A HOMOGENEOUS FORMULATION FOR LINES IN 3 SPACE

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ABSTRACT

Homogeneous coordinates have long been a standard tool of computer graphics. They afford a convenient representation for various geometric quantities in two and three dimensions. The representation of lines in three dimensions has, however, never been fully described. This paper presents a homogeneous formulation for lines in 3 dimensions as an anti-symmetric 4x4 matrix which transforms as a tensor. This tensor actually exists in both covariant and contravariant forms, both of which are useful in different situations. The derivation of these forms and their use in solving various geometrical problems is described.

Key Words and Phrases: geometric calculations, homogeneous coordinate's, computer graphics

CR Categories: 3.15, 5.14, 8.2

INTRODUCTION

We will assume the reader is somewhat familiar with the homogeneous representation of points and planes in 3 space. A good introduction may be found in [1]. Briefly, a point is represented as a four component vector, usually written as

$$\begin{pmatrix} x & y & z & w \end{pmatrix}$$

Any non-zero multiple of this row vector represents the same point. The "real" components of the point may be discovered by dividing by the fourth component to obtain the three components:

$$\begin{pmatrix} x/w & y/w & z/w \end{pmatrix}$$

A plane is represented as a four component column vector:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Any non-zero multiple of this column vector represents the same plane. The first three components describe a vector normal to the plane and the fourth is related to its distance from the origin.

The dot product of a point (row) vector and a plane (column) vector is proportional to the distance from the point to the plane.

$$\begin{pmatrix} x & y & z & w \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = ax + by + cz + dw \propto D$$

A special case of this is the fact that, if the dot product is zero, the point lies in the plane. If the dot product is non-zero, we can find the actual distance by the following means. Construct a three dimensional vector of unit length perpendicular to the plane. $(A\ B\ C) = (a\ b\ c)/\sqrt{a^2 + b^2 + c^2}$. Scale it up by D and add it to the position of the point. We should then have a point on the plane.

$$\begin{pmatrix} X & Y & Z \end{pmatrix} = \begin{pmatrix} (x/w)+DA & (y/w)+DB & (z/w)+DC \end{pmatrix}$$

Since this point is on the plane, its dot product with the plane vector will be zero. We now have an equation which can be solved for D.

$$0 = \begin{pmatrix} X & Y & Z & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x+DAw & y+DBw & Z+DZw & w \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$D = -\frac{ax+by+cz+dw}{w\sqrt{a^2+b^2+c^2}}$$

The sign of D indicates which side of the plane the point was on. It can be ignored if only the distance is required.

An object defined in terms of homogeneous points may be transformed by multiplication of its points by a 4x4 matrix.

$$\begin{pmatrix} x & y & z & w \end{pmatrix} T = \begin{pmatrix} x' & y' & z' & w' \end{pmatrix}$$

Any combination of scaling, translation, rotation, and perspective distortion may be represented by the matrix T. To determine the coordinates of a plane after it has undergone the same transformation we must pre-multiply by the inverse of T.

$$T^{-1} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \\ c' \end{pmatrix}$$

Thus the dot product of the transformed point and plane is the same as the dot product of the original point and plane. The relationship of a

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point lying on a plane is preserved.

Suppose we are given three points and we wish to determine the components of the plane vector through them. That is, we wish to solve for a, b, c, d in the equation:

$$\begin{pmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Consider a fourth point not in the plane of the other three. Its dot product with the desired plane vector will then be non-zero. We will call it q. The resulting equation is then:

$$\begin{pmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \\ x_4 & y_4 & z_4 & w_4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = M \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ q \end{pmatrix}$$

This equation may be solved by multiplying both sides by the adjoint of M. The adjoint is the transpose of the matrix formed from the co-factors of the original matrix. The co-factor of an element of a matrix is found by erasing the row and column containing the element and computing the determinant of the remaining smaller matrix, finally flipping the sign if the sum of the row and column indices of the element is odd. Thus the co-factor of the x4 term of M is:

$$\mathrm{cof}(x_4) = -\det \begin{pmatrix} y_1 & z_1 & w_1 \\ y_2 & z_2 & w_2 \\ y_3 & z_3 & w_3 \end{pmatrix}$$

The product of a matrix and its adjoint is the identity matrix times the determinant of the original matrix. The product of the adjoint with the right side of the equation is just q times the right hand column. Our equation is now:

$$\det M \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = q \begin{pmatrix} \mathrm{cof}(x_4) \\ \mathrm{cof}(y_4) \\ \mathrm{cof}(z_4) \\ \mathrm{cof}(w_4) \end{pmatrix}$$

Now, since any non-zero multiple of a plane vector represents the same plane, we can neglect the q and det M terms above. Finally, note that the co-factors do not contain any components of the arbitrarily chosen fourth point. This whole process can be represented in a short-hand notation:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} & \vec{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{pmatrix}$$

where

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{l} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

This is simply a generalization of the more familiar shorthand notation of the cross product of two vectors in ordinary three space. The only problem that could arise is if the matrix M were singular. This only occurs if the three original points are co-linear, whereupon there is no solution. In this case, the four co-factors are all zero. We can take the appearance of four zeros when looking for a plane through three points as an indication that the three points were co-linear.

There is a similar mechanism for determining the point of intersection of three planes. That is, the homogeneous coordinates of the point of intersection is:

$$\begin{pmatrix} x & y & z & w \end{pmatrix} = \det \begin{pmatrix} a_1 & a_2 & a_3 & \vec{i} \\ b_1 & b_2 & b_3 & \vec{j} \\ c_1 & c_2 & c_3 & \vec{k} \\ d_1 & d_2 & d_3 & \vec{l} \end{pmatrix}$$

where, here:

$$\vec{i} = (1000) \quad \vec{j} = (0100) \quad \vec{k} = (0010) \quad \vec{l} = (0001)$$

Again, the appearance of four zeros when solving for the point of intersection indicates that the three planes to not have a single common point. They, in fact, intersect on a line.

THE HOMOGENEOUS LINE REPRESENTATION

We shall now construct a homogeneous representation of lines in 3D taking the form of a 4x4 matrix we shall call L. It will have the property that any scalar multiple of it represents the same line. In addition, if a point vector is multiplied by L, a result of four zeros indicates that the point is on the line. The inspiration for this formulation comes from the Grassmann coordinate systems described in [2].

First re-consider the problem of finding the plane through three points. If the four co-factors in the solution are all zero then the three points were co-linear. We can re-interpret this as a condition upon a third point which will make it co-linear with two others. Thus for two given points P1 and P2, a third point is co-linear if:

$$\det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} & \vec{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x & y & z & w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

That is, we must have

$$-y \det \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} + z \det \begin{pmatrix} y_1 & w_1 \\ y_2 & w_2 \end{pmatrix} - w \det \begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \end{pmatrix} = 0$$

$$x \det \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} - z \det \begin{pmatrix} x_1 & w_1 \\ x_2 & w_2 \end{pmatrix} + w \det \begin{pmatrix} x_1 & z_1 \\ x_2 & z_2 \end{pmatrix} = 0$$

$$-x \det \begin{pmatrix} y_1 & w_1 \\ y_2 & w_2 \end{pmatrix} + y \det \begin{pmatrix} x_1 & w_1 \\ x_2 & w_2 \end{pmatrix} - w \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = 0$$

$$x \det \begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \end{pmatrix} - y \det \begin{pmatrix} x_1 & z_1 \\ x_2 & z_2 \end{pmatrix} + z \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = 0$$

Now defining the six new coordinates:

$$p=\det\begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} \qquad q=\det\begin{pmatrix} y_1 & w_1 \\ y_2 & w_2 \end{pmatrix} \qquad r=\det\begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \end{pmatrix}$$

$$s=\det\begin{pmatrix} x_1 & w_1 \\ x_2 & w_2 \end{pmatrix} \qquad t=\det\begin{pmatrix} x_1 & z_1 \\ x_2 & z_2 \end{pmatrix} \qquad u=\det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$

we can write the four equations in matrix form:

$$\begin{pmatrix} x & y & z & w \end{pmatrix} \begin{pmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

The above anti-symmetric matrix is then our desired line representation, L. Any non-zero multiple of L will still represent the same line. If a point is multiplied by L and four zeros result then the point is on the line. Furthermore, if the point is not on the line, the four coordinates obtained will be the same values obtained if all three points were solved for their common plane. That is, they will be the components of the plane common to the point and the line:

$$\begin{pmatrix} x & y & z & w \end{pmatrix} L = \begin{pmatrix} a & b & c & d \end{pmatrix}$$

We need only to transpose the row vector to get the plane vector in its more familiar column format.

There is an analagous process for generating the matrix representing the line formed by intersecting two planes. Given planes 1 and 2, the condition that a third plane contains their line of intersection is:

$$\det\begin{pmatrix} a_1 & a_2 & a & \vec{i} \\ b_1 & b_2 & b & \vec{j} \\ c_1 & c_2 & c & \vec{k} \\ d_1 & d_2 & d & \vec{l} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

That is, the four equations must be satisfied:

$$b \det\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} - c \det\begin{pmatrix} b_1 & b_2 \\ d_1 & d_2 \end{pmatrix} + d \det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} = 0$$

$$-a \det\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} + c \det\begin{pmatrix} a_1 & a_2 \\ d_1 & d_2 \end{pmatrix} - d \det\begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} = 0$$

$$a \det\begin{pmatrix} b_1 & b_2 \\ d_1 & d_2 \end{pmatrix} - b \det\begin{pmatrix} a_1 & a_2 \\ d_1 & d_2 \end{pmatrix} + d \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = 0$$

$$-a \det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} + b \det\begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} - c \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = 0$$

These can be written in matrix form:

$$K \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix K is an anti-symmetric matrix which is a homogeneous representation of the line of intersection of the two planes. Any non-zero multiple of K represents the same line. The product of K and any other plane vector will yield four zeros if the line is contained in the plane. If the line is not contained in the plane then the product will yield the homogeneous coordinates of the point of intersection of the line with the plane:

$$K \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

We need only to transpose the point vector to get it in the more familiar row form. There is one somewhat surprising fact, however. For a given line, the matrix L formed by two points on the line is not the same as the matrix K formed by two planes intersecting on the line. We will now show this.

THE DUAL LINE REPRESENTATION

We first take note of another interpretation of the matrix L. Since each column yields a zero when multiplied by a point oh the line we can think of it as a plane containing the line. Similarly each row of K can be thought of as a point on the line which it represents. Thus L consists of four planes containing the line represented by L and K consists of four points on the line represented by K. Let us take any three planes of L and attempt to find the point common to them. Since we know that the planes intersect, not at a single point, but at a line we expect to get four zeros.

$$\det\begin{pmatrix} 0 & p & -q & \vec{i} \\ -p & 0 & s & \vec{j} \\ q & -s & 0 & \vec{k} \\ -r & t & -u & \vec{l} \end{pmatrix} =$$

$$\vec{i}\times\det\begin{pmatrix} -p & 0 & s \\ q & -s & 0 \\ -r & t & -u \end{pmatrix} \quad -\vec{j}\times\det\begin{pmatrix} 0 & p & q \\ q & -s & 0 \\ -r & t & -u \end{pmatrix}$$

$$+\vec{k}\times\det\begin{pmatrix} 0 & p & -q \\ -p & 0 & s \\ -r & t & -u \end{pmatrix} \quad -\vec{l}\times\det\begin{pmatrix} 0 & p & -q \\ -p & 0 & s \\ q & -s & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = pu-qt+sr \begin{pmatrix} -s \\ -q \\ -p \\ 0 \end{pmatrix}$$

In order to make $x=y=z=w=0$, as we know must be the case, we are forced to the conclusion that either $s=q=p=0$ or $pu-qt+sr=0$. By a similar operation on other choices of columns of L we find that the latter choice is correct, Thus, to reiterate, for any matrix L constructed from two point vectors to represent the line connecting them, the six coordinates will always satisfy the relation:

$$pu - qt + sr = 0 \qquad (*)$$

Given this relation we can construct the following matrix product:

$$\begin{pmatrix} 0 & -u & -t & -s \\ u & 0 & -r & -q \\ t & r & 0 & -p \\ s & q & p & 0 \end{pmatrix} \begin{pmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The middle matrix is just L. The product is all zeros either identically or by virtue of relation

(*). How can we interpret the left hand matrix? Since each row multiplied by L yields four zeros each row must be a point on the line. The left hand matrix must then be the same as K, that is, four points on the line stacked into a 4x4 matrix. The matrix K thus contains the same numbers as the matrix L, they are just arranged differently. We can now match the names of the coordinates with their values if calculated as the intersection of two planes:

$$u=-\det\begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \quad t=\det\begin{pmatrix} b_1 & b_2 \\ d_1 & d_2 \end{pmatrix} \quad s=-\det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}$$

$$p=-\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \quad q=\det\begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} \quad r=-\det\begin{pmatrix} a_1 & a_2 \\ d_1 & d_2 \end{pmatrix}$$

Thus the homogeneous representation of a line exists in two dual forms generated by joining two points and by intersecting two planes. The six coordinate points generated in each case satisfy equation (*).

DISTANCE MEASUREMENTS

To further increase intuitive feel for the meaning of these six coordinates let us see where a given line intersects the plane at infinity. We multiply the K form of the line with the plane at infinity and get:

$$\begin{pmatrix} 0 & -u & -t & -s \\ u & 0 & -r & -q \\ t & r & 0 & -p \\ s & q & p & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -s \\ -q \\ -p \\ 0 \end{pmatrix}$$

The intersection is the point at infinity (-s -q -p 0). That means that the 3D vector (s q p) points parallel to the line. Now let us determine the plane containing the line and the origin. We multiply the L form of the line with the origin and get:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} L = \begin{pmatrix} -r & t & -u & 0 \end{pmatrix}$$

This means that the 3D vector (-r t -u) points perpendicular to this plane. The dot product of these two vectors is zero: this is just relation (*). Thus (s q p) lies in the plane containing the line and the origin. If we compute the cross product of the two vectors we will get a third vector which is perpendicular to the line and pointing directly toward it.

$$\begin{pmatrix} tp+uq & rp-us & -rq-st \end{pmatrix} = T$$

By making use of (*) it can be shown that the length of T is

$$|T| = \sqrt{(r^2+t^2+u^2)(s^2+q^2+p^2)}$$

We can now compute the perpendicular distance, D, from the origin to the line. Place the normalized T at the origin and scale it up by the factor D. We should now be at the point on the line which is closest to the origin.

$$\frac{1}{|T|}\begin{pmatrix} D(tp+uq) & D(rp+us) & D(-rq-st) & 1 \end{pmatrix} L = 0$$

Multiplying out and solving for D we get:

$$D = \sqrt{\frac{r^2+t^2+u^2}{s^2+q^2+p^2}}$$

This is the perpendicular distance from the origin to the line L.

TRANSFORMING LINES

A homogeneous point is transformed by post-multiplying by a 4x4 matrix. A homogeneous plane is transformed by pre-multiplying by the inverse of the point'transformation matrix. We shall now derive the process whereby a homogeneous line is transformed. This procedure should preserve dot products just as the plane transformation does. That is, given the relationship:

$$\begin{pmatrix} x & y & z & w \end{pmatrix} L = \begin{pmatrix} a & b & c & d \end{pmatrix}$$

we wish the transformed quantities to also satisfy the relationship:

$$\begin{pmatrix} x' & y' & z' & w' \end{pmatrix} L' = \begin{pmatrix} a' & b' & c' & d' \end{pmatrix}$$

We can express the primed point and plane in terms of the unprimed by

$$\begin{pmatrix} x & y & z & w \end{pmatrix} T = \begin{pmatrix} x' & y' & z' & w' \end{pmatrix}$$

$$\begin{pmatrix} a & b & c & d \end{pmatrix} \left(T^{-1}\right)^t = \begin{pmatrix} a' & b' & c' & d' \end{pmatrix}$$

Combining these

$$\begin{pmatrix} x & y & z & w \end{pmatrix} T \; L' = \begin{pmatrix} a & b & c & d \end{pmatrix} \left(T^{-1}\right)^t$$

$$\begin{pmatrix} x & y & z & w \end{pmatrix} T \; L' \; T^t = \begin{pmatrix} a & b & c & d \end{pmatrix}$$

Comparing this with the original point, line, plane relation we can state that a solution is:

$$L = T \; L' \; T^t$$

or

$$T^{-1} \; L \; \left(T^{-1}\right)^t = L'$$

Matrices which represent quantities which transform in this way are called tensors. In addition, since the transformation matrix used is the inverse of the point transformation matrix, it is a contra-variant tensor.

By applying the analagous process to the K form of the line we get

$$T \; K \; T^t = K'$$

This is another tensor. This time the transformation matrix is the same as the point transformation matrix so it is a covariant tensor.

INTERSECTING LINES

We have so far examined the problem of whether a point is on a line and whether a line is in a plane. There remains the question of whether two lines intersect, and, if so, where.

This can be solved by taking the point form of one line and multiplying it by the plane form of the other.

$$K_1 \quad L_2 \quad = \quad N$$

Each row of K_1, being a point of line 1, will generate a plane through that point and through line 2. If the two lines intersect, each of these will be the same plane. The plane containing the two lines. Likewise each column of L2, being a plane containing line 2, will generate a point at the intersection of that plane and line 1. If the two lines intersect, each of these will be the same point, the point of intersection of the lines. Thus each row of N is a plane vector for the plane common to the lines. Each column of N is a point vector for the intersection of the lines. N is the outer product of the point and the plane:

$$N = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \begin{pmatrix} a & b & c & d \end{pmatrix} = \begin{pmatrix} ax & bx & cx & dx \\ ay & by & cy & dy \\ az & bz & cz & dz \\ aw & bw & cw & dw \end{pmatrix}$$

Since the point of intersection always lies in the plane of intersection the inner product will be zero. This can be calculated as the trace of N. In terms of the components of K_1 and L_2 the trace of N has the value

$$\text{trace } N = p_1u_2-q_1t_2+s_1r_2 + p_2u_1-q_2t_1+s_2r_1$$

Note the similarity to relation (*).

For lines which do not intersect (skew lines) the trace of N will be proportional to the perpendicular distance between them. This can be seen in the following manner. First consider the cross product of the direction vectors of the two lines.

$$\begin{pmatrix} s_1 & q_1 & p_1 \end{pmatrix} \times \begin{pmatrix} s_2 & q_2 & p_2 \end{pmatrix} = \begin{pmatrix} s_3 & q_3 & p_3 \end{pmatrix}$$

This vector will be perpendicular to both lines. A plane having $(s_3\ q3\ p_3)$ as its (a b c) components will be parallel to both line 1 and line 2. We can find the particular such plane which contains line 1 by solving for d_1 in

$$K_1 \begin{pmatrix} s_3 \\ q_3 \\ p_3 \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This yields four equations all of which can be shown to have the common solution

$$d_1 = -p_1u_2 + q_1t_2 - s_1r_2$$

Similarly, the plane parallel to line 1 which contains line 2 has

$$d_2 = p_2u_1 - q_2t_1 + s_2r_1$$

The perpendicular distance of each of these planes to the origin is

$$D_1 = \frac{d_1}{\sqrt{s_3^2+q_3^2+p_3^2}} \qquad D_2 = \frac{d_2}{\sqrt{s_3^2+q_3^2+p_3^2}}$$

The perpendicular distance between the two planes and the perpendicular distance between the lines is

$$D_1-D_2 = \frac{d_1-d_2}{\sqrt{s_3^2+q_3^2+p_3^2}} = \frac{\text{trace } N}{\sqrt{s_3^2+q_3^2+p_3^2}}$$

If the trace is zero, the lines intersect. If the trace is non-zero, the perpendicular distance is as shown.

What, then, are the six homogeneous coordinates for the line along which this distance is measured? We already have the direction of the line as $(s_3\ q_3\ p_3)$. It remains to find r3, t3, and u_3. This can be accomplished by using the three facts that line 3 intersects line 1, line 3 intersects line 2, and the coordinates of line 3 must satisfy relation (*).

$$\text{trace } (K_3\ L_2) = 0$$

$$\text{trace } (K_1\ L_3) = 0$$

$$p_3u_3 - q_3t_3 + s_3r_3 = 0$$

These three equations may then be solved for r_3, t_3, and u_3.

CONCLUSION

The line representation developed here can be used to solve many geometric problems in three dimensions. Its form, however, does lead to much redundant calculation for many problems of interest. Its main use may therefore be as a conceptual tool to generate formulas for desired geometrical quantities which are then simplified based on other knowledge of the problem.

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