5. Examples

A number of examples were programmed on the CDC 1604 at the University of Wisconsin and the two described below are typical. Detailed aspects of the programming are given in [18].

Example I. Let region R be the quarter circle bounded by $x^2 + y^2 = 1$, $x = 0$ and $y = 0$. Consider problem D_1 with $k = -5$, $\phi = x + y^6$. Method D_1 was applied with $\sigma = .03$, $(\bar{x}, \bar{y}) = (0,.03)$, $h' = .025$, $h'' = .01$. The resulting set of 2947 linear algebraic equations was solved by over-relaxation [17]. The running time was 9 minutes. Selected, but typical, results are recorded under u^* in Table I.

Example II. Let R be a rectangular region whose boundary has consecutive vertices $A(0,0)$, $B(1,0)$, $C(1,2)$ and $D(0,2)$. Consider problem D_2 with $k = 1$, $\phi = -2x^2 + y^2$. Method D_2 was applied with $\sigma = .02$, $(\bar{x}, \bar{y}) = (0,.02)$, $h' = .02$, $h'' = .04$. The resulting set of 2401 linear algebraic equations was solved by over-relaxation. The running time was 10 minutes. Selected, but typical, results are recorded under u^* in Table II.

It should also be noted that Methods D_1 and D_2 yielded good approximations to certain problems in which u_{yy} became infinite as y approached zero.

RECEIVED NOVEMBER, 1963

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Generation of Test Matrices by Similarity Transformations

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A method for obtaining test matrices with a prescribed distribution of characteristic roots is given. The process consists of using particularly simple similarity transformations to generate full matrices from canonical forms. The matrices generated also have known characteristic vectors, inverses and determinants.

There are several well-known methods for generating matrices whose characteristic roots and vectors are known (see, e.g. [1–5]). If one wishes a matrix with a prescribed distribution of characteristic roots, however, it is natural to resort to the similarity transformation $A = CRC^{-1}$ where the characteristic roots of R are known. The purpose of this note is to describe a simple way of generating matrices by this transformation that does not seem to be as well known as it should be. Although the emphasis here is on matrices with known characteristic roots, it should be mentioned that since $A^{-1} = CR^{-1}C^{-1}$ one can easily construct the inverses of these matrices also.

Let $C = I + uv^*$ where u and v are $n \times 1$ and * denotes conjugate transpose.[1] Only the fact that $C^{-1} = I - (1 + v^*u)^{-1}uv^*$ is needed, but it is also easily verified that any vector orthogonal to v is a characteristic vector of C corresponding to the root 1 and u is a characteristic

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[1] It should be noted that the matrices discussed recently by Pei, LaSor, Rodman and Newberry in the Pracniques section of the *Communications of the ACM* are of the form $\gamma(I + uu^*)$, γ a scalar. For a discussion of these and a more general class of matrices see [2].

vector corresponding to $1+v^*u$. If $v^*u = 0$, C has a nondiagonal canonical form and v is a principal vector, i.e., $(C-I)^2v = 0$. Because of the characteristic root of multiplicity $n-1$, these matrices themselves have limited use as test matrices for characteristic value problems although they are quite useful as test matrices for inversion programs.

With $\alpha = (1+v^*u)^{-1}$, the similarity transformation then becomes

$$\begin{aligned} A &= (I+uv^*)R(I-\alpha uv^*) \\ &= R+uv^*R-\alpha Ruv^*-\alpha(v^*Ru)uv^*, \end{aligned}$$

which can be carried out with $O(n^2)$ operations. For testing accuracy of programs, however, it is imperative that A be generated exactly and we now consider some special choices of u, v and R that facilitate the computation. For simplicity we consider only real u, v and R and thus generate only real matrices A. The examples are meant only to be illustrative and there is obviously a great flexibility available in choosing u and v.

Symmetric Matrices

Let $\sum v_i^2 = 1$, $u = -2v$ and $R = D = \operatorname{diag}(d_1, \cdots, d_n)$. Then $I-2vv^T$ is orthogonal and

$$\begin{aligned} A &= (I-2vv^T)D(I-2vv^T) \\ &= D-2vv^TD-2Dvv^T+4(v^TDv)vv^T \end{aligned}$$

is symmetric with characteristic roots $d_1, \cdots, d_n$ and characteristic vectors which are the columns of $I-2vv^T$. In particular, if $v^T = (n^{-\frac{1}{2}}, \cdots, n^{-\frac{1}{2}})$ then

$$A = n^{-1}(nd_i\delta_{ij}-2d_i-2d_j+2r),$$

where $r = 2n^{-1}\sum d_k$ and δ_{ij} is the Kronecker symbol. Note that because of the divisions by n a little care is required to generate this matrix exactly.

Nonsymmetric Matrices with Real Roots

Again choose $R = D = \operatorname{diag}(d_1, \cdots, d_n)$. There is considerable freedom in choosing u and v although the restriction $u^Tv = 0$ affords some simplification. For example, if $n = 2k$, $u^T = c(1,1,\cdots,1)$, $v^T = (1,\cdots,1,-1,\cdots,-1)$ and $\sigma = v^TDu$, then

$$A = D+uv^TD-Duv^T-\sigma uv^T = (a_{ij}),$$

where

$$a_{ij} = \begin{cases} d_i\delta_{ij}-c(d_i-d_j+\sigma), & 1 \leqq j \leqq k, \\ d_i\delta_{ij}+c(d_i-d_j+\sigma), & k+1 \leqq j \leqq n. \end{cases}$$

This matrix has real roots $d_1,\cdots,d_n$ and characteristic vectors which are the columns of $I+uv^T$. It is easy to generate exactly since only additions of the d_i and multiplications by c are involved; if $c = 1$, only additions are involved. The parameter c allows some control over the "condition" of the problem since the condition number of the mth root, defined as the Euclidean length of the mth row of $(I+uv^T)^{-1}$ times the length of the mth column of $(I+uv^T)$, is $[1+c^4n^2+2c^2(n-2)]^{\frac{1}{2}}$.

Another choice of u and v that maintains the relation $u^Tv = 0$ but gives varying condition numbers for the roots is $u^T = (1,2,\cdots,k,\ 1,2,\cdots,k)$ and $v^T = (1,2,\cdots,k,\ -1,-2,\cdots,-k)$. The corresponding matrix is easily constructed and it suffices to remark that the condition number of the mth and $(m+k)$-th roots is $\frac{1}{3}\{[3+m^2k(k+1)\cdot(2k+1)]^2-36m^4\}^{\frac{1}{2}}$.

Real Matrices with Complex Roots

Let $R = \operatorname{diag}(R_1,\cdots R_p)$ be a block-diagonal matrix where the R_i have known characteristic roots and vectors. For example, the R_i may be 1×1 and 2×2 although we do not preclude the use of larger blocks. Partition $u^T = (u_1^T, \cdots, u_p^T)$ and $v^T = (v_1^T, \cdots, v_p^T)$ to correspond to the R_i and assume that $v^Tu = 0$. Then if $B_{ij} = u_iv_j^TR_j$, $C_{ij} = R_iu_iv_j^T$ and $\sigma = u^TRv$, A can be written in partitioned form as

$$A = R + \begin{bmatrix} B_{11} \cdots B_{1p} \\ \vdots \quad\quad \vdots \\ B_{p1} \cdots B_{pp} \end{bmatrix} - \begin{bmatrix} C_{11} \cdots C_{1p} \\ \vdots \quad\quad \vdots \\ C_{p1} \cdots C_{pp} \end{bmatrix} - \sigma \begin{bmatrix} u_1v_1^T \cdots u_1v_p^T \\ \vdots \quad\quad \vdots \\ u_pv_1 \cdots u_pv_p^T \end{bmatrix}.$$

If, for example, the elements of u and v are -1, 0 and 1, then the elements of A are just sums and differences of the elements of R and thus are easily generated. The characteristic and principal vectors of A are the columns of

$$I + \begin{bmatrix} u_1v_1^TQ_1 \cdots u_1v_p^TQ_p \\ \vdots \quad\quad \vdots \\ u_pv_1^TQ_1 \cdots u_pv_p^TQ_p \end{bmatrix},$$

where $R_i = Q_iJ_iQ_i^{-1}$ and J_i is the Jordan form of R_i.

Acknowledgment. I would like to thank Dr. Wallace Givens of Argonne National Laboratory for first pointing out to me some years ago the properties of matrices of the form $I+uv^*$.

RECEIVED NOVEMBER, 1963

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