proof considerably. When a process executes the traceable-read function, it does not explicitly inform the other processes which of their private values it is going to use. On the other hand, the executers of the traceable-write procedure correctly find which private values of which processes are in use in the system. Another important point is that, in our construction, a Scan writes a limited amount of information, only O(n) 1-writer 1-reader bits. Also, each local pool of

³ The concept of vector clock is used in many areas of distributed computing, all in related contexts, to keep track of execution evolution in distributed systems. (Cf. The articles by Mattern[32, 33].)

private values contains fewer than $2n^2$ values. We use a total of $n^2 O(n \log n)$ bit size 1-reader 1-writer regular order variables, requiring a total of $O(n^3 \log n)$ safe 1-reader 1-writer bits at the fundamental level. Both the scan and labeling operation executions require O(n) steps in terms of the shared variables used. But in our construction, a Scan reads at most (n - 1)1-writer 1-reader regular order variables, whereas in their construction it is (2n - 2) 1-writer *n*-reader atomic ones. Thus, at the fundamental level they scan order of magnitude more bits than we do.

Our construction is not optimal in terms of the usage of shared space (Cf. Table 1 in Section 5). It is perhaps possible to use a bounded set of global values and to recycle them instead of using private values. Recycling of global values could lead to an optimal construction.

The remainder of this paper is organized as follows. Section 2 discusses the system model and presents the problem statement precisely. A new construction of concurrent timestamp systems is presented in Section 3, and its correctness proof in Section 4. Section 5 concludes the paper.

2 Model, Problem Definition, and some Notations

A concurrent bounded timestamp system (CTS, in short) is an abstract communication system for *n* completely asynchronous processes P_1, \ldots, P_n . It consists of *n* objects $\mathcal{O}[1..n]$, each of finite space representation, and supports two operations, namely *labeling* and *scan(ing)*. A labeling operation execution (Labeling, in short) of process P_p assigns a new label to object $\mathcal{O}[p]$. It may use all existing labels of $\mathcal{O}[1..n]$, but it is not allowed to change the labels of components other than $\mathcal{O}[p]$. A scan operation execution (Scan, in short) enables a process to determine the ordering in which all the objects are written, that is, it returns a set of labeledobjects ordered temporally⁴. It returns a pair (\bar{l}, \prec) , where \bar{l} is a set of current labels, one for each object-component, and \prec is a total order on \bar{l} . Operation executions of each process are sequential. However, operation executions of different processes need not be sequential, i.e., they might overlap.

Let us denote the k th operation execution (Labeling or Scan) of a process P_p by $O_p^{[k]}$, $k \ge 1$. If it is a Scan (Labeling), we denote it explicitly by $S_p^{[k]}$ $(L_p^{[k]})$. The label written by a labeling operation execution $L_p^{[k]}$ is denoted by $l_p^{[k]}$.

⁴We ignore, in this paper, the data values of the objects.

For operation executions A and B on a shared variable, $A \dashrightarrow B$ means that the execution of A starts before that of B finishes. That is, if $A \dashrightarrow B$, then either $A \longrightarrow B$ or A overlaps B; in other words, $B \not\longrightarrow A$. We also assume that if $B \not\longrightarrow A$, then $A \dashrightarrow B$. That is, we assume the global time model [29].

A concurrent timestamp system must ensure the following properties [9, 14].

- P1. Ordering: There exists an irreflexive total order \Rightarrow on the set of all labeling operation executions, such that the following two conditions hold.
 - Precedence: For every pair of Labelings $L_p^{[k]}$ and $L_q^{[k']}$, if $L_p^{[k]} \longrightarrow L_q^{[k']}$ then $L_p^{[k]} \Rightarrow L_q^{[k']}$.
 - Consistency: For every Scan $S_i^{[j]}$ returning (\bar{l}, \prec) , for every two labels $l_p^{[k]}$ and $l_q^{[k']}$ in $\bar{l}, l_p^{[k]} \prec l_q^{[k']}$ iff $L_p^{[k]} \Rightarrow L_q^{[k']}$.
- P2. Regularity: For every label $l_p^{[k]}$ in \overline{l} returned by a Scan $S_i^{[j]}$, $L_p^{[k]}$ begins before $S_i^{[j]}$ terminates, i.e., $L_p^{[k]} \dashrightarrow S_i^{[j]}$, and there is no Labeling $L_p^{[k']}$ such that $L_p^{[k]} \longrightarrow L_p^{[k']} \longrightarrow S_i^{[j]}$.
- P3. Monotonicity: Let $S_i^{[j]}$ and $S_{i'}^{[j']}$ be a pair of Scans returning sets \overline{l} and $\overline{l'}$, respectively, which contain labels $l_p^{[k]}$ and $l_p^{[k']}$, respectively. If $S_i^{[j]} \longrightarrow S_{i'}^{[j']}$, then $k \leq k'$.
- P4. Extended Regularity: Let $l_p^{[k]}$ be a label returned by a Scan $S_i^{[j]}$. For each Labeling $L_q^{[k']}$, if $S_i^{[j]} \longrightarrow L_q^{[k']}$, then $L_p^{[k]} \Rightarrow L_q^{[k']}$.

The intuitive meaning of the above four properties is as follows. The ordering property says that all the labeling operation executions can be totally ordered which is an extension of their real-time precedence order " \rightarrow ". Moreover, if two different Scans return labels l and l', then both Scans will have the same order on the labels. The regularity property says that labels returned by a Scan are not obsolete. The monotonicity property says that for every two Scans ordered by " \rightarrow ", it is not the case that the preceding Scan returns a new label of a process P_p and the succeeding Scan an old label of P_p . The monotonicity property does not imply that labeling and scan operation executions of all processes are linearizable [20]. It does imply the linearizability of the Scans of all processes and labeling operation executions of a single process [9]. The extended regularity property says that if a Scan precedes a labeling operation execution L, then all labels returned by the Scan were assigned by labeling operation executions that precede L in \Rightarrow .

We are interested in those CTSs in which operation executions are *wait-free*, that is, each operation execution will take a bounded number of its own steps (a step is a read/write of a shared variable), irrespective of the presence of other operation executions and their relative speeds. This paper is concerned with implementing wait-free CTSs from basic 1-writer 1-reader shared variables.

3 The Construction

For the sake of convenience and better understanding, we first present an intuitive informal description of a construction that uses unbounded shared space [42] (the same idea is used in [10]). Each process maintains a separate local pool of private values that are natural numbers with the standard order relations on them.

A label is a vector of n values ('tickets' in [42]); its p th component holds a private value of process P_p . The current label of $\mathcal{O}[p]$ is denoted by $l_p[1..n]$ or simply l_p . The current private value of process P_p is $l_p[p]$. Initially, $l_p[p] = 1$ and $l_p[q] = 0$, for all $q \neq p$. To determine a new label for $\mathcal{O}[p]$, process P_p reads all current private values of other processes P_q , namely $l_q[q]$, and increments its own private value $l_p[p]$ by one to obtain the new private value. The new label vector contains these n values, and it is written atomically in $\mathcal{O}[p]$. Since the same private value is not used twice in labeling operation executions, no two labels ever produced in the system are the same. The ordering of two label vectors is done by using the standard lexicographic (dictionary) order \prec : for every two labels, $l_p \neq l_q$, the *least significant index* in which they differ is the lowest k such that $l_p[k] \neq l_q[k]$; then, $l_p \prec l_q$ iff $l_p[k] < l_q[k]$. This lexicographic order \prec is a total order on the set of all possible labels [13], and this fact is a static common knowledge to the processes. (In fact, \prec is an elementary example of a well-ordered relation.) A Scan simply reads all the current labels and orders them using the lexicographic order. This unbounded construction satisfies all the properties required for a concurrent timestamp system (Cf. [10]).

In the unbounded construction discussed above, every time a process P_k executes a new labeling operation, it uses a new private value greater than the previously used ones. In a bounded construction, each process has only a bounded number of private values, and hence, it needs to use the same private value at different times, that is, it needs to recycle its own private values. The following observation (which is a synthesis of the text in [42, page 236]) by Dwork and Waarts helps doing the recycling in some possible way. We quote them verbatim:

... for a system to be a concurrent timestamp system, every time a new private value chosen by process P_k need not be the one that was never used by P_k beforehand; roughly speaking, instead of increasing its private value, it is enough for P_k to take as its new private value any value v of its private values that does not appear in any labels, with one proviso: P_k must inform the other processes that v is to be considered larger than all its other private values currently in use.

Consequently, we cannot use the standard ordering relations on the natural numbers any more, for the numbers may be recycled repeatedly. One has now to consider these numbers as mere symbols with no standard ordering relations defined on them. We define for every two different private values v and v' of process P_k currently in use in the system, $v \prec_k v'$ iff v is issued before v' by P_k . Thus, in the bounded construction, the ordering relation among the private values changes in time, and hence it cannot be *a priori* common knowledge. Note that at any point in time, the relation \prec_k on the values in use is a total order as the values are produced in sequences, and in fact, it is well-ordered. For every two labels, $l_p \neq l_q$, obtained by a Scan, if k is the least significant index such that $l_p[k] \neq l_q[k]$, then we define $l_p \prec l_q$ iff $l_p[k] \prec_k l_q[k]$. Then, \prec is also a well-ordered relation [13]. Now, we are concerned with two things in a bounded construction. First, to make the relations \prec_k useful, processes P_k cannot recycle a private value if some other processes are using it. Second, for every two private values v and v' of P_k currently in use, if $v \prec_k v'$ then all other processes should (get to) know this ordering before using these values. Note that the meaning of < on the natural numbers is a static common knowledge, but the meaning of \prec_k changes continually. Thus, every time P_k changes the ordering of two different private values, it should inform all the other processes well in advance. Then, for all labels read by a Scan, the labels are ordered lexicographically, based on the orderings \prec_k of all processes P_k . Then, the correctness of the bounded system trivially follows from that of the unbounded system mentioned above (given in [42, 10]).

In the following paragraphs, we present a novel construction, based on [42, 18], to achieve the afore mentioned two objectives. The construction is given in Figure 1.

We now introduce some terminology. The description of the construction has five parts:

shared variables declaration, TRACEABLE-WRITE procedure, TRACEABLE-READ function, LABELING procedure and SCAN function. The procedures and the functions are written in a Pascal-type language. To avoid too many 'begin's and 'end's, some blocks are shown just by indentation. All the statements in the four routines are numbered only for reference purposes.

A base shared variable x is read (respectively, written) by executing an instruction 'read *local-variable* from x' (respectively, 'write *local-variable* in x'), where the *local-variable* is local to the function or the procedure. The read-instruction assigns the value of x to the *local-variable*, and the write-instruction writes the value of the *local-variable* in x. The writer (owner) of a shared variable can retain the value of the variable in its local storage and refer to it later on if needed, that is, it need not read the shared variable to determine the current value of the variable. Nevertheless, for the sake of convenience and to avoid using many local variables, we let the writer also read its own shared variable. It also uses some private (local, non-shared) variables for each process. We assume that the private variables are persistent.

Let us consider operation executions of a particular process P_p . Process P_p executes the LABELING procedure to obtain and assign a new label to $\mathcal{O}[p]$, and executes the SCAN function to report the temporal ordering of the labels of $\mathcal{O}[1..n]$. In a labeling operation execution, it selects a presently unused private value from its local pool of values (Statements 1–2 in the LABELING procedure), collects the current private values of all other processes (Statements 5–6), and then writes these n values atomically in $\mathcal{O}[p]$ as its new label (Statement 7). The selection of a new private value is done in such a way that there is no trace of this value in the system at present. In a scan operation execution, process P_p first reads the current labels of all the processes (Statement 1 in the SCAN function), and then determines their temporal ordering using the latest ordering information available from some ordering shared variables (Statement 2).

The collection of the current private values of other processes is done by executing the TRACEABLE-READ function, and the writing of the new label is done by executing the TRACEABLE-WRITE procedure.⁵ These two routines collectively implement atomic reading and writing of labels from and into objects $\mathcal{O}[p]$. (In rest of the paper, an execution of the TRACEABLE-READ function (TRACEABLE-WRITE procedure) will be called a traceable Read (traceable Write).) Note that these two routines are not parts of the interface to the

⁵These two routines resemble pretty closely the READ and WRITE routines in [17, 19, 40, 42].

CTS, and the processes cannot directly invoke them. They directly invoke the LABELING and SCAN routines in which they, in turn, invoke traceable Read (Write) to read (write) labels.

A process P_p uses shared variables w[p, 1..n], r[p, 1..n], c[p], label[p, 0..1] and copylabel[p, 1..n]to atomically read and write new labels from and into object $\mathcal{O}[p]$. The label and copylabel variables are used to hold labels of $\mathcal{O}[p]$. w and r are handshake variables used to detect overlapping of traceable Reads and Writes. The variable c is used to atomically declare writings of new labels in $\mathcal{O}[p]$. Process P_p uses the shared variables order[p, 1..n] to inform all the processes of the latest ordering relation \prec_p . The shared variables lend[p, 1..n] are used to inform all the processes which of their private values might be in use in the system. The component lend[p, j] contains all the private values of process P_j that P_p may have lent to other processes. Process P_p also uses static private variables: cl_p , $myLend_p$, \prec_p , and $old-label_p$. cl_p and $myLend_p$ always store the values of c[p] and lend[p, 1..n], respectively, locally. \prec_p contains the latest ordering information of all the private values in use in the system. $old-label_p$ stores the label of the on-going or the recently completed Labeling operation execution.

The traceable Writes of process P_p use two *n*-reader safe main label variables, label[p, 0]and label[p, 1], and a 1-reader safe copy label variable for each process, copylabel[p, 1..n]. The main label variables are used alternately for writing successive new labels. Immediately after writing a new label in a main label variable, the process records that variable index in the 1-writer multireader boolean atomic variable c[p]. (This writing atomically 'declares' the current label of component $\mathcal{O}[p]$.) Then the process checks for each *i* whether a new traceable Read of process P_i started since the last traceable Write (of P_p). This is done by using a pair of boolean 1-writer 1-reader (handshaking) atomic variables r[i, p] and w[p, i].⁶ Process P_i sets these values different, by assigning the complement of w[p,i] to r[i,p], at the beginning of each traceable Read (Statements 1–2 in TRACEABLE-READ), and process P_p makes sure that they are the same, at the end of each traceable Write (Statements 4.1) and 4.2.3 in TRACEABLE-WRITE). By this way the processes P_p and P_i can find if there are overlappings of their traceable Writes and Reads. Hence if the two values are different when the process P_p checks them, a new traceable Read of P_i must have started by then. In that case, P_p writes the new label value in copylabel[p, i] also, and then sets the above values the same, by assigning the r[i, p] value to w[p, i]. (This way it is guaranteed that a reading

⁶This strategy of detecting overlapping operation execution is pioneered by Peterson [34].

and a writing on *copylabel* variables do not overlap each other, and contains a valid value for the traceable Read [42, 17, 40].) For each such process P_i , P_p takes a note which of the private values of processes P_j could be used by P_i (Statement 4.2.2). Finally, P_p informs all the processes P_j which of their private values could be in use (all that P_p knows of) through 1-writer 1-reader regular variables lend[p, j] (Statement 6).

Each traceable Read of process P_p , from a process P_i , after reading w[i, p] and writing its complement in r[p, i] as mentioned above (Statements 1–2 in TRACEABLE-READ), finds out from c[i] the main label variable that has been written by P_i most recently, and reads from that variable. Then it reads w[i, p] again and compares with r[p, i]. If the two values continue to be different, then the reading of the main label variable does not overlap any writings of the label variable and hence it returns the value just read from the main label variable. Otherwise, there is a possibility that the reading of the label variable overlaps with some writing of the same variable, and hence, it reads copylabel[i, p] and returns that value. Note that in the latter case, a traceable Write by P_i must have finished (with respect to P_p , that is, P_i must have done loop iteration p at Statement 4 in TRACEABLE-WRITE) after the traceable Read started, and that Write would have written in copylabel[i, p].

In selecting a new (currently unused) private value, process P_p does not use any of the values stored in lend[1..n, p] (Statements 1–2 in LABELING). After selecting the new private value, say v, P_p informs all processes P_i that v is the most recent private value through 1-writer 1-reader regular variables order[p, i] (Statements 3–4) which are used by the Scans of P_i .

4 Correctness Proof

Proposition 1 [29] For operation executions B and C on a shared variable, and all operation executions A and D, if $A \longrightarrow B \dashrightarrow C \longrightarrow D$, then $A \longrightarrow D$.

Proof: The implication follows by the transitivity of (i) A finishes before B starts, (ii) B starts before C finishes and (iii) C finishes before D starts. \Box

Definition. For operation executions A and B executed on the same atomic variable x, we say $A \Longrightarrow_x B$ if A precedes B in the total ordering imposed on the operation executions by the atomic variable. The subscript x is omitted when it is clear from the context. \Box

Proposition 2 For operation executions B and C on an atomic variable x, and all operation executions A and D, if $A \longrightarrow B \Longrightarrow_x C \longrightarrow D$, then $A \longrightarrow D$.

Proof: The relation $B \Longrightarrow_x C$ implies B precedes or overlaps C (since the total order imposed on the operation executions by the atomic variable is an extension of the precedence relation), that is, $B \dashrightarrow C$. Then the implication follows by Proposition 1. \Box

The following notations are used in the presentation of the correctness proofs.

- N1. The kth operation execution of a process P_p is denoted, as stated in Section 2, by $O_p^{[k]}(\mathcal{O}), k \geq 1$; if it is a Scan (alternatively, a Labeling), we denote it explicitly by $S_p^{[k]}(\mathcal{O})$ (alternatively, $L_p^{[k]}(\mathcal{O})$). The ' (\mathcal{O}) ' part in the notation is omitted when it is clear from the context. All the operation executions of P_p are totally ordered. That is, for $k > 2, O_p^{[k-1]} \longrightarrow O_p^{[k]}$.
- N2. For a shared variable x, the Read (respectively, Write) of x by $O_p^{[k]}$ is denoted by $R_p^{[k]}(x)$ (respectively, $W_p^{[k]}(x)$). If x is referred more than once, then the superscript [k, j] is used for the jth access.
- N3. Each operation execution $O_p^{[k]}$ $(L_p^{[k]} \text{ or } S_p^{[k]})$ of process P_p executes the TRACEABLE-READ function for every other process P_i ; the whole function execution is denoted by a traceable Read $TR_{p,i}^{[k]}$.
- N4. Each labeling operation execution $L_p^{[k]}$ of process P_p executes the TRACEABLE-WRITE procedure; the whole procedure execution is denoted by a traceable Write $TW_p^{[k]}$.
- N5. For the sake of convenience, the variables r[p, i] and w[p, i] are abbreviated to $r_{p,i}$ and $w_{p,i}$, respectively.

Definition. For a shared variable x, we define a *reading mapping* π_x for Reads of x as follows: if a Read R returns the value written by a Write W, then $\pi_x(R)$ is W; otherwise $\pi_x(R)$ is undefined. (Note, for safe x, π_x is a partial mapping.) We omit the subscript x when it is clear from the context. \Box

Lemma 1 (a) No two consecutive labeling operation executions of a process have the same private value.

(b) No two consecutive traceable Writes of a process have the same private value.

Proof: Part (a) follows from the select statement (Statement 2) in the LABELING procedure. Part (b) follows from Part (a) as each Labeling executes one and the only one traceable Write. \Box

Lemma 2 Each time the value written in $w_{p,i}$ is the complement of the previous value of $w_{p,i}$.

Proof: Immediate from Statements 4.1, 4.2 and 4.2.3 in the TRACEABLE-WRITE procedure. \Box

Lemma 3 Any traceable Write $TW_p^{[k]}$ (actually, $L_p^{[k]}$) that writes $w_{p,i}$ sets $w_{p,i} = r_{i,p}$, and if $R_i^{[l,1]}(w_{p,i}) \Longrightarrow W_p^{[k]}(w_{p,i}) \Longrightarrow R_i^{[l,2]}(w_{p,i})$ for some traceable Read $TR_{i,p}^{[l]}$ (actually, $O_i^{[l]}$) of process P_i , then the equality continues to hold until the execution of $TR_{i,p}^{[l]}$ is complete, in fact until the next traceable Read $TR_{i,p}^{[l+1]}$ writes $r_{i,p}$.

Proof: Initially, $w_{p,i} = r_{i,p}$, since both of them are initialized to 0. Among the traceable Writes of the process P_p , some will write $w_{p,i}$, and some will not. Let $TW_p^{[k_j]}$, $j \ge 1$, $k_j \ge 1$, be the *j* th traceable Write that writes $w_{p,i}$.

Consider $TW_p^{[k_1]}$. By Lemma 2, it writes 1 in $w_{p,i}$. This implies, by Statements 4.1 and 4.2.3 in TRACEABLE-WRITE, that it read 1 from $r_{i,p}$. Since the initial value of $r_{i,p}$ is 0, some traceable Read of P_i must have written 1 in $r_{i,p}$. Let $TR_{i,p}^{[l_1]}$ be the first such traceable Read. Then $W_i^{[l_1]}(r_{i,p}) \Longrightarrow R_p^{[k_1]}(r_{i,p})$. Note that $TR_{i,p}^{[l_1]}$ reads 0 from $w_{p,i}$ and hence writes 1 in $r_{i,p}$ (Statements 1–2 in TRACEABLE-READ). Also each subsequent traceable Read $TR_{i,p}^{[l'_1]}$, if any, such that $R_i^{[l'_1,1]}(w_{p,i}) \Longrightarrow W_p^{[k_1]}(w_{p,i})$, would read 0 from $w_{p,i}$, and hence will write 1 in $r_{i,p}$. Hence, irrespective of whether $W_i^{[l'_1]}(r_{i,p}) \Longrightarrow R_p^{[k_1]}(r_{i,p})$ or $R_p^{[k_1]}(r_{i,p}) \Longrightarrow W_i^{[l'_1]}(r_{i,p})$, on $W_p^{[k_1]}(w_{p,i}), w_{p,i} = r_{i,p}$, and if $R_i^{[l,1]}(w_{p,i}) \Longrightarrow W_p^{[k_1]}(w_{p,i}) \Longrightarrow R_i^{[l,2]}(w_{p,i})$ for some traceable Read $TR_{i,p}^{[l]}$, then the equality continues to hold until $TR_{i,p}^{[l]}$ is complete, in fact until the next traceable Read $TR_{i,p}^{[l+1]}$ writes $r_{i,p}$, since $w_{p,i}$ will not be changed by any traceable Write $TW_p^{[k'_1]}$, for $k'_1 > k_1$, that may occur before $TR_{i,p}^{[l]}$ is complete.

Assuming as induction hypothesis that the assertion holds for $TW_p^{[k_j]}$, for some j, we show that the assertion holds for $TW_p^{[k_{j+1}]}$. By the statement of the lemma, $TW_p^{[k_j]}$ sets $w_{p,i} = r_{i,p}$ by writing value, say $b \in \{0,1\}$ in $w_{p,i}$. Then, by Lemma 2, $TW_p^{[k_{j+1}]}$ writes $\neg b$ in $w_{p,i}$.⁷ This implies by Statements 4.1 and 4.2.3 in TRACEABLE-WRITE, it read $\neg b$ from

 $^{^{7}\}neg b$ is defined as 1-b.

 $r_{i,p}$. As the value of $r_{i,p}$ is b when $TW_p^{[k_j]}$ reads it, there must be a traceable Read that writes $\neg b$ in $r_{i,p}$ after $TW_p^{[k_j]}$ sets $w_{p,i} = r_{i,p}$. Let $TR_{i,p}^{[l]}$ be the first such traceable Read. Then, $W_i^{[l]}(r_{i,p}) \Longrightarrow R_p^{[k_{j+1}]}(r_{i,p})$, and $TR_{i,p}^{[l]}$ writes $\neg b$ in $r_{i,p}$. Each subsequent traceable Read $TR_{i,p}^{[l']}$, if any, such that $R_i^{[l',1]}(w_{p,i}) \Longrightarrow W_p^{[k_{j+1}]}(w_{p,i})$, would read b from $w_{p,i}$, and hence will write $\neg b$ in $r_{i,p}$. Hence, irrespective of whether $W_i^{[l']}(r_{i,p}) \Longrightarrow R_p^{[k_{j+1}]}(r_{i,p})$ or $R_p^{[k_{j+1}]}(r_{i,p}) \Longrightarrow W_i^{[l']}(r_{i,p})$, on $W_p^{[k_{j+1}]}(w_{p,i})$, $w_{p,i} = r_{i,p}$. If $R_i^{[l]}(w_{p,i}) \Longrightarrow W_p^{[k_{j+1}]}(w_{p,i}) \Longrightarrow R_i^{[l,2]}(w_{p,i})$ for some traceable Read $TR_{i,p}^{[l]}$, then the equality continues to hold until $TR_{i,p}^{[l]}$ is complete, in fact until the next traceable Read $TR_{i,p}^{[l+1]}$ writes $r_{i,p}$, since $w_{p,i}$ will not be changed by any traceable Write $TW_p^{[k']}$, for $k' > k_{j+1}$, that may occur before $TR_{i,p}^{[l]}$ is complete. \Box

Lemma 3 implies the following property.

Lemma 4 Let $TR_{i,p}^{[l]}$ be a traceable Read. There can be at most one traceable Write, say $TW_p^{[k]}$, such that $R_i^{[l,1]}(w_{p,i}) \Longrightarrow W_p^{[k]}(w_{p,i}) \Longrightarrow R_i^{[l,2]}(w_{p,i})$. The traceable Read $TR_{i,p}^{[l]}$ on $R_i^{[l,2]}(w_{p,i})$ will find $r_{i,p} = w_{p,i}$ if there is such a traceable Write, and $r_{i,p} \neq w_{p,i}$ otherwise. \Box

In the following we use a typical kind of notation for labeling operation executions.

N6. The labeling operation executions of process P_p are sometimes denoted by $L_p^{[k_j]}$, where k is some alphabet and j is a natural number, $j \ge 1$, $k_j \ge 1$. Thus, for j > 1, $L_p^{[k_{j-1}]}$ and $L_p^{[k_j]}$ are two consecutive labeling operation executions of P_p such that $L_p^{[k_{j-1}]} \longrightarrow L_p^{[k_j]}$. They need not be two consecutive operation executions, that is, $k_j \ge k_{j-1} + 1$.

In the following two lemmas, we show that traceable Reads return valid label values. We also define their reading mapping function π . Lemmas 5 and 6 deal with the case traceable Reads return values from *label* and *copylabel* variables, respectively.

Lemma 5 Let $TR_{i,p}^{[l]}$ be a traceable Read that finds $r_{i,p} \neq w_{p,i}$ on $R_i^{[l,2]}(w_{p,i})$. Suppose $\pi(R_i^{[l]}(c[p]))$ is $W_p^{[k_j]}(c[p])$ (of the traceable Write $TW_p^{[k_j]}$ of $L_p^{[k_j]}$), and label[p, x] is the main label variable from which $TR_{i,p}^{[l]}$ returns the label value.

(a) If j' is the least index such that $R_i^{[l,2]}(w_{p,i}) \Longrightarrow W_p^{[k_{j'}]}(w_{p,i})$, then j' equals j or j+1. (b) $\pi(TR_{i,p}^{[l]})$ is $TW_p^{[k_j]}$.

(c) The traceable Read $TR_{i,p}^{[l]}$ reading label[p, x] does not conflict with any traceable Write writing that label variable.

Proof:

(a) Let j'' be the greatest index such that j'' < j' and $TW_p^{[k_j'']}$ writes $w_{p,i}$. Then by (i) the choice of j', (ii) the assumption that $TR_{i,p}^{[l]}$ finds $r_{i,p} \neq w_{p,i}$ on $R_i^{[l,2]}(w_{p,i})$ and (iii) Lemma 4, it follows that $W_p^{[k_{j''}]}(w_{p,i}) \Longrightarrow R_i^{[l,1]}(w_{p,i})$. That is, $W_p^{[k_{j''}]}(w_{p,i}) \Longrightarrow R_i^{[l,1]}(w_{p,i}) \longrightarrow R_i^{[l,2]}(w_{p,i}) \Longrightarrow W_p^{[k_{j'}]}(w_{p,i})$. The traceable Write $TW_p^{[k_{j''}]}$ sets $w_{p,i}$ equal to $r_{i,p}$, $TR_{i,p}^{[l]}$ sets $r_{i,p}$ not equal to $w_{p,i}$, and hence $TW_p^{[k_{j'}]}$ is the first traceable Write, after $TW_p^{[k_{j''}]}$, that finds $r_{i,p} \neq w_{p,i}$.

From $W_i^{[l]}(r_{i,p}) \longrightarrow R_i^{[l]}(c[p]) \Longrightarrow W_p^{[k_{j+1}]}(c[p]) \longrightarrow R_p^{[k_{j+1}]}(r_{i,p})$, we have $W_i^{[l]}(r_{i,p}) \longrightarrow R_p^{[k_{j+1}]}(r_{i,p})$. That is, the traceable Write $TW_p^{[k_{j+1}]}$ will find $r_{i,p} \neq w_{p,i}$, the inequality set by $TR_{i,p}^{[l]}$, unless an earlier traceable Write has found the inequality and set $w_{p,i}$ equal to $r_{i,p}$. We claim that such an earlier traceable Write, if one exists, can only be $TW_p^{[k_j]}$. Suppose, on the contrary, that it is $TW_p^{[k_j''']}$, for j''' < j. Then, by the choice of j'' and Lemma 4, we have $W_p^{[k_{j''}]}(w_{p,i}) \Longrightarrow R_i^{[l,1]}(w_{p,i}) \longrightarrow R_i^{[l]}(c[p]) \longrightarrow R_i^{[l,2]}(w_{p,i}) \Longrightarrow W_p^{[k_{j'''}]}(w_{p,i}) \longrightarrow W_p^{[k_j]}(c[p])$. This implies $R_i^{[l]}(c[p]) \longrightarrow W_p^{[k_j]}(c[p])$, contradicting the assumption that $\pi(R_i^{[l]}(c[p]))$ is $W_p^{[k_j]}(c[p])$. The assertion follows.

(b and c) Let label[p, x'] be the variable in which $TW_p^{[k_j]}$ writes.

For j' described in part (a), we have $R_i^{[l]}(label[p,x]) \longrightarrow R_i^{[l,2]}(w_{p,i}) \Longrightarrow W_p^{[k_{j'}]}(w_{p,i}) \longrightarrow TW_p^{[k_{j+2}]}$. That is, $TR_{i,p}^{[l]}$ finishes reading label[p,x] before the traceable Write $TW_p^{[k_{j+2}]}$ starts its execution. From (i) the assumption that $\pi(R_i^{[l]}(c[p]))$ is $W_p^{[k_j]}(c[p])$, (ii) the property that $TW_p^{[k_{j+1}]}$ does not write in the same main label variable that $TW_p^{[k_j]}$ writes, (iii) $W_p^{[k_j]}(label[p,x']) \longrightarrow W_p^{[k_j]}(c[p]) \Longrightarrow R_i^{[l]}(c[p]) \longrightarrow R_i^{[l]}(label[p,x])$, and (iv) Statements 1–3 in TRACEABLE-WRITE, it follows that x = x', and $TW_p^{[k_j]}$ finishes writing label[p,x] before $TR_{i,p}^{[l]}$ starts reading it. The assertions follow. \Box

Lemma 6 Let $TR_{i,p}^{[l]}$ be a traceable Read that finds $r_{i,p} = w_{p,i}$ on $R_i^{[l,2]}(w_{p,i})$. Suppose $TW_p^{[k_j]}$ is the traceable Write such that $R_i^{[l,1]}(w_{p,i}) \Longrightarrow W_p^{[k_j]}(w_{p,i}) \Longrightarrow R_i^{[l,2]}(w_{p,i})$.

(a) The traceable Read $TR_{i,p}^{[l]}$ reading copylabel[p,i] does not conflict with any traceable Write writing it.

(b)
$$\pi(TR_{i,p}^{[l]}) = TW_p^{[k_j]}$$

Proof: (a and b) By Lemma 4, $TW_p^{[k_j]}$ is the only traceable Write such that $R_i^{[l,1]}(w_{p,i}) \Longrightarrow W_p^{[k_j]}(w_{p,i}) \Longrightarrow R_i^{[l,2]}(w_{p,i})$. It is clear from the TRACEABLE-WRITE procedure that $TW_p^{[k_j]}$

writes the value in copylabel[p, i] (Statement 4.2.1) before setting the $w_{p,i}$ and $r_{i,p}$ values equal (Statement 4.2.3). This equality will not be changed until P_i starts the next traceable Read. Thus, the traceable Write $TW_p^{[k_{j+1}]}$ and subsequent traceable Writes of P_p , if they find $r_{i,p} = w_{p,i}$, will not write the copy label variable. From $W_p^{[k_j]}(copylabel[p,i]) \longrightarrow W_p^{[k_j]}(w_{p,i}) \Longrightarrow R_i^{[l,2]}(w_{p,i}) \longrightarrow R_i^{[l]}(copylabel[p,i])$, we have $W_p^{[k_j]}(copylabel[p,i]) \longrightarrow R_i^{[l]}(copylabel[p,i])$. The assertions follow. \Box

Now we would like to show that private values of processes P_p are traceable. If a process P_i in its current label uses a private value v of another process P_p , P_i informs this "using of" v by setting lend[i, p][1][i] to v at the end of the corresponding traceable Write (Statements 5–6). Thus, all the private values in the existing labels are traceable by their respective owners. The following lemma shows that the private values used by Scans are also traceable.

Lemma 7 Let a Scan $S_i^{[l]}$ of a process P_i use a private value v of a process P_p that has written the value v in a traceable Write $TW_p^{[k_j]}$. Then, P_p does not recycle v until $S_i^{[l]}$ is complete.

Proof: We need to consider the following two cases.

Case 1: $S_i^{[l]}$ got v directly from P_p .

We need to consider two subcases.

Subcase a. If the traceable Read $TR_{i,p}^{[l]}$ returns the value v from copylabel[p, i], then, by Lemma 6 and 4, the traceable Write $TW_p^{[k_j]}$ has executed the *if*-statement body (Statement 4.2) for process P_i . There it has set $myLend_p[p][1][i]$ to v (Statement 4.2.2). The successive traceable Writes of P_p that occur before $S_i^{[l]}$ is complete will not execute the ifstatement, and hence, will not change the $myLend_p[p][1][i]$ value. (Statement 5 does not change the value too.) As the labeling operation executions of P_p do not reuse the values referred to in lend[1..n, p], v will not be reissued at least until $S_i^{[l]}$ is complete (Statements 1–2 in LABELING).

Subcase b. If the traceable Read $TR_{i,p}^{[l]}$ returns the value v from a main label variable, then by Lemma 5(a), traceable Write $TW_p^{[k_j]}$ or $TW_p^{[k_{j+1}]}$ executes the *if*-statement for process P_i . In the case of $TW_p^{[k_j]}$, $myLend_p[p][1][i]$ is set to v, and in the case of $TW_p^{[k_{j+1}]}$, $myLend_p[p][0][i]$ is set to v (Statements 4.2.2 and 7). The successive traceable Writes of P_p that occur before $S_i^{[l]}$ is complete will not execute the if-statement, and hence, will not change the $myLend_p[p][0..1][i]$ values. (Statement 5 does not change the values too.) By Lemma 1, $TW_p^{[k_{j+1}]}$ uses a private value different from v. So, by the argument given in the Subcase a, v will not be reissued as a new private value until $S_i^{[l]}$ is complete.

Case 2: $S_i^{[l]}$ got v from another process P_q .

Claim. Process P_q has obtained v directly from P_p .

Proof: Note $S_i^{[l]}$ got v by reading a label from P_q . That is, P_q writes v in the pth component of the label. To form a new label, P_q uses the jth component of the labels it reads from processes P_j (Statements 5–6 in LABELING). Hence, P_q obtains v directly from P_p . \Box

Let $L_q^{[m_o]}$ be the corresponding labeling operation execution. Note that each labeling operation execution also executes traceable Reads (Statement 5). Then $\pi(TR_{q,p}^{[m_o]})$ is $TW_p^{[k_j]}$ and $\pi(TR_{i,q}^{[l]})$ is $TW_q^{[m_o]}$. As argued in Case 1, either $TW_p^{[k_j]}$ or $TW_p^{[k_{j+1}]}$ stores v in $myLend_p[p][0..1][q]$ This value will not be changed until $L_q^{[m_o]}$ is complete, in fact until P_q starts its next operation execution $O_q^{[m_o+1]}$. Let $TW_p^{[k_{j'}]}$, $j' \ge j+1$, be the first traceable Write that changes the $myLend_p[p][0..1][q]$ values different from v. Then, it must have found $L_q^{[m_o]}$ is complete and the next operation execution of P_q , namely $O_q^{[m_o+1]}$, has started. From $W_q^{[m_o]}(lend[q,p]) \longrightarrow$ $O_q^{[m_o+1]}(\mathcal{O}) \longrightarrow L_p^{[k_{j'}]}(\mathcal{O}) \longrightarrow L_p^{[k_{j'+1}]}$, we have $W_q^{[m_o]}(lend[q,p]) \longrightarrow L_p^{[k_{j'+1}]}$. That is, $L_p^{[k_{j'+1}]}$ and successive labeling operation executions of P_p would not reissue v if v is found in lend[q, p](Statements 1–2). Note that $TW_q^{[m_o]}$ will write v in lend[q, p][1][q] at the end of its execution (Statements 5–6 in TRACEABLE-WRITE). Also note that the traceable Write $TW_p^{[k_{j'}]}$ (actually $L_p^{[k_{j'}]}$) does not issue v. Now, from $\pi(TR_{i,q}^{[l]})$ is $TW_q^{[m_o]}$ it follows, by Lemmas 5 and 6, that either $TW_q^{[m_o]}$ or $TW_q^{[m_{o+1}]}$ would execute the *if*-statement for P_i , and write v in $myLend_q[p][0..1][i]$ indicating that the private value v of P_p is being used by P_i , and this will not be changed until $S_i^{[l]}$ is complete; in fact, until the next operation execution $O_i^{[l+1]}$ of P_i starts. Hence $L_p^{[k_{j'+1}]}$ and successive labeling operation executions of P_p that may occur before $S_i^{[l]}$ is complete are able to trace v in lend[q, p], and hence, will not reissue v. \Box

Corollary 1 It is clear from the proof of Lemma 7 that if a Scan $S_i^{[l]}$ uses a private value vof P_p which is written in labeling operation execution $L_p^{[k_j]}$, then $TW_p^{[k_j]}(\mathcal{O}[p]) \dashrightarrow TR_{i,p}^{[l]}(\mathcal{O}[p])$ for direct reading and $TW_p^{[k_j]}(\mathcal{O}[p]) \dashrightarrow TR_{q,p}^{[m_o]}(\mathcal{O}[p]) \longrightarrow TW_q^{[m_o]}(\mathcal{O}[q]) \dashrightarrow TR_{i,q}^{[l]}(\mathcal{O}[q])$ for indirect reading of v via process P_q . For the latter relation, by the axioms of Anger[3], $TW_p^{[k_j]}(\mathcal{O}[p]) \dashrightarrow TR_{i,q}^{[l]}(\mathcal{O}[q]) \square$

The following lemma shows that Scans can determine the correct temporal order of the private values of all processes.

Lemma 8 Let $S_i^{[l]}$ be a Scan that uses private values v and v' of a process P_p . Then, $S_i^{[l]}$ can determine the correct temporal order between the values v and v'.

Proof: Assume Scan $S_i^{[l]}$ uses the two different private values v and v' of process P_p that has written them in traceable Writes $TW_p^{[k_j]}$ and $TW_p^{[k_{j'}]}$, respectively, where j < j', and hence, $v \prec_p v'$ (as defined in Section 3). By Lemma 7, P_p does not recycle v and v' until $S_i^{[l]}$ is complete. To guarantee the correctness of the timestamp system, we need to make sure that $S_i^{[l]}$ can correctly determine the order $v \prec_p v'$ in case these values are used in ordering some of the scanned labels. From the LABELING and SCAN routines and Corollary 1, we have $W_p^{[k_{j'}]}(order[p,i]) \longrightarrow TW_p^{[k_{j'}]}(\mathcal{O}[p]) \dashrightarrow TR_{i,q}^{[l]}(\mathcal{O}[q]) \longrightarrow R_i^{[l]}(order[p,i])$, where q is as defined in Corollary 1. That is, $W_p^{[k_{j'}]}(order[p,i]) \longrightarrow R_i^{[l]}(order[p,i])$. Now, we need to make sure that $L_p^{[k_{j'}]}$ can correctly determine that the private value v is being used by the process P_i , before writing order[p,i]. Off course, it would assume v' could be used by P_i too. As it knows $v \prec_p v'$, to inform this ordering to P_i , it writes v at a lower indexed entry in order[p,i]than v'. The successive labeling operation executions do not change this ordering. Thus, P_i can determine the order of v and v' correctly after reading order[p,i], by the regularity of order variables.

Now we answer the question how $L_p^{[k_{j'}]}$ finds that v might be used by P_i . Note that P_p does not know precisely which of its private values P_i is going to use. So, it guesses a subset of its private values, which contains the values actually being used by P_i . There are two cases to be considered.

Case 1. P_i obtains v directly from P_p . Either $TW_p^{[k_j]}$ or $TW_p^{[k_{j+1}]}$ will reserve v for P_i by storing v in lend[p,p][0..1][i], and hence the use of v by P_i is traceable.

Case 2. P_i obtains v indirectly through another process P_q , for some q. From the claim in the proof of Lemma 7, we know that P_q has obtained v directly from P_p . Let the corresponding labeling operation execution be $L_q^{[m_0]}$. Either $TW_p^{[k_j]}$ or $TW_p^{[k_{j+1}]}$ will set lend[p,p][0..1][q]to v, and P_p assumes v could be used by any process P_i through $\mathcal{O}[q]$ (one level of indirect propagation of a private value). At the end of $L_q^{[m_o]}$, in $TW_q^{[m_o]}$, P_q informs P_p that v is in $\mathcal{O}[q]$ by setting lend[q,p][1][q] to v (Statements 5–6), and this value could be used by any process P_i . Alternatively, if P_q detects that the v is being used by P_i , it informs "this using" through lend[q,p][0..1][i] (Statements 4.2.2 and 6).

Hence, if $L_p^{[k_{j'}]}$ finds v in lend[p, p][0..1][i] or lend[p, p][0..1][q] or lend[q, p][1][q] or lend[q, p][0..1][i],

for some q, it will assume that v is being used by P_i (Statements 1 and 4.1 in LABELING procedure).

The assertion follows. \Box

Claim 1 Each order variable is of size at most 5n.

Proof: As discussed in the proof of Lemma 8, P_p needs to reserve its private values referred to in lend[q,p][0..1][i], lend[q,p][1][q] and lend[p,p][0..1][q] for all q, that is, at most 5n values for process P_i . The claim follows. \Box

Corollary 2 The set of private values is bounded. In fact, by Statements 1–2 in the LABEL-ING procedure, the size of the set is less than $2n^2$. \Box

By the discussion at the end of 3rd paragraph, Section 3, the correctness of the proposed construction is immediate. However, for the sake of completeness, we give the proof in Theorem 1. Before that a technical lemma follows.

Lemma 9 Let $TR_{i,p}^{[l]}$ and $TR_{i',p}^{[l']}$ be two traceable Reads such that $TR_{i,p}^{[l]} \longrightarrow TR_{i',p}^{[l']}$ and $\pi(TR_{i,p}^{[l]})$ be $TW_p^{[k_j]}$. Then,

(a)
$$W_p^{[k_j]}(c[p]) \Longrightarrow R_{i'}^{[l']}(c[p]),$$

(b) $\pi(TR_{i',p}^{[l']})$ is $TW_p^{[k_{j'}]},$ where $j' \ge j, k_{j'} \ge k_j.$

Proof: We have the following two cases.

Case 1: $TR_{i,p}^{[l]}$ finds $r_{i,p} \neq w_{p,i}$ on $R_i^{[l,2]}(w_{p,i})$.

Lemma 5(b) implies that $\pi(R_i^{[l]}(c[p]))$ is $W_p^{[k_j]}(c[p])$. Then, we have $TW_p^{[k_{j-1}]} \longrightarrow W_p^{[k_j]}(c[p]) \Longrightarrow$ $R_i^{[l]}(c[p]) \longrightarrow R_{i'}^{[l',1]}(w_{p,i'}) \longrightarrow R_{i'}^{[l']}(c[p]).$ *Case 2:* $TR_{i,p}^{[l]}$ finds $r_{i,p} = w_{p,i}$ on $R_i^{[l,2]}(w_{p,i}).$

By Lemma 6, we have $TW_p^{[k_{j-1}]} \longrightarrow W_p^{[k_j]}(c[p]) \longrightarrow W_p^{[k_j]}(w_{p,i}) \Longrightarrow R_i^{[l,2]}(w_{p,i}) \longrightarrow R_{i'}^{[l',1]}(w_{p,i'}) \longrightarrow R_{i',p}^{[l']}(c[p]).$

For both the cases we have $W_p^{[k_j]}(c[p]) \Longrightarrow R_{i'}^{[l']}(c[p])$; part (a) follows. If $TR_{i',p}^{[l']}$ finds $r_{i',p} \neq w_{p,i'}$ on $R_{i'}^{[l',2]}(w_{p,i'})$, then part (b) follows by Lemma 5. Assume $TR_{i',p}^{[l']}$ finds $r_{i',p} = w_{p,i'}$ on $R_{i'}^{[l',2]}(w_{p,i'})$. From the above two cases, we have $TW_p^{[k_{j-1}]} \longrightarrow R_{i'}^{[l',1]}(w_{p,i'})$. Then part (b) follows by Lemmas 4 and 6. \Box

Theorem 1 The construction of Figure 1 is a correct implementation of wait-free bounded concurrent timestamp systems.

Proof: The wait-freedom property is immediate from the structure of the four routines in Figure 1. The boundedness follows from Corollary 2. We now show that the construction satisfies all the four properties P1–P4 described in Section 2.

Ordering: Consider two labeling operation executions $L_p^{[k]}$ and $L_q^{[k']}$ with labels $l_p^{[k]}$ and $l_q^{[k']}$, respectively. Let m be the least significant index such that $l_p^{[k]}[m] \neq l_q^{[k']}[m]$. Assume these private values $l_p^{[k]}[m]$ and $l_q^{[k']}[m]$ are written by P_m at labeling operation executions $L_m^{[s_o]}$ and $L_m^{[s_{o'}]}$, respectively. We define $L_p^{[k]} \Rightarrow L_q^{[k']}$ iff $L_m^{[s_o]} \longrightarrow L_m^{[s_{o'}]}$.

- Precedence: Without loss of generality we assume $L_p^{[k]} \longrightarrow L_q^{[k']}$. By Lemmas 5 and 6, we have $\pi(TR_{p,m}^{[k]})$ is $TW_m^{[s_o]}$ and $\pi(TR_{q,m}^{[k']})$ is $TW_m^{[s_{o'}]}$. Then, from $TR_{p,m}^{[k]} \longrightarrow TR_{q,m}^{[k']}$ and Lemma 9(b), we have $s_{o'} \ge s_o$. As $l_p^{[k]}[m] \ne l_q^{[k']}[m]$, we have $s_{o'} \ne s_o$, and hence, $s_{o'} > s_o$. That is, $L_m^{[s_o]} \longrightarrow L_m^{[s_{o'}]}$. The precedence property follows.
- Consistency: For any two labels $l_p^{[k]}$ and $l_q^{[k']}$ (returned by a Scan) such that m is the least significant index for which $l_p^{[k]}[m] \neq l_q^{[k']}[m]$. We define $l_p^{[k]} \prec l_q^{[k']}$ iff $l_p^{[k]}[m] \prec_m l_q^{[k']}[m]$ iff $L_m^{[s_o]} \longrightarrow L_m^{[s_{o'}]}$. The consistency property follows by Lemma 8 and the definition of \Rightarrow given above.

Regularity: Consider a Scan $S_i^{[j]}$ that returns a label $l_p^{[m_o]}$ that is written by a labeling operation execution $L_p^{[m_o]}$, that is, $\pi(TR_{i,p}^{[j]})$ is $TW_p^{[m_o]}$. By Lemmas 5 and 6, we can say $TW_p^{[m_o]} \dashrightarrow TR_{i,p}^{[j]}$, and hence, $L_p^{[m_o]} \dashrightarrow S_i^{[j]}$. The second part of the regularity property follows from: (i) if $TR_{i,p}^{[j]}$ finds $r_{i,p} \neq w_{p,i}$ on $R_i^{[j,2]}(w_{p,i})$, then, by Lemma 5, $\pi(TR_{i,p}^{[j]})$ is $TW_p^{[m_o]}$, where $\pi(R_i^{[j]}(c[p]))$ is $W_p^{[m_o]}(c[p])$, and so, $TW_p^{[m_o+1]} \not\to TR_{i,p}^{[j]}$, and hence $L_p^{[m_o+1]} \not\to$ $S_i^{[j]}$; (ii) if $TR_{i,p}^{[j]}$ finds $r_{i,p} = w_{p,i}$ on $R_i^{[j,2]}(w_{p,i})$, then, by Lemma 6, $\pi(TR_{i,p}^{[j]})$ is $TW_p^{[m_o]}$, where $R_i^{[j,1]}(w_{p,i}) \Longrightarrow W_p^{[m_o]}(w_{p,i}) \Longrightarrow R_i^{[j,2]}(w_{p,i})$, and so, $TW_p^{[m_o+1]} \not\to TR_{i,p}^{[j]}$, and hence $L_p^{[m_o+1]} \not\to S_i^{[j]}$.

Monotonicity: Consider two Scans $S_i^{[j]} \longrightarrow S_{i'}^{[j']}$. Let $S_i^{[j]}$ return label $l_p^{[m_o]}$ from a process P_p . By Lemmas 5 and 6, we have $\pi(TR_{i,p}^{[j]})$ is $TW_p^{[m_o]}$. From $S_i^{[j]} \longrightarrow S_{i'}^{[j']}$, we have $TR_{i,p}^{[j]} \longrightarrow TR_{i',p}^{[j']}$. The monotonicity property follows by Lemma 9.

Extended regularity: Consider a Scan $S_i^{[j]}$ that returns a label $l_p^{[m_o]}$ that is written by a labeling operation execution $L_p^{[m_o]}$, that is, $\pi(TR_{i,p}^{[j]})$ is $TW_p^{[m_o]}$. For each labeling operation execution

$$\begin{split} L_q^{[m']}, & \text{if } S_i^{[j]} \longrightarrow L_q^{[m']}, \text{ then } TR_{i,p}^{[j]} \longrightarrow TR_{q,p}^{[m']}. \text{ Then, by Lemma 9}(a), \text{ we have } W_p^{[m_o]}(c[p]) \Longrightarrow \\ R_q^{[m']}(c[p]) \text{ and hence, } \pi(TR_{q,p}^{[m']}) \text{ is } TW_p^{[m_o]} \text{ or a successor, by Lemma 9}(b). \text{ Also by Lemma 5} \\ & \text{and 6 and the LABELING procedure, we have } TR_{p,s}^{[m_o]} \longrightarrow TW_p^{[m_o]} \dashrightarrow TR_{i,p}^{[j]} \longrightarrow TR_{q,s}^{[m']} \text{ for } \\ & \text{all } s \neq p, \text{ that is, } TR_{p,s}^{[m_o]} \longrightarrow TR_{q,s}^{[m']}. \text{ Hence, } L_q^{[m']} \text{ reads more recent (at least equal) private} \\ & \text{values of all processes than } L_p^{[m_o]}. \text{ Also, we have } l_p^{[m_o]}[q] \prec_q l_q^{[m']}[q]. \text{ Hence } L_p^{[m_o]} \Rightarrow L_q^{[m']}. \text{ The extended regularity property follows. } \Box \end{split}$$

5 Concluding Remarks

This paper combines the preliminary [42, 18]. The former paper is the first to characterize multiwriter shared variables, and provides a bounded construction of multiwriter multireader multivalued atomic variable from 1-writer variables. However, it was later found that the proposed construction doesn't satisfy some properties of atomic shared variables [43]. The technical report [18] corrected and extended [42] to a construction of a concurrent timestamp system using an idea from [10]. The final result is very close to the incorrect construction of [42]. It uses $O(n \log n)$ bit size shared variables (order and lend variables), where n is the number of processes. Scan and labeling operation executions require O(n) steps. The construction uses less shared space than that of [10] at the fundamental level, and is orders of magnitude more efficient in terms of scanning bits at the fundamental level.

5.1 Comparison with Related Work

In [10], they have defined three routines, namely, traceable-read, traceable-write and garbage collection. When the traceable-read function is executed to read a label, the executing process explicitly informs the other processes which of their private values it is going to use. The traceable-write procedure is executed to write a new label. To determine which of its private values are currently in use, a process executes the garbage collection routine. This routine helps processes to safely recycle their respective private values. This is the most intricate routine.

In our construction, we have used a separate implementation technique for a weaker form of the traceable-read and the traceable-write routines. We do not need a garbage collection routine. When a process executes the traceable-read function, it does not explicitly inform the other processes which of their private values it is going to use. On the other hand, the executers of the traceable-write procedure correctly finds which private values of which processes are in use.

Every process needs a separate pool of private values, whose size is fewer than $2n^2$. In their construction, the pool size is $22n^2$. All the ordering shared variables used in our construction are of 1-writer 1-reader regular ones, whereas they are 1-writer *n*-reader atomic ones in their construction. In our construction, a Scan reads at most n-1 1-writer 1-reader regular ordering shared variables, whereas in their construction it is 2n - 2 1-writer *n*-reader atomic ones. In our construction all but one bit are nonatomic 1-writer 1-reader variables. Table 1 presents some comparison results briefly.

Construction	Shared variable size	Shared space(bits)	Labeling	Scan
[9]	O(n)	$O(n^3)$	O(n)	$O(n^2 \log n)$
[14]	$O(n^2)$	$O(n^4)$	$O(n \log n)$	$O(n \log n)$
[22]	$O(n^2)$	$O(n^4)$	O(n)	O(n)
[10]	$O(n \log n)$	$O(n^5 \log n)$	O(n)	O(n)
[11]	O(n)	$O(n^3)$	O(n)	O(n)
This paper	$O(n \log n)$	$O(n^3 \log n)$	O(n)	O(n)

Table 1: Comparison Results.

Of all proposed constructions of bounded concurrent timestamp systems we are aware of, the construction in this paper is the 'simplest'. The correctness proof, though involved, is easier to follow. It is used as a basis in the reference text [4] to describe bounded concurrent timestamp system.

Although we have used a notion of vector clocks for our construction, as in [42], we may not really need the full power of vector clock concept developed later by Mattern [32]. In CTSs, we are not interested in determining causal 'independence' of various labeling operation executions. The ordering property of CTSs infers that the causal orders among labeling operation executions matter most. We need to have a total order on all labeling operation executions, and the total order must extend their original causal relation. This is akin to the logical time of Lamport [27]. We suspect that there might be a way to eliminate the vector clock altogether, by an efficient way of recycling of global values, instead of using n sets of private values.

The construction presented here should not be considered as an alternative implementation

of the traceable use abstraction, for it restricts the value propagation at indirection level one. It is not clear to the authors how this strategy could be extended for a general implementation of the abstraction.

5.2 A Brief Early History

The development of bounded wait-free shared variables and timestamp systems has been quite problematic and error-prone. It may be useful at this point to present a brief early history of the area: who did what, when, and where, and which solutions are known to be incorrect. In a series of papers [25, 26, 27, 29] starting in 1974, Lamport explored various notions of concurrent reading and writing of shared variables culminating in the seminal 1986 paper [29]. It formulates the notion of wait-free implementation of an atomic shared variable written by a single writer and read by (another) single reader—from safe 1-writer 1-reader 2-valued shared variables, being mathematical versions of physical *flip-flops*. Predating the latter paper, in 1983 Peterson [34] published an ingenious wait-free construction of an atomic 1-writer, *n*-reader *m*-valued atomic shared variable from n+2 safe 1-writer *n*-reader *m*-valued registers, 2n 1-writer 1-reader 2-valued atomic shared variables, and 2 1-writer n-reader 2valued atomic shared variables. He presented also a proper notion of wait-freedom property. Lamport [28] gave an example that appeared to contradict a possible interpretation of the informal statement of a theorem in [34], which, as Peterson apparently retorted to Lamport, was not intended. In his paper, Peterson didn't tell how to construct the *n*-reader boolean atomic variables from flip-flops, while Lamport mentioned the open problem of doing so, and, incidentally, uses a version of Peterson's construction to bridge the algorithmically demanding step from atomic shared bits to atomic shared multivalues. Based on this work, N. Lynch, motivated by concurrency control of multi-user data-bases, posed around 1985 the question of how to construct wait-free multiwriter atomic variables from 1-writer multireader atomic variables. Her student Bloom [5] found in 1985 an elegant 2-writer construction, which, however, has resisted generalizations to multiwriter. Vitányi and Awerbuch [42] were the first to define and explore the complicated notion of wait-free constructions of general multiwriter atomic variables. They presented a proof method, an unbounded solution from 1-writer 1-reader atomic variables, and a bounded solution from 1-writer n-reader atomic variables. The unbounded solution was made bounded in [31]. It is optimal for the implementation of n-writer *n*-reader atomic variables from 1-writer 1-reader ones. "Projections" of the construction also

give specialized constructions for the implementation of 1-writer *n*-reader atomic variables from 1-writer 1-reader ones, and for the implementation of n-writer n-reader atomic variables from 1-writer *n*-reader ones. As noted in [30], the first "projection" is optimal, while the last "projection" may not be optimal since it uses O(n) control bits per writer while only a lower bound of $\Omega(\log n)$ was established. Taking up this challenge, the construction in [23] apparently achieves this lower bound. The earlier bounded solution in [42] (corresponding in fact to the problem correctly solved by the last "projection" above) turned out not to be atomic, but only achieved regularity [43]. Nonetheless, [42] introduced important notions and technique in the area, like (bounded) vector clocks. These were inspired by the celebrated "Bakery" algorithm of Lamport [25], which can be viewed as a global bounded "clock" determining the order among queued processes much like the ticket dispenser in a bakery serves to determine the order of servicing waiting customers. The multiwriter situation has stronger requirements than apparently can be satisfied by a global ticket dispenser. The solution in [42] was the construction of a bounded "vector clock": a private ticket dispenser for each process, the storing and updating of a vector of latest tickets held by all processes, together with a semantics to determine the order between vectors. Moreover, a complex mechanism primitive traceable read/write—is presented to keep track of which tickets of what processes could still be present in the system, with the objective of bounding the private ticket pool of each process by recycling obsolete tickets. Following the appearance of [42], Peterson who had been working on the multiwriter problem for a decade, together with Burns, revamped the construction retaining the vector clocks, but replaced the primitive traceable read/write elements by repeated scanning as in [34]. The result [35] was found to be nonetheless erroneous, in the technical report [36]. This makes the multiwriter problem perhaps the only one for which two consecutive wrong solutions were published in the highly selective FOCS conferences. Neither the re-correction in [36], nor the claimed re-correction by the authors of [35] has appeared in print. The present paper constitutes a correction of the original [42] by the extension of [18]: by implementing the stronger concurrent timestamp system it also solves the atomic multiwriter problem. Apart from the already mentioned [31], the only other multiwriter multireader atomic shared variable construction that appeared in journal version seems to be of Abraham [1]. Also in 1987 there appeared at least five purported solutions for the implementation of 1-writer *n*-reader atomic shared variable from 1-writer 1-reader ones: [24, 38, 6, 37] and the conference version of [21], of which [6] was shown to be incorrect in [15]

and only [37] appeared in journal version. The only other 1-writer *n*-reader atomic shared variable construction appeared in journal version is of Haldar and Vidyasankar [16]. A. Is-raeli and M. Li were attracted to the area by the work in [42], and, in an important paper [21], they raised and solved the question of the more general and universally useful notion of bounded timestamp system to track the order of events in a concurrent system. Their sequential timestamp system was published in journal version, but the preliminary concurrent timestamp system in the conference proceedings, of which a more detailed version has been circulated in manuscript form, has not been published in final form.

The difficulty of wait-free atomic multireader-, multiwriter-, and timestamp system constructions, and the many errors in purported and published solutions, have made it hard to publish results in print. Of the major pioneering papers, the first correct multiwriter construction of 1987 [31] was rejected at five consecutive conferences until it was published in ICALP, 1989. The final journal version was handled by three consecutive editors, scrutinized by three consecutive sets of referees, and lasted from 1989 until publication in 1996. The pioneering timestamp paper, [21], was submitted in 1987/88 to this journal, after a couple of years rejected since a stronger result [9] had appeared in conference version, submitted to another journal and finally appeared in 1993, but only the part containing the simpler sequential timestamp construction. The first generally accepted concurrent timestamp construction [9] appeared in conference version in 1989, but its journal version appeared only in 1997. As stated before, the concurrent timestamp construction in the present paper is based on the 1986 paper [42] supplemented by the 1993 technical report [18]. For further remarks see [31] in this journal and the Introduction to present paper.

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Declarations

Constants:

n = number of processes;

Type:

label-type: array [1..n] of natural number; {represents vector clock} boolean: 0..1;

Shared variables and their initial values:

w: array [1n, 1n] of boolean <i>atomic</i> ;	{all initially 0} { P_p writes $w[p, 1n]$ and P_i reads $w[1n, i]$ }
r: array [1 n , 1 n] of boolean <i>atomic</i> ;	{all initially 0} { P_p writes $r[p, 1n]$ and P_i reads $r[1n, i]$ }
c: array [1 n] of boolean <i>atomic</i> ;	{ initially 0} { P_p writes $c[p]$, and the others read}
label: array [1n, 01] of label-type $safe$;	{all initially 0, except $label[p, 0][p] = 1$ for all p } { P_p writes $label[p, 01]$ and the others read}
copylabel: array $[1n, 1n]$ of label-type a	safe; $\{P_p \text{ writes } copylabel[p, 1n] \text{ and } P_i \text{ reads } copylabel[1n, i]\}$
lend: array $[1n, 1n]$ of regular array $[0]$	1] of label-type; { all initially 0} { P_p writes $lend[p, 1n]$ and P_i reads $lend[1n, i]$ }
order: array $[1n, 1n]$ of regular array $[1$	5n] of natural number; { initially $order[1n, 1n][1] = 0$ and $order[1n, 1n][2] = 1$ } { P_p writes $order[p, 1n]$ and P_i reads $order[1n, i]$ }

Private variables for process P_p , p = 1, 2, ..., n:

 $\begin{array}{ll} cl_p: \mbox{ boolean;} & \{\mbox{initially } 0\} \\ myLend_p: \mbox{ array } [1..n] \mbox{ of array } [0..1] \mbox{ of label-type; } \{\mbox{all initially } 0\} \\ old-label_p: \mbox{ label-type; } & \{\mbox{all initially } 0, \mbox{ except } old-label_p[p] = 1\} \\ \prec_p: \mbox{ total order relation; } & \{\mbox{ (0..1) } \} \\ \end{array}$

Figure 1: Shared variables.

Procedure TRACEABLE-WRITE(p: 1..n; new-label: label-type); { P_p writes new-label in $\mathcal{O}[p]$ } var

 $i, j: 1..n; \{\text{loop index}\}$ lr: boolean; begin 1. $cl_p := \neg cl_p;$ 2. write *new-label* in *label* $[p, cl_p]$; 3. write cl_p in c[p]; 4. for i := 1 to n do begin {could be done in parallel} read lr from r[i, p]; 4.14.2if $lr \neq w[p, i]$ then 4.2.1write *new-label* in copylabel[p, i]; 4.2.2for j := 1 to n do $myLend_p[j][0..1][i] := \langle old-label_p[j], new-label[j] \rangle$; $\{w[p,i] = r[i,p]\}$ 4.2.3write lr in w[p, i]; endif; endfor; 5. for j := 1 to n do $myLend_p[j][1][p] := new-label[j];$ 6. for j := 1 to n do write $myLend_p[j]$ in lend[p, j]; {could be done in parallel} 7. $old-label_p := new-label;$

end; {of procedure}

Function TRACEABLE-READ(p: 1..n, i: 1..n): label-type; { P_p reads a label from P_i } var

lw: boolean; lc: boolean; savelabel: label-type;

begin

- 1. read lw from w[i, p]; 2. write $\neg lw$ in r[p, i]; 3. read lc from c[i];
- 4. read savelabel from label[i, lc];
- 5. read lw from w[i, p];
- 6. if $(r[p, i] \neq lw)$ then return(savelabel) 7. else

$$\{r[p,i] = w[i,p]\}$$

 $\{r[p,i] \neq w[i,p]\}$

read and return(copylabel[i, p])

endif;

end; {of function}

Figure 1: Construction for process P_p . (Cont'd.)

Procedure LABELING(p: 1..n);

var

j, k: 1..n; temp: array [1..n] of array [0..1] of label-type; lab: array [1..n] of label-type; new-label: label-type; private-value: natural number;

begin

- 1. for j := 1 to n do {could be done in parallel}
- read temp[j] from lend[j, p]; {we do not need temp[j][0][j]}
- 2. select a new *private-value* not in temp[1..n] and the current private value; {use the axiom of choice here}
- 3. put the new *private-value* in \prec_p as the largest element;
- 4. for j := 1 to n do {could be done in parallel}
- 4.1 order the elements of (temp[1..n][0..1][j]),

temp[k][1][k] and

temp[p][0..1][k] for all k,

and the new *private-value*) consistent with \prec_p

- 4.2 and write them in order[p, j];
- 5. for j := 1 to $n, j \neq p$, do $lab[j] := TRACEABLE-READ(p, j); \{could be done in parallel\}$
- 6. $new-label := \langle lab[1][1], lab[2][2], \dots, lab[p][p] := private-value, \dots, lab[n][n] \rangle;$
- 7. TRACEABLE-WRITE(*p*, *new-label*);

end;

Function SCAN(p: 1..n): (\overline{l}, \prec) ;

var

i, j, k: 1..n;lab: array [1..n] of label-type;

begin

```
    for j := 1 to n do lab[j] :=TRACEABLE-READ(p, j); {could be done in parallel}
    for i := 1 to n do
```

- 2.1 for j := 1 to n do
- 2.1.1 let k be the least significant index in which lab[i] differs from lab[j];
- 2.1.2 if order[k, p] (which is a subset of \prec_k) is not read yet, then read it;
- 2.1.3 determine the order between lab[i] and lab[j] using \prec_k ;

```
end;
```

Figure 1: Construction for process P_p . (Cont'd.)