

# Planar Minimally Rigid Graphs and Pseudo-Triangulations

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## Abstract

Pointed pseudo-triangulations are planar minimally rigid graphs embedded in the plane with *pointed* vertices (adjacent to an angle larger than  $\pi$ ). In this paper we prove that the opposite statement is also true, namely that planar minimally rigid graphs always admit pointed embeddings, even under certain natural topological and combinatorial constraints. The proofs yield efficient embedding algorithms. They also provide - to the best of our knowledge - the first algorithmically effective result on graph embeddings with oriented matroid constraints other than convexity of faces. These constraints are described by *combinatorial pseudo-triangulations*, first defined and studied in this paper. Also of interest are our two proof techniques, one based on Henneberg inductive constructions from combinatorial rigidity theory, the other on a generalization of Tutte's barycentric embeddings to directed graphs.

## 1 Introduction

In this paper we bring together two classical topics in graph theory, planarity and rigidity, to answer the fundamental question (posed in [42]) of *characterizing the class of planar graphs which admit pointed pseudo-triangular embeddings*. Our main result is that this coincides with the class of all *planar minimally rigid graphs* (*planar Laman graphs*). Furthermore we extend the result in several directions, attacking the same type of question for other (not necessarily pointed) classes of pseudo-triangulations and for *combinatorial pseudo-triangulations*, a new class of objects first introduced and studied in this paper.

**Novelty.** As opposed to traditional planar graph embeddings, where all the faces are designed to be convex, ours have interior faces which are as non-convex as possible (pseudo-triangles). Planar graph embeddings with *non-convex* faces have not been systematically studied. Our result links them to rigidity theoretic and matroidal properties of planar graphs. To the best of our knowledge, this is the first result holding for an interesting family of graphs on algorithmically efficient graph embeddings with oriented matroid constraints other than convexity of faces. In contrast, the universality theorem for pseudo-line arrangements of Mnëv [28] implies that the general problem of embedding graphs with oriented matroid constraints is as hard as the existential theory of the reals.

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**Proof Techniques and Algorithmic Results.** We present two proof techniques of independent interest. The first one is of a *local* nature, relying on incremental (inductive) constructions known in rigidity theory as *Henneberg constructions*. The second one is based on a *global* approach making use of a directed version of *Tutte's barycentric embeddings*, which - to the best of our knowledge - is first proven in this paper. Both proofs are constructive, yield efficient algorithms, emphasize distinct aspects of the result and lead into new directions of further investigation: combinatorial versus geometric embeddings, local versus global coordinate finding.

**Laman Graphs and Pseudo-Triangulations.** Let  $G = (V, E)$  be a graph with  $n = |V|$  vertices and  $m = |E|$  edges.  $G$  is a *Laman graph* if  $m = 2n - 3$  and every subset of  $k$  vertices spans at most  $2k - 3$  edges.<sup>1</sup> An embedding  $G(P)$  of the graph  $G$  on a set of points  $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$  is a mapping of the vertices  $V$  to points in the Euclidean plane  $i \mapsto p_i \in P$ . The edges  $ij \in E$  are mapped to straight line segments  $p_i p_j$ . We say that the vertex  $i$  of the embedding  $G(P)$  is *pointed* if all its adjacent edges lie on one side of some line through  $p_i$ . Equivalently, some consecutive pair of edges adjacent to  $i$  (in the circular counter-clockwise order around the vertex) spans a *reflex* angle. An embedding  $G(P)$  is *non-crossing* if no pair of segments  $p_i p_j$  and  $p_k p_l$  corresponding to non-adjacent edges  $ij, kl \in E, i, j \notin \{k, l\}$  have a point in common. A graph  $G$  is *planar* if it has a non-crossing embedding.

A *pseudo-triangle* is a simple planar polygon with exactly three convex vertices. A *pseudo-triangulation* of a planar set of points is a non-crossing embedded graph  $G(P)$  whose outer face is convex and all interior faces are pseudo-triangles. In a *pointed pseudo-triangulation* all the vertices are pointed. See Figure 1.

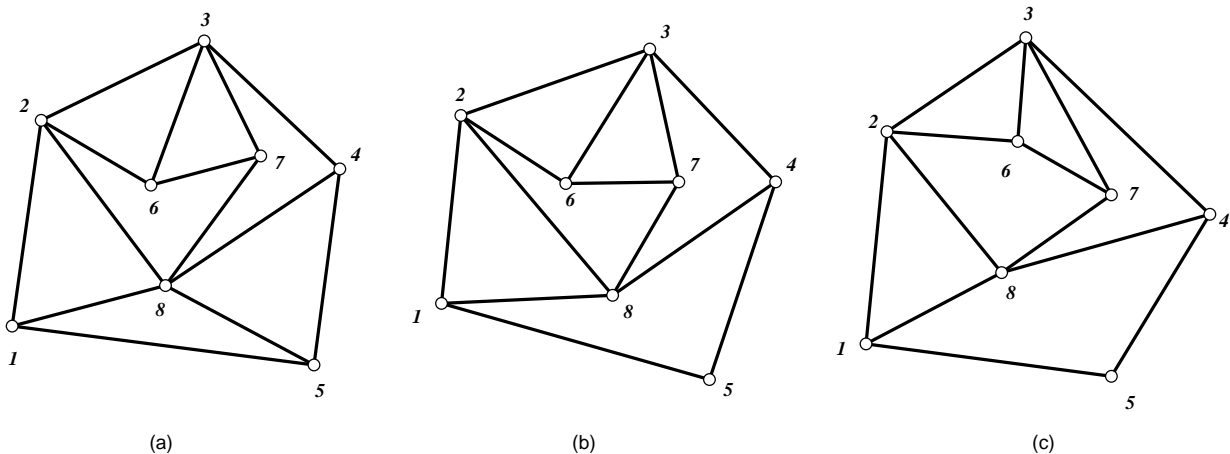


Figure 1: (a) A pseudo-triangulation (necessarily non-pointed, since the underlying graph is a circuit, not a Laman graph) and two embeddings of a planar Laman graph: (b) is a pointed pseudo-triangulation, (c) is not: the faces 2876 and 1548 are not pseudo-triangles and the vertices 6 and 8 are not pointed.

Pseudo-triangulations are thus planar graphs with a special embedding. As shown in [41], the graphs of pointed pseudo-triangulations are Laman graphs. They have very useful rigidity theoretic properties and a wealth of combinatorial properties. They have emerged via several applications in Computational Geometry, where they are used in Kinetic Data Structures for collision detection and in certain motion planning problems.

<sup>1</sup>This is called the definition *by counts* of Laman graphs. An equivalent definition via Henneberg constructions will be given in Section 2.

**Historical Perspective.** Techniques from Rigidity Theory have been recently applied to problems such as collision free robot arm motion planning [10, 41], molecular conformations [23, 51, 46] or sensor and network topologies with distance and angle constraints [14].

Laman graphs are *the* fundamental objects in 2-dimensional Rigidity Theory. Also known as *isostatic* or *generically minimally rigid graphs*, they characterize combinatorially the property that a graph, embedded on a generic set of points in the plane, is infinitesimally rigid (with respect to the induced edge lengths). See [25], [18], [52]. The most famous open question in Rigidity Theory (the *Rigidity Conjecture*, see [18]) is finding their 3-dimensional counterpart.

Pseudo-triangulations are relatively new objects, introduced and applied in Computational Geometry for problems such as visibility [34], [33], [39], kinetic data structures [4] and motion planning for robot arms [41]. They have rich combinatorial, rigidity theoretic and polyhedral properties ([41], [38], [29]), many of which have only recently started to be investigated ([35], [24], [3], [6], [1], [7], [2]). In particular, the fact that they are Laman graphs which become *expansive* mechanisms when one of their convex hull edges is removed, has proven to be crucial in designing efficient motion planning algorithms for planar robot arms, see [41]. Finding their 3-dimensional counterpart, which is perhaps the main open question about pseudo-triangulations and expansive motions, may lead to efficient motion planning algorithms for certain classes of 3-dimensional linkages, with potential impact on understanding protein folding processes.

Graph Drawing is a field with a distinguished history, and embeddings of planar graphs have received substantial attention in the literature ([16], [48], [47], [9], [17], [43], [13]). Extensions of graph embeddings from straight-line to pseudo-line segments have been recently considered (see e.g. [32]). It is natural to ask which such embeddings are stretchable, i.e. whether they can be realized with straight-line segments while maintaining some desired combinatorial substructure. Indeed, the primordial planar graph embedding result, Fary's Theorem [16], is just an instance of answering such a question. Graph embedding stretchability questions have usually ignored oriented matroidal constraints, allowing for the free reorientation of triplets of points when not violating other combinatorial conditions. The notable exception concerns the still widely open *visibility graph recognition problem*, approached in the context of pseudo-line arrangements (oriented matroids) by [31]. In [40] it is shown that it is not always possible to realize with straight-lines a pseudo-visibility graph, while maintaining oriented matroidal constraints.

In contrast, *this paper gives the first non-trivial stretchability result on a natural graph embedding problem with oriented matroid constraints*. It adds to the already rich body of surprisingly simple and elegant combinatorial properties of pointed pseudo-triangulations by proving a natural connection.

**Main Result.** We are interested in *planar Laman graphs*. Not all Laman graphs fall into this category. For example,  $K_{3,3}$  is Laman but not planar. But the underlying graphs of all pointed pseudo-triangulations are planar Laman. We prove that the converse is always true:

**Theorem 1.1 (Main Theorem)**

*Every planar Laman graph can be embedded as a pointed pseudo-triangulation.*

The following characterization follows then from well known properties of Laman graphs:

**Corollary 1.2** *Given a planar graph  $G$ , the following conditions are equivalent:*

- (i)  $G$  is a Laman graph
- (ii) Generically,  $G$  is minimally rigid
- (iii)  $G$  can be embedded as a pointed pseudo-triangulation

We prove in fact several stronger results, allowing the *a priori* choice of the facial structure (Theorem 3.1) and even of the combinatorial information regarding which vertices are convex in each face (Theorem 5.1). This last result needs the apparatus of *combinatorial pseudo-triangulations*, first defined and studied in this paper.

Finally, we answer a natural question related to the underlying matroidal structure of planar rigidity and extend the result to planar rigidity circuits, which are minimal dependent sets in the rigidity matroid where the bases (maximally independent sets) are the Laman graphs. By adding edges to a pointed (minimum) pseudo-triangulation while maintaining planarity, the graph has increased dependency level (in the rigidity matroid) and can no longer be realized with all vertices pointed, but it can always be realized with straight edges. Our concern is to maintain the minimum number of non-pointed vertices, for the given edge count. For circuits, this number is one, and we show that it can be attained.

**Organization.** The paper is organized as follows. In Section 2 we give the basic definitions needed for an independent reading of Section 3. For increased readability, additional technical definitions are later included in the sections that use them. The first proof of the Main Theorem is presented in Section 3, which is further devoted to all the proofs (combinatorial or geometric) making use of the inductive Henneberg technique: planar Laman graphs, combinatorial pseudo-triangulations, pointed pseudo-triangulations and Laman-plus-one combinatorial and geometric pseudo-triangulations. Section 4 is devoted to combinatorial pseudo-triangulations and to the perfect matching technique for assigning combinatorial pseudo-triangular labelings to plane graphs. Section 5 focuses on the second proof technique based on Tutte embeddings and contains our most general result on plane graph embeddings compatible with *given* combinatorial pseudo-triangulations. We conclude in Section 6 with a list of further directions of research and open questions.

## 2 Preliminaries

For the standard graph and rigidity theoretical terminology used in this paper we refer the reader to [18] and [53]. For relevant facts about pointed pseudo-triangulations, see [41]. In this section we continue what we started in the Introduction and give most of the definitions needed for reading Sections 3 and 4. The technically denser Section 5 contains its own additional concepts.

**Notation and Abbreviations.** Throughout the paper we will abbreviate *counter-clockwise* as *ccw*. To emphasize that a graph has  $n$  vertices we may denote it by  $G_n$ . We will occasionally abbreviate *combinatorial pseudo-triangulation* as *cpt* and *pointed combinatorial pseudo-triangulation* as *pointed cpt*.

**Plane Graphs.** A non-crossing embedding of a connected planar graph  $G$  partitions the plane into faces (bounded or unbounded), edges and vertices. Their incidences are fully captured by the vertex *rotations*: the ccw circular order of the edges incident to each vertex in the embedding. A *sphere embedding* of a planar graph refers to a choice of a system of rotations (and thus of a facial structure), and is oblivious of an outer face. It is well-known (Whitney [54]) that a 3-connected planar graph induces a unique set of rotations, but 2-connected ones may induce several. A *plane graph* is a spherical graph with a choice of a particular face as the outer face. Every simple plane graph can be realized with straight-line edges in the plane (Fary's theorem [16]).

A (combinatorial) *angle* (incident to a vertex or a face in a plane graph) is a pair of consecutive edges (consecutive in the order given by the rotations) incident to the vertex or face.

**Pseudo-Triangulations.** We have defined pseudo-triangles, pseudo-triangulations and pointed pseudo-triangulations in the Introduction. In addition, we will use the following related concepts. The *corners* of a pseudo-triangle are its three convex angles, and its *side chains* are the pieces of the boundary between two corners (vertices and edges). The *extreme edges* of a pointed vertex are the two edges incident with its unique incident reflex angle.

**Minimally Rigid (Laman) Graphs and Henneberg constructions.** Besides the definition by *counts* given in the Introduction, Laman graphs can be characterized in a variety of ways. In particular, a Laman graph on  $n$  vertices has an inductive construction as follows (see [21, 52]). Start with an edge for  $n = 2$ . At each step, add a new vertex in one of the following two ways:

- **Henneberg I** (vertex addition): the new vertex is connected via two new edges to two old vertices.

- **Henneberg II** (edge splitting): a new vertex is added on some edge (thus splitting the edge into two new edges) and then connected to a third vertex. Equivalently, this can be seen as removing an edge, then adding a new vertex connected to its two endpoints and to some other vertex.

See Figure 2, where we show drawings with crossing edges, to emphasize that the Henneberg constructions work for *general*, not necessarily *planar* Laman graphs.

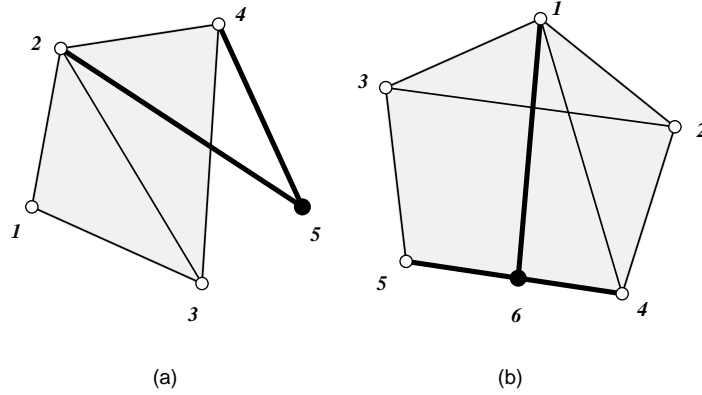


Figure 2: Illustration of the two types of steps in a Henneberg sequence, with vertices labelled in the construction order. The shaded part is the old graph, to which the black vertex is added. (a) Henneberg I for vertex 5, connected to old vertices 3 and 4. (b) Henneberg II for vertex 6, connected to old vertices 3, 4 and 5.

We will make heavy use of the following result, essentially stated by Henneberg [21], and of its proof, due to Tay and Whiteley [45].

**Lemma 2.1** *A graph is Laman if and only if it has a Henneberg construction.*

The proof of Lemma 2.1 proceeds inductively to show that there always exists a vertex of degree 2 or 3 which can be removed in the reverse order of a Henneberg step while maintaining the Laman property. It is instructive to give a slightly more general proof. We will make use of it in Section 3.

**Lemma 2.2** *A Laman graph has a Henneberg construction starting from any prescribed subset of two vertices. Moreover, if there exist three vertices of degree 3 mutually connected in a triangle, then we can prescribe them as the first three vertices of the Henneberg construction.*

**Proof:** Let  $G_n = (V, E)$  be a Laman graph on  $n = |V|$  vertices and let  $V_2 \subset V$  be any subset of two vertices. We show that as long as  $n > 2$  we can always remove a vertex not in  $V_2$  in the opposite direction of a Henneberg step. In the actual Henneberg construction this amounts to starting the induction from this prescribed pair.

Vertices of degree 0 or 1 do not exist in Laman graphs, otherwise the Laman property would be violated on a subset of  $n - 1$  vertices. Since  $G_n$  has  $2n - 3$  edges, a simple count shows that there exist at least three vertices of degree at most 3, hence either of degree 2 or 3. At least one of them (call it  $v$ ) is not in  $V_2$ . This will be the vertex we choose to remove, in a backwards application of a Henneberg step.

If  $v$  has degree 2, we remove  $v$  and its incident edges: the resulting graph on  $n - 1$  vertices and  $2(n - 1) - 1$  edges is clearly Laman. If  $v$  has degree 3, let its neighbors be  $v_1$ ,  $v_2$  and  $v_3$ . The removal of  $v$  and of its three adjacent edges produces a graph  $G'_{n-1}$  with a deficit of one edge:  $n - 1$  vertices but only  $2(n - 1) - 4$  edges. We must put back one edge joining one of the three pairs of vertices in  $v_1, v_2, v_3$ .

Consider the *rigid components* of  $G'_{n-1}$ : maximal subsets of some  $k$  vertices spanning  $2k - 3$  edges. The three endpoints  $v_1, v_2$  and  $v_3$  cannot belong to the same rigid component (otherwise the Laman count would be violated in  $G_n$  on the subset consisting of this component and  $v$ ). Two rigid components share at most one vertex, otherwise their union would be a larger Laman subgraph. Suppose that  $v_1$  and  $v_2$  are in distinct rigid components. Then adding an edge between  $v_1$  and  $v_2$  doesn't violate the Laman condition on any subset and completes  $G'_{n-1}$  to a Laman graph  $G_{n-1}$ .

If  $G_n$  contains a subset  $V_3$  of three degree 3 vertices connected in a triangle, then a similar counting argument shows that there is an additional vertex of degree at most 3. This fourth vertex can be removed in such a way that the invariant (of having three vertices of degree 3 connected in a triangle) is maintained. Hence the vertices of  $V_3$  may be prescribed as the three starting vertices. To finish, let's show that the invariant is maintained. Let  $G'$  be the subgraph induced on the vertices in  $V \setminus V_3$ : if it contains more than two elements, then it spans  $2n - 3 - 6 = 2(n - 3) - 3$  edges, hence it is Laman. Let  $N(V_3)$  be the neighbors of  $V_3$  in  $G_n$ : none of these vertices can be of degree 2, otherwise  $G'$  would not be Laman. If there is a vertex  $v'$  of degree 3 in  $N(V_3)$  which is removed at some Henneberg step, one must put back an edge incident to two of its neighbors, and one of them must be in  $V_3$ : otherwise, the induced subgraph  $G'$  on  $V \setminus V_3$  (after performing the reverse Henneberg step) would violate the Laman counts. Since at any reverse Henneberg step we remove either vertices of degree 2 (which are not incident to  $V_3$ ) or of degree 3, which do not change the degree of their neighbors in  $V_3$ , it follows that the vertices in  $V_3$  maintain their degrees and the property of being connected in a triangle throughout the construction (in fact, until  $n = 5$ , from which point it is easy to see that  $V_3$  can still be prescribed).  $\square$

**Laman-plus-one Graphs and Rigidity Circuits.** A *Laman-plus-one* graph is a Laman graph with one additional edge. It has  $2n - 2$  edges and every subset of  $k$  edges induces at most  $2k - 2$  edges. A *rigidity circuit* (shortly, a *circuit*) is a graph with the property that removing *any* edge produces a Laman graph. It is therefore a special Laman-plus-one graph. In a rigidity circuit  $G$  with  $n$  vertices, the number  $n$  of vertices is at least 4, the number  $m$  of edges is  $2n - 2$  and every subset of  $k$  vertices spans at most  $2k - 3$  edges. Moreover, the minimum degree in a circuit is 3. It is straightforward to prove that a Laman-plus-one graph contains a unique rigidity circuit: take the maximal subgraph satisfying the circuit counts. It is unique, because otherwise the union of two circuits would violate the Laman-plus-one counts.

These concepts are motivated by the matroid view of Rigidity Theory, see [18]: Laman graphs correspond to bases (maximal *independent* sets of edges) in the generic rigidity matroid, while the circuits are the minimally *dependent* sets.

It has been proven recently that 3-connected rigidity circuits admit an inductive (Henneberg-type) construction (using only Henneberg II steps and starting from  $K_4$ ), where all intermediate graphs are themselves circuits. All rigidity circuits are 2-connected, hence they can be obtained by making use of Tutte's Theorem on the structure of 3-connected graphs in terms of 2-connected components (see [49], [50]).

We show now (and use later) that Laman-plus-one graphs also admit a simple Henneberg construction. This proof structure (much easier than [5] because of a simpler inductive invariant) will be used in Section 3 to show stretchability of planar Laman-plus-one graphs, and thus of planar rigidity circuits.

**Lemma 2.3** *A graph  $G$  is Laman-plus-one if and only if it has a Henneberg construction starting from a  $K_4$ .*

**Proof:** The proof is similar to that of Lemma 2.1 and uses the  $2n - 2$  counts. There must be at least two vertices in  $G$  of degree at most 3. Vertices of degree 2 are outside the circuit and are removed just as in the Laman case. We now show how to handle a vertex  $v$  of degree 3. If there are no vertices of degree 2, there are at least four of degree 3.

Let  $C = (V_c, E_c)$  be the unique induced subgraph which is the circuit of  $G$ ,  $V_o = V \setminus V_c$  the vertices outside the circuit and  $V_b \subset V_c$  the *boundary* of  $V_c$ , i.e. the set of vertices in  $V_c$  incident to a vertex in  $V_o$ . It is easy to see (from the  $2n - 2$  counts) that between  $V_b$  and  $V_o$  there must be at least two edges if  $|V_o| = 1$  or three edges if  $|V_o| \geq 2$ .

If  $v \in V_o$  then its neighbors cannot all three belong to the circuit, because otherwise the subgraph induced on  $V_c \cup \{v\}$  would violate the  $2n - 2$  counts. Remove (temporarily) an edge  $ab$  of  $G$  from inside  $C$ : the resulting graph is Laman, containing  $C$  without this edge as a rigid block (subset on which the Laman count is satisfied with equality). By Lemma 2.2 there is a well defined way of removing  $v$  and placing back an edge to perform a Henneberg II step in reverse: the added edge is not between two vertices of  $V_c$ . Therefore we can put back the temporarily removed edge  $ab$  to get a Laman-plus-one graph.

If  $v \in V_c$ , notice first that it cannot be on the boundary  $V_b$ , otherwise its degree in  $C$  would be at most 2, contradicting the fact that  $C$  is a circuit. So all its three neighbors  $v_1, v_2$  and  $v_3$  are in  $V_c$ . Removing  $v$  and its incident edges produces a Laman graph. Either all of the edges  $v_1v_2, v_1v_3, v_2v_3$  are present in  $G$  or not. If not (say,  $v_1v_2$  is missing), then we add  $v_1v_2$ , get a Laman-plus-one graph and continue the induction. Otherwise, the circuit was a  $K_4$ . If there are no vertices outside the circuit, we are done. Otherwise, there are at least two edges out of  $V_b$ , increasing the degree of at least one vertex in the circuit  $K_4$ . If there are no vertices of degree 2, then there must be at least one other vertex of degree 3 in  $V_o$ . We will perform the Henneberg step on it (and thus not touch  $K_4$  until the end).  $\square$

**Combinatorial Pseudo-Triangulations.** Let  $G$  be a plane connected graph<sup>2</sup>. A *combinatorial pseudo-triangulation* (cpt) of  $G$  is an assignment of labels *big* (or *reflex*) and *small* (or *convex*) to the angles of  $G$  such that:

- (i) Every face except the outer face gets *three* vertices marked *small*. These will be called the *corners* of the face.
- (ii) The outer face gets only *big* labels (has no corners).
- (iii) Each vertex is incident to at most one angle labeled *big*. If it is incident to a big angle, it is called *pointed*.
- (iv) A vertex of degree 2 is incident to one angle labeled *big*.

By analogy with pseudo-triangulations, we also define extreme edges, side-chains and non-pointed vertices of combinatorial pseudo-triangulations.

Combinatorial pseudo-triangulations share many combinatorial properties with pseudo-triangulations. The following lemma follows easily from the definition.

**Lemma 2.4** *A combinatorial pseudo-triangulation on  $n$  vertices has at least  $2n - 3$  edges. If a cpt has  $m \geq 2n - 3$  edges, then it contains exactly  $m - (2n - 3)$  non-pointed vertices.*

**Proof:** Let  $V_K$  be the set of non-pointed vertices. Let  $m$  be the number of edges,  $f$  the number of faces,  $k$  the size of  $V_K$  and  $d_v$  the degree of a vertex  $v$ . We count the number of small angles in two ways and apply Euler's formula. Summing over the faces we get  $3(f - 1)$ . Summing over vertices we get  $\sum_{v \notin V_K} (d_v - 1) + \sum_{v \in V_K} d_v = \sum_{v \in V} d_v - (n - k) = 2m - n + k$ . This solves to  $m = 2n - 3 + k$  and proves the statement.  $\square$

A cpt with exactly  $2n - 3$  edges will have all vertices pointed: we'll call it a *pointed combinatorial pseudo-triangulation* (pointed cpt). Another case of interest in this paper is when exactly one vertex is *combinatorially non-pointed*, i.e. has no incident big angle: we call it a *pointed-plus-one* cpt. Notice that in this case the non-pointed vertex has degree at least 3, is interior, i.e. not incident to the outer face and that the cpt has  $2n - 2$  edges.

In Section 3 we will prove that all planar Laman graphs and all planar Laman-plus-one graphs have cpt assignments. The previous lemma implies that such cpt assignments must be pointed for Laman graphs, resp. pointed-plus-one for Laman-plus-one graphs (defined below).

**Pointed-plus-one and Circuit Pseudo-triangulations.** A *pointed-plus-one pseudo-triangulation* is a pseudo-triangulation with exactly one non-pointed vertex. It is easy to see that it has  $2n - 2$  edges, and is

<sup>2</sup>The definition is valid in a more general setting than what we use in this paper, and works even with multiple edges, vertices of degree one and loops. In such a case, a vertex of degree one is incident to a unique angle, labeled *big*.

in fact just a planar Laman-plus-one graph embedded as a pseudo-triangulation. A *pseudo-triangulation circuit* is a planar rigidity circuit embedded as a pseudo-triangulation. A reminder that by a *pseudo-triangulation* we mean any decomposition into pseudo-triangles, which may not be pointed. In fact, any such pseudo-triangulation with more than  $2n - 3$  edges is necessarily *non-pointed*<sup>3</sup>, because pointed pseudo-triangulations are *maximal* pointed sets edges on any planar set of points and must have exactly  $2n - 3$  edges (cf. [41]). See Figure 1 for an example.

### 3 Main result: Inductive Proof via Henneberg construction

We are now ready to give our first proof of the Main Theorem, in the following slightly more general form, and extend it to planar Laman-plus-one graphs.

**Theorem 3.1** *Any plane Laman graph has a pointed pseudo-triangular embedding.*

**Theorem 3.2** *Any plane Laman-plus-one graph has a pointed-plus-one pseudo-triangular embedding.*

Both proofs have the same structure and are divided into three steps: topological, combinatorial and geometric. They are based on the corresponding Henneberg constructions, the common theme of this section. The easy topological lemmas shows that any Henneberg construction on a (Laman or Laman-plus-one) plane graph can be carried out in a manner compatible with the plane embedding. The combinatorial lemmas construct a combinatorial pseudo-triangulation while performing a topological Henneberg construction. Finally, in the geometric lemmas we show how to perform geometrically, with straight-line edges, the Henneberg extension (rather than just combinatorially). All the ideas are already contained in the proof of the first theorem: the second one will only be sketched, with indications of where the differences lie.

#### 3.1 Pseudo-Triangular Embeddings of Plane Laman Graphs

The proof of Theorem 3.1 is a consequence of the four lemmas stated and proven below. Lemma 3.3 reduces the construction to the case when the outer face is a triangle. Lemma 3.4 provides the framework for a Henneberg induction on plane graphs. This is then used in Lemma 3.5 to compute a combinatorial pseudo-triangulation assignment and in Lemma 3.6 to realize the same thing geometrically. Theorem 3.1 follows from Lemma 3.6. We remark that Lemma 3.5 is not needed for the proof of Theorem 3.1. It is however natural to include it here because it makes use of the same Henneberg technique (ubiquitous in this section). It also gives a better intuition about the combinatorial structure of the many possibilities involved in a complete proof by case analysis of Lemma 3.6.

**Lemma 3.3 (Fixing the Outer Face)** *Embedding a plane Laman graph as a pseudo-triangulation reduces to the case when the outer face is a triangle.*

**Proof:** Let  $G$  be a plane Laman graph with an outer face having more than three vertices. We construct another Laman graph  $G'$  of  $n + 3$  vertices by adding 3 vertices on the outer face and connecting them into a triangle including the original graph. Then we add an edge from each of the three new vertices to three distinct vertices on the exterior face of  $G$ . See Figure 3. We now realize  $G'$  as a pseudo-triangulation with the new triangle as the outer face. The graph  $G$ , as a subgraph of  $G'$ , must be realized with *its* outer face convex by the following argument. The three new interior edges of  $G'$  provide two corners each at their end-point incident to the outer face and at least one corner in the interior one. Since the three faces incident to them have nine corners in total, the boundary of  $G$  provides no corner to the three new interior faces of  $G'$ .  $\square$

Notice that a planar Laman graph (on  $n$  vertices) always has at least two triangular faces: the dual planar graph has  $n - 1$  vertices (including the vertex corresponding to the outer face) and  $2(n - 1) - 1$

<sup>3</sup>In general, those with  $k$  non-pointed vertices (all interior) have  $2n - 3 + k$  edges.



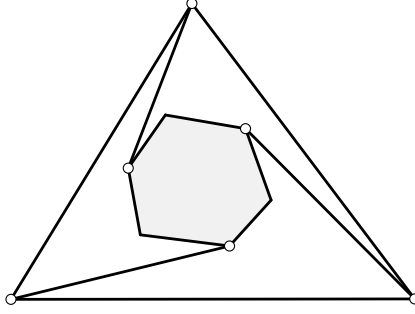


Figure 3: Reducing to an embedding with a triangular outer face.

edges, hence there exists at least two of degree three. The construction in the previous lemma makes it possible to use the stronger invariant of the Henneberg construction from Lemma 2.2 and to start any geometric embedding with a triangular outer face, then to insert only on interior faces. This feature is not needed in the proof of the topological or combinatorial lemmas below.

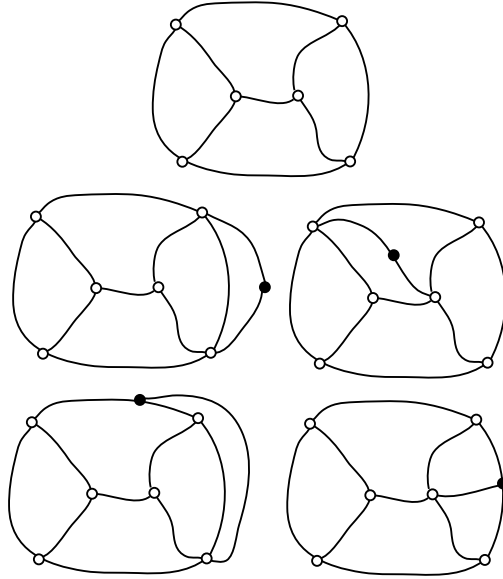


Figure 4: A plane Henneberg construction. Top row:  $G_n$ , to which a new vertex will be added. Middle row: Henneberg I on the outer, resp. an interior face. Bottom row: Henneberg II on the outer, resp. an interior face.

**Lemma 3.4 (The Topological Lemma)** *Every plane Laman graph has a plane Henneberg construction in which:*

1. *All intermediate graphs are plane*
2. *At each step, the topology is changed only on edges and faces involved in the Henneberg step: either*

a new vertex is added inside a face of the previous graph (Henneberg I), or inside a face obtained by removing an edge between two faces of the previous graph (Henneberg II).

In addition, if the outer face of the plane graph is a triangle, we may perform the Henneberg construction starting from that triangle. The Henneberg steps will never insert vertices on the outer face.

**Proof:** We follow the structure of the basic Henneberg construction from Lemma 2.2. Find an appropriate vertex of degree 2 or 3. Removing it, and its incident edges, merges two (resp. three) faces into one. The other endpoints of the removed edges are incident to this face, hence the added edge in the Henneberg II step simply splits this face and maintains the planarity of the embedding.  $\square$

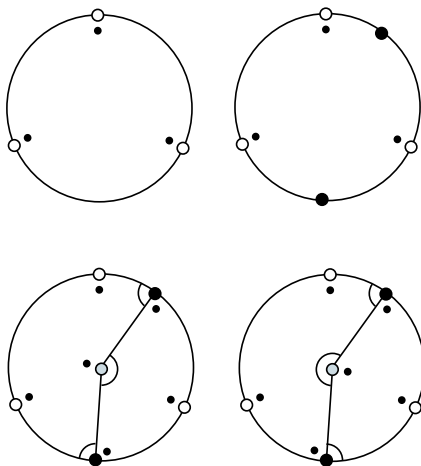


Figure 5: Extending a combinatorial pseudo-triangulation in a Henneberg I step. Top left: the combinatorial face is represented as a circle with its three corners, denoted by white vertices, marked *small* (a small black dot). Top right, a representative situation: the two endpoints of the newly added edges (in black) are distributed on two distinct side chains of the face. Two distinct labelings are possible in this case (bottom row): the newly created angles after the insertion of the new vertex (grey) are labeled with a small black dot for *small* (or *convex*) and with a large arc for *big* (or *reflex*).

**Lemma 3.5 (The Combinatorial Lemma)** *Every plane Laman graph admits a combinatorial pseudo-triangulation assignment.*

**Proof:** Let  $G_n$  be a plane Laman graph on  $n$  vertices. We may assume that the outer face is a triangle. We proceed with a plane Henneberg construction guaranteed by Lemma 3.4, which will insert only on interior<sup>4</sup> faces. The base case is a triangle and has a unique cpt labeling. At each step we have, by induction, a cpt labeling which we want to extend. The proof will guarantee that each one can be extended (so there is no need to backtrack).

In a Henneberg I step, the new vertex  $v$  is inserted on a face  $T$  (already labeled as a pseudo-triangle), and joined to two old vertices  $v_1$  and  $v_2$ . The new edges  $vv_1$  and  $vv_2$  partition the face  $F$  and its three corners into two. Three cases may happen. Either the three corners fall one in one face and two in another face, or one of them is split by a new edge (say  $v_1$  is a corner) and the other two are either both in the same new face or are separated, or two corners are split, and the third corner is in one of the two newly created faces. In either case, the assignment of *big* and *small* labels is what one would expect: a

<sup>4</sup>This is just a technical simplification reducing the size of our case analysis. The reader may verify that the Henneberg steps work as well for insertions on the outer face.

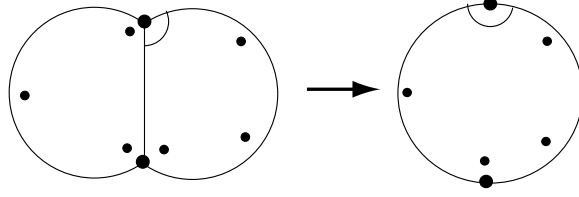


Figure 6: Merging two faces into one: a representative case for the analysis of a combinatorial Henneberg II step. The markings of *small* and *big* angles follow the conventions from Figure 5. The black vertices are the endpoints of the removed edge  $v_1v_2$ .

small angle is split into two small angles, a big angle is split into a big and a small angle, and the new point gets exactly one big angle. We illustrate one representative case in Figure 5 and leave the rest of the straightforward details of this case analysis to the reader.

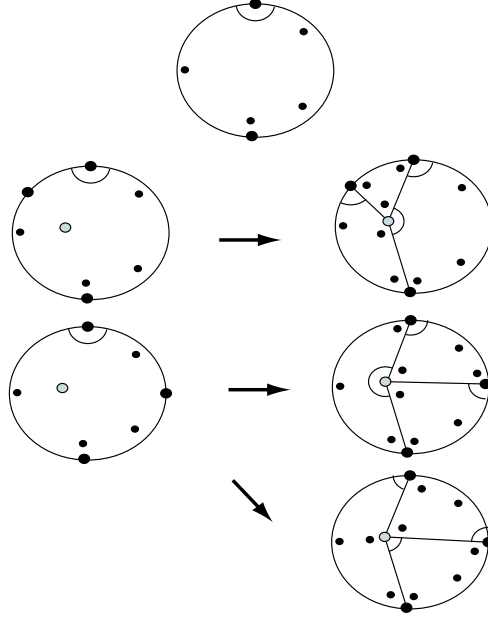


Figure 7: Extending a combinatorial pseudo-triangulation in a Henneberg II step. Top: a representative case of a combinatorial face with four corners. Left: two (out of three) possible placements of the third vertex. Right: the possible labelings of the induced faces as combinatorial pseudo-triangles.

In a Henneberg II step, an edge  $v_1v_2$  is first removed, merging two faces labeled as combinatorial pseudo-triangles into one face  $T$ . In this process, some angles are merged into one: their labels must be reassigned, but we make no changes to the labels of the other angles. The rules for assigning labels to merged angles are simple, mimicking what one would expect to happen in a straight-line situation: if one old angle was *big*, the merged angle is marked *big*, otherwise *small*. The face  $T$  thus gets exactly four small angles. Its boundary is separated by the vertices  $v_1$  and  $v_2$  into two chains: each contains at least one corner. Four cases may happen:  $v_1$  and  $v_2$  are both *small* (corners), separating the other two corners; only one is *small* (say,  $v_1$ ), and the other corners are distributed as 1-2 on the chains; or both  $v_1$  and  $v_2$  are *big*, and the four corners are distributed as either 2-2 or 1-3. See Figure 6 for a representative case

(the other are similar and are left to the reader). Notice that it is impossible to have all four corners on only one chain induced by  $v_1$  and  $v_2$ .

The new vertex  $v$  is now inserted inside this face  $T$ , and joined to the old vertices  $v_1$  and  $v_2$ , and to some other vertex  $v_3$  on  $T$ . The new edges  $vv_1$ ,  $vv_2$  and  $vv_3$  partition the face  $T$  and its four corners into three parts, which can be assigned the labels in several ways. See Figure 7 for a representative case: the systematic verification of all the cases is straightforward and is left to the reader.  $\square$

Notice that in general the pointed combinatorial pseudo-triangulation guaranteed by Lemma 3.5 is not unique. The Lemma shows in fact how to systematically generate all of them in a non-backtracking manner: if we succeeded in finding a pointed cpt at step  $n$ , we simply extend it at the next Henneberg step. We next prove that at least one of them is realizable with straight-lines via a similar Henneberg extension technique.

**Lemma 3.6 (The Geometric Lemma)** *Every plane Laman graph  $G$  can be embedded as a pseudo-triangulation.*

**Proof:** Let  $G_n$  be a plane Laman graph on  $n$  vertices with a triangular outer face. Assume we have a plane Henneberg construction for  $G_n$  starting with the outer face and adding vertices only on interior faces. We basically follow the same analysis as in Lemma 3.5. This time, however, we will not *choose* the big/small labels of the angles, but rather show that *there exists* a way of placing a point  $p_n$  inside a face which realizes a compatible partitioning of the face into pseudo-triangles as prescribed by the Henneberg step on the vertex  $v_n$  of degree 2 or 3.

As in Lemma 3.5, the Henneberg I step is straightforward on an interior face (which is what we do here). As an exercise pointing out to the difference between the combinatorial and the geometric case, we leave it to the reader to verify that this is not the case on an outer face, where the placement of a vertex at step  $n + 1$  may be constrained by the realization up to step  $n$ , and thus may not directly allow an embedding with a certain prescribed outer face.

The analysis of a Henneberg II step is identical to that performed in the combinatorial lemma, and leads to several cases to be considered. We illustrate here only a representative case (but have verified them all). The important fact is that *it is always possible* to realize with straight-lines *at least one* of the possible Henneberg II combinatorial pseudo-triangular extensions.

Consider the (embedded) interior face  $F$  with four corners obtained by removing an interior edge  $p_i p_j$ , and let  $p_k$  be a vertex on the boundary of  $F$ . We must show that there exists a point  $p$  inside  $F$  which, when connected to  $p_i$ ,  $p_j$  and  $p_k$  partitions it into three pseudo-triangles and is itself pointed. The three line segments  $pp_i$ ,  $pp_j$  and  $pp_k$  must be tangent to the side chains of  $F$ . We define the *feasibility region* of an arbitrary point  $p_a$  on the boundary of  $F$  as the (single or double) wedge-like region inside  $F$  from where tangents to the boundary of  $F$  at  $p_a$  can be taken. The feasibility region of several points is the intersection of their feasibility regions. An important fact is that the feasibility region of  $p_i$  and  $p_j$  always contains the part of the supporting line of the removed edge  $p_i p_j$ , and that the feasibility region of *any* other vertex  $p_k$  cuts an open segment on it. In fact, the feasibility region of  $p_k$  intersects the feasibility region of  $p_i$  and  $p_j$  in a non-empty feasible 2-dimensional region on one side or the other (or both) of this segment. One can easily see that not only is this region non-empty, but it contains a subregion where a placement of  $p$  as a *pointed vertex* is possible (we call that a *pointed-feasible region*). We skip the rest of the straightforward details. See Figure 8 for a representative case.  $\square$

**Proof of Theorem 3.1.** Let  $G$  be a plane Laman graph. If its outer face is not a triangle, apply Lemma 3.3 to get a new graph  $G_n$  which will contain  $G$  in its embedding. Follow the plane geometric Henneberg construction described in Lemma 3.6 to embed  $G_n$  starting from a triangle and always inserting new vertices in some interior face.  $\square$

**Algorithmic analysis.** The proof of Theorem 3.1 can be turned into an efficient polynomial time algorithm. Given a Laman graph, verifying its planarity and producing a plane embedding (stored as a quad-edge data structure [19] with face information) can be done in linear time [22]. One then chooses an

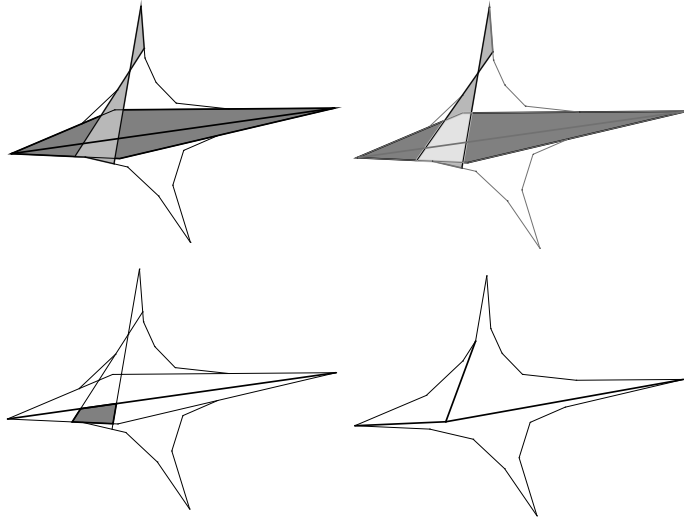


Figure 8: Henneberg II step on an interior face. One sees that the feasible region of  $p_k$  (light grey) intersects the feasible region of the two endpoints  $p_i$  and  $p_j$  (dark grey) of the removed edge. We show the two feasible regions (of the pair  $p_i, p_j$ , resp.  $p_k$ ), their intersection, the final pointed-feasible region and a placement of a pointed vertex  $p$  and its three tangents.

outer face and in linear time one can perform the construction from Lemma 3.3 to get a triangular outer face. For producing a topological Henneberg construction, we'll keep an additional field in the vertex data structure, storing the degree of the vertex. We will keep the vertices in a min-heap on the degree field. To work out the Henneberg steps in reverse we need to do efficiently the following operations: a) detect a vertex of minimum degree (which will be 2 or 3), b) if the minimum degree is 3, corresponding to a vertex  $v$ , we must find an edge  $e$  that will be put back in after the removal of the neighbors of  $v$ , and c) restore the quad-edge data structure. Step a) can be done in  $O(\log n)$  time. Step c) can be done in constant time. Step b) requires deciding which of the three possibilities for  $e$  among  $v_1v_2$ ,  $v_1v_3$  and  $v_2v_3$  (where  $v_1, v_2$  and  $v_3$  are the neighbors of the vertex removed in a reverse Henneberg II step) produces a Laman graph. Testing for the Laman condition on a graph with  $2n - 3$  edges can be done by several algorithms (the algorithms of Imai and Sugihara, via reductions to network flow or bipartite matching, or via matroid (tree) decompositions, see [53] and the references given there), and takes  $O(n^2)$ . Therefore the time for performing a reverse Henneberg step is dominated by b), which gives a total running time of  $O(n^3)$ .

The embedding is done now by performing the Henneberg steps, starting with the outer triangular face embedded on an arbitrary initial triple of points. It is straightforward to see that each step takes constant time to determine a position for the new vertex, and the whole embedding takes linear time once the Henneberg sequence is known.

The time complexity would be improved by a positive answer to the open questions 3 and 4 listed in the concluding Section 6.

### 3.2 Pseudo-Triangular Embeddings of Plane Laman-plus-one Graphs

We now turn to a proof of Theorem 3.2 using Henneberg constructions for plane Laman-plus-one graphs. It is very similar to the proof of Theorem 3.1. It is instructive though to see the differences, which lie in the combinatorial (and hence also geometric) pseudo-triangulation assignment, where we must keep track of the non-pointed vertex. We have two items which may in principle be prescribed: the outer face

and the vertex to become the unique non-pointed one. The non-pointed vertex may only be interior to the circuit. In a Henneberg construction, we will see that it is easy to prescribe either the outer face or the interior vertex to be non-pointed, but the analysis becomes more complicated for the prescription of both. In Section 4 we use a different, global argument to do the simultaneous prescription of the outer face and of the non-pointed vertex, in the case of a rigidity circuit.

The next two lemmas are straightforward extensions of the Laman case.

**Lemma 3.7 (Fixing the Outer Face)** *Embedding a plane Laman-plus-one graph as a pseudo-triangulation reduces to the case when the outer face is a triangle.*

Notice that this would fail if we only planned to deal with a circuit.

**Lemma 3.8 (The Topological Lemma)** *Every plane Laman-plus-one graph has a plane Henneberg construction.*

A planar Laman-plus-one graph (on  $n$  vertices) always has at least two triangular faces: the dual planar graph has  $n$  vertices (including the vertex corresponding to the outer face) and  $2n - 2$  edges, hence there exists at least two of degree three. The previous construction allows to prescribe the outer face in a geometric embedding, should we want to do so, and is not needed in the proof of the combinatorial lemma below.

**Lemma 3.9 (The Combinatorial Lemma)** *Every plane Laman-plus-one graph admits a pointed-plus-one combinatorial pseudo-triangulation assignment.*

**Proof:** The proof has the same basic structure (but more cases to analyze) as Lemma 3.5, and relies on the details of the Henneberg construction from Lemma 2.3. The base case is  $K_4$  which has a unique cpt assignment for a choice of an outer face. It is easy to see that Henneberg I steps cause no problem, and the Henneberg II steps work as before when the vertex of degree 3 is not inside the circuit, is not the pointed vertex, and it is not incident to it.

Let  $v_i v_j$  be the removed edge and  $v_k$  the third vertex involved in the Henneberg II step. The only problematic case is when the edge  $v_i v_j$  is incident to the unique non-pointed vertex. In this case, the resulting face after the removal of  $v_i v_j$  is a pseudo-triangle: it has three, not four corners. We must argue that *in at least one combinatorial pseudo-triangulation* compatible with the information so far, the three vertices  $v_i$ ,  $v_j$  and  $v_k$  cannot lie all three on the same side-chain of this face, otherwise the extension to a cpt is impossible.

There is a way around this, which would guarantee that both the outer face and the non-pointed vertex could be prescribed. We will describe this, in a more general setting, in a forthcoming paper. For the time being, it suffices to notice that if this happens during the Henneberg construction, we can always pick up one of the other three vertices guaranteed to have degree three (when there are no degree two vertices), and continue from there. Notice that this may change the outer face assignment, though.  $\square$

**Lemma 3.10 (The Geometric Lemma)** *Every plane Laman-plus-one graph  $G$  can be embedded as a pseudo-triangulation.*

This proof, and the proof of Theorem 3.2 are now straightforward extensions of those done for the Laman case. Notice that it may not be possible in general to guarantee a certain outer face or non-pointed vertex.

**Remarks.** The inductive technique described in this section works in fact for *rigid* graphs on  $n$  vertices and fewer than  $2n$  edges (i.e. Laman graphs with at most two extra edges). Indeed, the only ingredient that is needed is the existence of a vertex of degree at most 3 whose removal either leaves a rigid graph or a graph with one degree of freedom. Given a rigid graph with a combinatorial pseudo-triangulation assignment we can add edges and combinatorially assign the big and small labels to the angles, preserving at each step the property that every face has exactly three small angles and every vertex has at most one big angle and the outside face has only big angles. However, not all of these combinatorial angle assignments are geometrically realizable.

The **algorithmic analysis** is similar to the case of plane Laman graphs.

## 4 Combinatorial pseudo-triangulations

In this section we extend the results from Section 3 on pointed and pointed-plus-one combinatorial pseudo-triangulations. We present a global, non-inductive technique for generating cpt assignments for planar Laman graphs and planar circuits. It is based on a reduction to finding perfect matchings in a certain associated bipartite graph. By showing that Hall's condition is satisfied, we are guaranteed to have a solution (and hence a cpt) for both plane Laman graphs and circuits. We also show that the existence of a pointed combinatorial pseudo-triangulation assignment is not restricted to plane Laman graphs or circuits.

Let  $G = (V, E, F)$  be a plane graph with vertices  $V$ , edges  $E$  and faces  $F$ . Assume  $|V| = n$  and  $|E| = 2n - 3$ . Euler's relation implies that  $|F| = n - 1$ . Denote by  $F'$  the set of interior faces and by  $f_o$  the outer face (with  $h$  vertices),  $F = F' \cup \{f_o\}$ . We define a bipartite graph  $H$  with the two sets of the bipartition labeled  $V$  and  $W$ .  $V$  stands for the set of vertices  $V$  of  $G$  and has  $n$  elements. The set  $W$  corresponds to the faces  $F$  of  $G$  taken with certain multiplicities. For an interior face  $f \in F'$  of degree (number of edges on the face)  $d_f$ , we will put  $d_f - 3$  vertices in  $W$ . For the outer face  $f_o$  we will put  $h = d_{f_o}$  nodes in  $W$ . The total number of elements in  $W$  is thus  $\sum_{f \in F'} (d_f - 3) + h = \sum_{f \in F} d_f - 3|F'| = 2|E| - 3(n - 2) = 2(2n - 3) - 3(n - 2) = n$ .

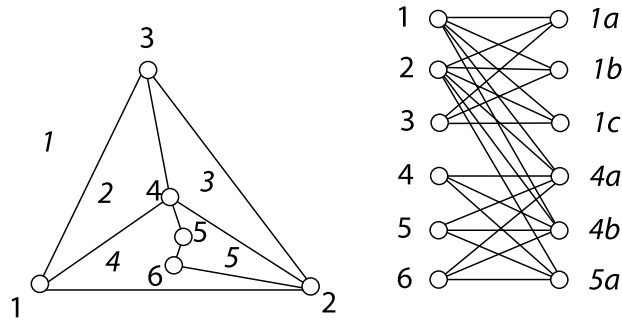


Figure 9: A plane graph with  $2n - 3$  edges and its associated bipartite graph  $H$ .

A vertex  $v \in V$  is connected in  $H$  to the vertices in  $W$  corresponding to the interior faces  $f$  of degree larger than 3 to which it belongs in  $G$ , and to the vertices corresponding to the outer face (if it belongs to it). Hence if  $v$  belongs to three faces  $f_1, f_2, \dots$ , and these faces have multiplicities  $d_1, d_2, \dots$  in  $W$ , then  $v$  is connected to  $d_1$  copies of the vertex for  $f_1$ ,  $d_2$  copies for  $f_2$ , etc. See Figure 9. The 6 vertices and 5 faces of degrees 3 (outer face 1), 3 (faces 2 and 3), 4 (face 5) and 5 (face 4) lead to the bipartition sets  $V = \{1, 2, 3, 4, 5, 6\}$  and  $W = \{1a, 1b, 1c, 4a, 4b, 5a\}$ , connected by edges as in the figure.

The connections (edges) in the bipartite graph  $H$  represent potential assignments of *big* angles, where an *angle* is viewed as a pair  $(vertex, face)$ . Since each vertex must receive a big angle, we want a perfect matching. Since each interior face receives all but three big angles, and the outer face receives all big angles, the choice of multiplicities reflects just that. These considerations lead to the following Lemma.

**Lemma 4.1** *There is a one-to-one correspondence between the combinatorial pseudo-triangulations of a plane graph  $G$  with  $n$  vertices and  $2n - 3$  edges and the perfect matchings in the associated bipartite graph  $H$ .*

In general, plane graphs satisfying the conditions of the previous Lemma may or may not have combinatorial pseudo-triangulation assignments. See Figure 10 for examples. But for Laman graphs, we are guaranteed a solution. The main result of this section is:

**Theorem 4.2** *If  $G$  is a Laman graph, then  $H$  has a perfect matching. Hence  $G$  has a pointed combinatorial pseudo-triangulation.*

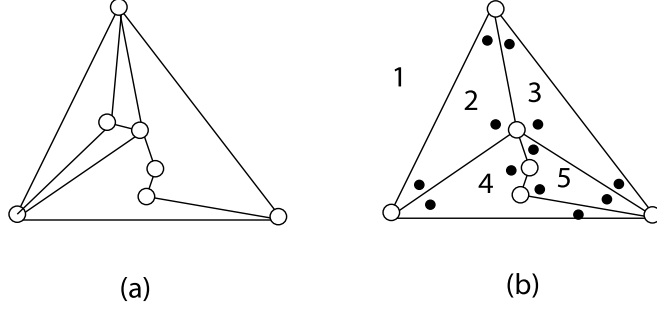


Figure 10: (a) A plane graph with  $2n - 3$  edges and no combinatorial pseudo-triangulation assignment. (b) A plane non-Laman graph with a cpt assignment.

**Proof:** We will check Hall's condition to guarantee the existence of a perfect matching. Let  $A \subset V$  be a subset of  $|A| = a$  vertices. Let  $F_A$  of size  $|F_A| = f_a$  be the set of faces incident to the vertices in  $A$ , and let  $D = \sum_{f \in F_A} d_f$ . We need to show that  $a \leq D - 3f_A$ .

It suffices to carry out the analysis on different face-connected components of  $F_A$  separately. See Figure 11. One face-connected component is a polygon with (say)  $b$  boundary edges,  $b' \leq b$  boundary vertices,  $h \geq 0$  holes and  $e_i$  interior edges. We have  $D = 2e_i + b$ . By Euler's relation  $(a + b') + (f_A + h + 1) = (b + e_i) + 2$ . Hence  $f_A = e_i - a + 1 + b - b' - h$ , where  $\Delta := b - b' - h \geq 0$ . Laman's condition implies that  $b + e_i \leq 2(a + b') - 3$ , hence  $e_i \leq 2a + 2b' - b - 3$ . Now to show  $a + 3f_A \leq D$  we need  $a + 3(e_i - a + 1 + b - b' - h) \leq 2e_i + b$ , i.w.  $e_i \leq 2a - 3 + 3b' - 2b + 3h$ . Since we know  $e_i \leq 2a + 2b' - b - 3$ , it remains to show  $2b' - b \leq 3b' - 2b + 3h$ , i.e.  $b \leq b' + 3h$ , which is obviously true.  $\square$

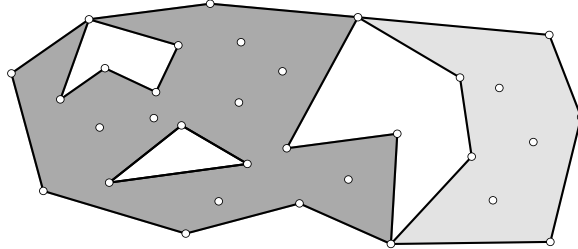


Figure 11: The analysis in the proof of Theorem 4.2: it suffices to analyze separately the face-connected components of  $F_A$  (two in this case, shaded slightly differently).

The result of Theorem 4.2 extends to the case of plane circuits. Moreover, we will be able to show in this case a more general version of Theorem 3.8, by being able to prescribe both the outer face and the non-pointed vertex (which may be chosen as any vertex non-incident to the outer face). In this case, the associated bipartite graph is slightly different: the set  $V$  contains all vertices but one, namely the vertex prescribed to be the non-pointed one. The set  $W$  has the same description, but its size now is  $n - 1$  (because the number of faces is  $n$ ).

**Theorem 4.3** *If  $G$  is a plane circuit, then  $H$  has a perfect matching. Hence  $G$  has a pointed-plus-one combinatorial pseudo-triangulation with a prescribed outer face and prescribed non-pointed vertex.*

**Proof:** The analysis from proof of Theorem 4.2 still holds, because in the case of a circuit, the condition on subsets of size  $k < n$  is *exactly* the same as for Laman graphs: they span at most  $2k - 3$  edges.



Therefore the analysis works whenever  $F_A$  does not cover the whole polygon. Since at least one vertex is missing from  $A$  (the vertex prescribed to be non-pointed), this is always the case.  $\square$

**Algorithmic analysis.** To check whether a graph admits a combinatorial pseudo-triangulation (and to compute one) we will use the  $O(n^{3/2})$  time algorithm for the maximum flow problem of Dinits (see [44]) to solve the bipartite matching problem described above.

**Remark.** This set of degree-constrained subgraphs of a bipartite graph can be modelled as a network flow problem. Thus the set of combinatorial pseudo-triangulations of a given graph (with a given planar embedding, including a specification of the outer face) is in one-to-one correspondence with the vertices of a polytope, given by the equations and inequalities of the network flow.

## 5 Stretching combinatorial pseudo-triangulations

We have seen in the previous sections two proofs of the fact that every plane Laman graph can be assigned a combinatorial pseudo-triangulation labeling. The technique from Section 3 does not realize geometrically every such possible combinatorial structure. In this section we give the strongest version of the main result by proving the following theorem.

**Theorem 5.1** *For any plane Laman graph  $G$  and for any of its combinatorial pseudo-triangular assignments, there exists (and can be efficiently found) a compatible straight-line embedding. The same holds for plane circuit graphs.*

The proof is a consequence of two general results of independent interest. We first give in Theorem 5.4 two characterizations of stretchable combinatorial pseudo-triangulations. The stretchability proof relies on a directed version of Tutte's Barycentric Embedding Theorem, (Theorem 5.5). Finally, we show that the characterization in Theorem 5.4 is satisfied for Laman (Theorem 5.6) and circuit plane graphs (Theorem 5.7) with cpt assignments.

### 5.1 Two characterizations of stretchability

In this section we give two combinatorial characterizations of stretchability of combinatorial pseudo-triangulations in terms of the number of *corners* of planar subcomplexes and in terms of 3-connectivity properties of an associated directed graph.

Let  $G = (V, E)$  be a plane graph with a combinatorial pseudo-triangulation labeling. We do not impose any restrictions on its number of non-pointed vertices or rigidity properties. As a plane graph, every subgraph  $G_S = (S, E_S)$  induced by a subset of vertices  $S \subset V$  has an induced plane embedding and a well-defined unbounded region. The *boundary* of the unbounded region consists of cycles of vertices and edges, with one cycle for each connected component of  $G_S$ . Some edges and/or vertices may be repeated in these cycles. For example, if  $G_S$  is a tree then every edge appears twice.

**Corners of boundary cycles.** We have defined *corners* in combinatorial pseudo-triangulations as being the angles marked *small*. We extend the concept to the *vertices* on boundary cycles of induced subgraphs  $G_S$  by looking at the labels of angles in  $G$  incident to  $v$  on the outer face of  $G_S$ . We call  $v$  a *corner of type 1* if it contains a *big* label on the outer face, or a *corner of type 2* when  $v$  is non-pointed in  $G$  but contains two consecutive *small* labels on the outer face.

The following simple counting lemma will be useful later.

**Lemma 5.2** *Let  $G_S$  be a subgraph of a cpt induced by the subset  $S \subset V$ . Assume that  $G_S$  is connected and that it contains all the edges lying in the interior of its boundary cycle.*

*Let  $m$  be the number of edges,  $k$  the number of pointed vertices,  $l$  the number of non-pointed vertices in  $G_S$  and  $b$  the length of the boundary cycle in  $G_S$ . Then the number  $c_1$  of corners of the type 1 (big angles in the outer boundary) of  $G_S$  equals*

$$c_1 = m + 3 - 2k - 3l + b.$$

In this statement a vertex in  $G_S$  is called *pointed* if and only if it was pointed in  $G$ .

**Proof:** Let  $f$  is the number of interior pseudo-triangles. The number of interior angles in  $G_S$  is  $3f + k - c_1$ , because there are  $3f$  small interior angles and  $k - c_1$  interior big angles. But the number of interior angles also equals  $2m - b$  (since the total number of angles in any plane graph equals  $2m$ ). Hence,  $2m - b = 3f + k - c_1$ , or

$$3(m - f) = m + k + b - c_1.$$

Finally, Euler's formula applied to  $G_S$  (as it contains all its interior edges) is  $m - f = (k + l) - 1$ , which implies  $3k + 3l - 3 = m + k + b - c_1$  and thus the desired statement.  $\square$

**The partially directed auxiliary graph  $D$  of a combinatorial pseudo-triangulation  $G$ .** A *partially directed graph*  $D = (V, E, \vec{E})$  is a graph  $(V, E)$  together with an assignment of directions to some of its edges. Thus edges are allowed to get two directions, one direction only, or remain undirected. Formally,  $\vec{E}$  is a subset of  $E \cup (-E)$  where  $E$  is the set of directed edges of  $G$ .

A plane embedding of a partially directed graph  $(V, E, \vec{E})$  is *3-connected to the boundary* if from every interior vertex  $p$  there are at least three vertex-disjoint directed paths in  $\vec{E}$  ending in three different boundary vertices. Equivalently, if for any interior vertex  $p$  and for any pair of forbidden vertices  $q$  and  $r$  there is a directed path from  $p$  to the boundary not passing through  $q$  or  $r$ .

**Lemma 5.3** *For every combinatorial pseudo-triangulation, there exists a partially directed graph  $D$  satisfying the following conditions:*

1.  $D$  is planar and contains the underlying graph of  $G$ .
2. Every interior pointed vertex  $v \in V$  of  $G$  has three out-neighbors: its two neighbors in  $G$  along extreme edges and a neighbor along the interior of the pseudo-triangle containing the big angle at  $v$ .
3. For every pointed vertex  $v$  of  $G$  its out-neighbors in  $D$  are exactly its neighbors in  $G$ .

**Proof:** We extend the underlying graph of  $G$  to a (topological) triangulation by triangulating the pseudo-triangles of  $G$  with more than three vertices in such a way that every big angle of  $G$  is dissected by at least one new edge. This can be achieved by recursively dissecting each face with an edge joining a non-pointed vertex on the face to the opposite corner. Then the edges are oriented as required by the statement. See Figure 12 for an illustration of how a face is triangulated and how the edges incident to big angles are oriented.  $\square$

The main result of this section can now be stated.

**Theorem 5.4** *For a combinatorial pseudo-triangulation  $G$  with non-degenerate (simple polygonal) faces the following are equivalent:*

1.  $G$  can be stretched into a compatible pseudo-triangulation.
2. Every subgraph of  $G$  with at least three vertices has at least three corners.
3. Every partially directed graph satisfying the requirements of Lemma 5.3 is 3-connected to the boundary.

## 5.2 Proof of Theorem 5.4

The implication from part 1 to part 2 is trivial. If  $G$  is embedded as a pseudo-triangulation, there is no loss of generality in assuming that the embedding is in general position, so that every subgraph with at least three vertices has at least three convex hull vertices. And all convex hull vertices of a subgraph of  $G$  will be corners according to our definition.

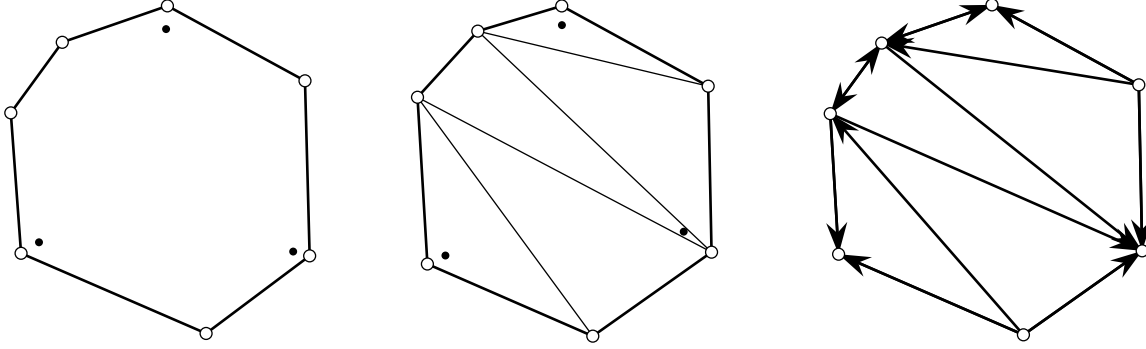


Figure 12: Left: A combinatorial pseudo-triangular face, with a small black dot indicating a small angle (big angles are not marked). Middle: a compatible triangulation of the face. Right: the edges of the auxiliary directed graph.

**Proof of  $2 \Rightarrow 3$  in Theorem 5.4:** To prove 3-connectedness, we will show that from every interior vertex  $a$  there is a directed path in  $D$  going to the boundary and not passing through two arbitrary (but fixed) vertices  $b$  and  $c$ .

Let us consider  $A$ , the *directed connected component* of vertex  $a$ , defined as the set of all vertices and directed edges of  $D$  that can be reached from  $v$  without passing through  $b$  or  $c$ . We define this component as not containing the forbidden points  $b$  and  $c$ , but it may contain edges *arriving at them*. Our goal is to prove that  $A$  contains a vertex on the boundary of  $D$ . We argue by contradiction. Suppose that all the vertices of  $A$  are interior to  $D$ .

For each interior pointed vertex  $v$ , let  $T_v$  be the unique pseudo-triangle of  $G$  containing the big angle at  $v$ . Thus  $v$  is in an edge-chain of  $T_v$  containing also the two extreme adjacent edges of  $v$ . Let  $G_S = (S, E_S)$  be the graph enclosing all the pseudo-triangles  $T_v$  associated to the pointed vertices  $v$  of  $A$  and all the pseudo-triangles incident to the non-pointed vertices. Clearly  $G_S$  contains  $A$ : indeed, every directed edge of  $G$  is contained in a pseudo-triangle associated to its source vertex.

We now use our hypothesis that  $G_S$  has at least three corners. We claim that at least one of them,  $d$ , belongs to  $A$ . This gives the contradiction, because then there is an edge of  $D \setminus G$  jumping out of that corner  $d$  (by the conditions imposed to the partial orientation in  $D$ ), which means that the pseudo-triangle(s) corresponding to  $d$  should have been contained in  $G_S$  and hence  $d$  is not a corner of  $G_S$  anymore.

To prove the claim, let  $v_1, \dots, v_k$  be the corners of  $G_S$  which are not in  $A$ . We want to prove that  $k \leq 2$ . For this let  $T_1, \dots, T_k$  be pseudo-triangles in  $G_S$ , each having the corresponding  $v_i$  as a corner (there may be more than one valid choice of the  $T_i$ 's; we just choose one). By definition, some non-corner pointed vertex or some corner non-pointed vertex of each  $T_i$  is in the component  $A$ . Were there no forbidden points, from a non-corner pointed vertex we could arrive to the three corners of  $T_i$  by three disjoint paths: two of them along the concave chain containing the initial point and the third starting with an edge of  $D \setminus G$ . From a non-pointed corner vertex we could arrive to the other two corners by two disjoint paths: moving out from the vertex to the two neighbors in the incident pseudo-edges and then following along them. In particular, since  $v_i$  is not in  $A$ , one of the forbidden points must obstruct to one of these paths, which implies that either  $v_i$  equals one of the forbidden points  $b$  and  $c$  or that  $T_i$  is the pseudo-triangle of one of the two forbidden points. And, clearly, each of the two forbidden points contributes to only one of the indices  $i$  (either as a corner of  $G_S$  or via its associated pseudo-triangle if it is not a corner, but not both). This shows that  $k \leq 2$  and completes the lemma.  $\square$

**Tutte's equilibrium condition.** To prove  $3 \Rightarrow 1$  we use a directed version of Tutte's Theorem on barycentric embeddings of graphs.

An embedding  $D(P)$  of a partially directed graph  $D = (V, E, \vec{E})$  on a set of points  $P = \{p_1, \dots, p_n\}$ , together with an assignment  $w : \vec{E} \rightarrow \mathbb{R}$  of weights to the directed edges is said to be in *equilibrium* at a vertex  $i \in V$  if

$$\sum_{(i,j) \in \vec{E}} w_{ij}(p_i - p_j) = 0.$$

**Theorem 5.5 (Directed Tutte Theorem)** *Let  $D = (\{1, \dots, n\}, E, \vec{E})$  be a partially directed plane graph, 3-connected to the boundary and whose boundary cycle has no repeated vertices. Let  $(k+1, \dots, n)$  be the ordered sequence of vertices in this boundary cycle and let  $p_{k+1}, \dots, p_n$  be the ordered vertices of a convex  $(n-k)$ -gon. Let  $p_1, \dots, p_k$  be the ordered vertices of a convex  $(n-k)$ -gon. Let  $w : \vec{E} \rightarrow \mathbb{R}$  be an assignment of positive weights to the internal directed edges. Then:*

- (i) *There are unique positions  $p_1, \dots, p_k \in \mathbb{R}^2$  for the interior vertices such that all of them are in equilibrium in the embedding  $D(P)$ ,  $P = \{p_1, \dots, p_n\}$ .*
- (ii) *In this embedding, all cells of  $D$  are then realized as non-overlapping convex polygons.*

**Proof:** The proof of Tutte's Theorem given in [36] (Theorem 12.2.2, pages 123–132) works with only minor modifications. First, in the definition of *good representation* (Definition 12.2.6, page 126), each point  $p_i$  is required to be in the relative interior of its *out-neighbors*, since this is what the directed equilibrium condition gives. Second, Claim 1 on page 126 proves that in a good representation it is not possible for a vertex  $p$  that  $p$  and all its neighbors lie in a certain line  $\ell$ , using 3-connectedness. The proof can be adapted to use *3-connectedness to the boundary* as follows: consider three vertex disjoint paths from  $p$  to the boundary. Call  $q$  any of the three end-points, assumed not to lie in the line  $\ell$ . Complete the other two paths to end at  $q$  using boundary edges in opposite directions. This produces three vertex-disjoint paths from  $p$  to a vertex  $q$  not lying on  $\ell$ . The rest needs no change.  $\square$

**Proof of  $3 \Rightarrow 1$  in Theorem 5.4:** Construct an auxiliary partially directed graph  $D$  in the conditions of Lemma 5.3, choose arbitrary positive weights for its directed edges, and apply the Directed Tutte Theorem to it. Since all weights are positive, the equilibrium condition on an interior vertex  $p$ , together with the convexity of faces that comes from Tutte's theorem implies that every interior vertex is in the relative interior of the convex hull of its out-neighbors. The conditions on  $D$  then imply that the straight-line embedding of  $G$  so obtained has big and small angles distributed as desired.  $\square$

**Time Analysis.** Suppose that we are given a cpt that can be stretched. Tutte's theorem actually gives an algorithm to find a stretching: construct the auxiliary graph  $D$  of Lemma 5.3, choose coordinates for the boundary cycle in convex position and arbitrary positive weights for the directed edges, and then compute the equilibrium positions.

Everything can be done in linear time, except for the computation of the equilibrium position for the interior vertices. In this computation one writes a linear equation for each interior vertex, which says that the position of the vertex is the average of its (out-)neighbors. The position of the boundary vertices is fixed. It has been observed [8, Section 3.4] that the planar *structure* of this system of equations allows it to be solved in  $O(n^{3/2})$  time, using the  $\sqrt{n}$ -separator theorem for planar graphs in connection with the method of Generalized Nested Dissection (see [26, 27] or [37, Section 2.1.3.4]), or even in time  $O(M(\sqrt{n}))$ , where  $M(n) = O(n^{2.375})$  is the time to multiply two  $n \times n$  matrices.

### 5.3 Laman and circuit combinatorial pseudo-triangulations can be stretched

Not all combinatorial pseudo-triangulations can be stretched: see for instance the first example in Figure 10. Its non stretchability can be proved either by showing that the graph is not Laman (while the graph

of every pointed pseudo-triangulation must be so) or by applying the characterization given in Theorem 5.4 and finding a subgraph with less than three corners.

Our next goal is to prove that if the underlying graph of a combinatorial pseudo-triangulation is Laman (or is a rigidity circuit) then it can be stretched. The proof uses the Laman counting condition to show that every subgraph has at least three corners. We recall that both Laman graphs and circuits have the property that a subgraph induced on a subset of  $k \geq 2$  vertices has at most  $2k - 3$  edges. By taking the complementary set of vertices (and edges) this is equivalent to:

- In a Laman graph with  $n$  vertices, every subset of  $k \leq n - 2$  vertices is incident to at least  $2k$  edges.
- In a rigidity circuit graph with  $n$  vertices, every subset of  $k \leq n - 2$  vertices is incident to at least  $2k + 1$  edges.

**Theorem 5.6** *Every subgraph  $G_S$  of a Laman combinatorial pseudo-triangulation  $G$  has at least 3 corners. Therefore  $G$  can be stretched.*

**Proof:** We show first that there is no loss of generality in assuming that  $G_S$  is *simply connected* (i.e. it is connected and contains all the edges of  $G$  interior to its contour) and that no edge appears twice in the boundary cycle.

If  $G_S$  has an edge which appears twice on the boundary cycle, its removal does not change the number of corners; indeed, each end-point of such an edge that is a corner after the removal must be a corner before as well. If  $G_S$  is not connected, either some connected component has at least three vertices or all the vertices of  $G_S$  are corners. If  $G_S$  is connected but not simply connected, then adding to  $G_S$  the pseudo-triangles, edges and vertices of  $G$  that fill in the holes does not change the number of corners.

We now observe that since  $G$  is pointed, the equation in Lemma 5.2 becomes

$$e = 2k - 3 - (b - c),$$

where  $e$ ,  $k$ ,  $b$  and  $c$  are the numbers of edges, vertices, boundary edges and corners of  $S$ .

Let  $b_0$  be the number of boundary vertices of  $G_S$  (which may be smaller than  $b$ , if a boundary vertex appears twice in the boundary cycle). We now consider the set of edges incident to vertices in the interior of  $G_S$ . Since there are  $k - b_0$  interior vertices, the (rephrased) Laman property tells us that there are at least  $2(k - b_0)$  such edges. On the other hand, these edges are all interior to  $G_S$ , and the total number of interior edges in  $G_S$  is  $e - k$ . Hence:

$$e - k = 2k + c - 3 - 2 \geq 2k - 2b_0$$

which implies the desired relation  $c \geq 3 + 2b - 2b_0 \geq 3$ . □

**Theorem 5.7** *Every subgraph  $G_S$  of a rigidity circuit  $G$  has at least 3 corners. Hence,  $G$  can be stretched.*

**Proof:** As in Theorem 5.6 we may assume without loss of generality that  $G_S$  is connected, contains all the edges of  $G$  enclosed by its boundary cycle and its boundary cycle has no repeated edges.

Let  $k$ ,  $l$ ,  $e$ , and  $b$  be the numbers of pointed vertices, non-pointed vertices, edges and boundary edges of  $S$ , respectively. By Lemma 5.2 we have

$$e = 2k + 3l + c_1 - b - 3.$$

If  $G_S$  has no interior edge then the statement is trivial: either  $G_S$  contains no pseudo-triangle and then all its vertices are corners, or it contains only one pseudo-triangle and the three corners of it are corners of  $G_S$ . If  $G_S$  has at least one interior edge then:

$$e \geq 2k + 2l - b + 1.$$

Indeed, let  $A \subset S$  be the set of vertices interior to  $G_S$ , so that its cardinality equals  $k+l-b_0$ , where  $b_0 \leq b$  is the number of boundary vertices in  $G_S$ . If  $A$  is empty then  $b_0 = k+l \leq b$  and our inequality becomes  $e \geq b+1-2(b-b_0)$ , which holds by the existence of at least one interior edge. If  $A$  is not empty we apply the Laman condition to  $G_A$ , which says that the number of interior edges of  $G_S$  is at least  $2(k+l-b_0)+1$ , hence the total number of edges in  $G_S$  is at least  $2(k+l-b_0)+1+b = 2k+2l-b+1+2(b-b_0) \geq 2k+2l-b+1$ .

The two formulas above imply that

$$l + c_1 \geq 4$$

and, since  $l \leq 1$  (because there is only one non-pointed vertex in  $G$ ),  $c_1 \geq 3$ . □

This completes the proof of Theorem 5.1.

## 6 Conclusions and Open Problems

We have shown that any combinatorial pseudo-triangulation of a plane Laman graph or of a plane rigidity circuit is stretchable. In this latter case, we may even prescribe the non-pointed vertex. In addition, Laman-plus-one and Laman-plus-two graphs are also stretchable, although we may not in general be able to prescribe the outer face or the non-pointed vertices.

The Main Result stated in the Introduction has thus been extended along several lines, leading to interesting combinatorial objects to study and several open questions, some solved in this paper, some left for the future. We end with a listing of the main directions for further investigations.

**Embeddability of planar generically rigid graphs as pseudo-triangulations.** The goal here is the clarification of the connection between minimum (pointed) pseudo - triangulations of a planar point set and triangulations (maximal planar graphs embedded on the same point set). Triangles are pseudo-triangles, and every triangulation is a pseudo-triangulation, but some or all of the vertices of the embedding may not be pointed. All planar graphs containing a Laman graph are rigid (although not minimally so). Stratifying by the number of additional edges (besides a minimally rigid substructure) added to a Laman graph, we want to investigate realizability as triangulations with some prescribed number of non-pointed vertices. In this paper, we solved the case of one additional edge (via the special case of rigidity circuits). We leave open the question of completing the characterization for the whole hierarchy. Such an investigation will shed light into new intrinsic properties of planar triangulations, some of the best studied and still elusive objects in Combinatorial Geometry. We make the following conjecture.

**Conjecture 1** *Given a plane graph  $G$ , the following conditions are equivalent:*

- (i)  $G$  is generically rigid
- (ii)  $G$  can be straightened as a pseudo-triangulation.

**Combinatorial pseudo-triangulations and embeddings on oriented matroids (pseudolines).**

We have seen that not all planar graphs admitting combinatorial pointed pseudo-triangular labelings are Laman graphs. But those which are Laman also have straight-line realizations. A further direction of research emerging from our work is to study the connection between the combinatorial pseudo-triangulations and realizations in the oriented matroid sense (on pseudo configurations of points).

**Grid size of pseudo-triangular embeddings.** Every planar graph can be embedded on a grid of size  $O(n) \times O(n)$ , see for example [17], [43], [15]. Here is a natural remaining problem.

**Open Problem 2** *Can a planar Laman graph be embedded as a pseudo triangulation on a  $O(n^k) \times O(n^k)$  size grid? What is the smallest such  $k$ ?*

**Reciprocal duals of pseudo-triangulations.** Planar graphs have combinatorial duals, obtained by replacing faces with vertices and vice-versa. Moreover, when an embedded planar graph supports a self-stress, it has a *geometric dual*, the so-called *reciprocal diagram* of Maxwell [12]. A natural question (which will be answered in a subsequent paper) concerns the connection between stressed pseudo-triangulations (necessarily not minimal) and the planarity of their reciprocal duals, see [30].

**Algorithmic questions.** We conjecture that our embedding algorithms can be improved from  $O(n^{\frac{3}{2}})$  to  $O(n \log n)$  time. In general, the time complexity would be improved by a positive answer to the following open questions.

**Open Problem 3** *Is it possible to decide the Laman condition in linear time for a planar graph?*

**Open Problem 4** *Is it possible to decide, faster than by testing the Laman condition, which edge to put back in a Henneberg II step for a planar graph? For a combinatorial pseudo-triangulation?*

**Acknowledgements** This research was initiated at the *Workshop on Rigidity Theory and Scene Analysis* organized by Ileana Streinu at the Bellairs Research Institute of McGill University in Barbados, Jan. 11-18, 2002 and partially supported by NSF grant CCR-0203224. A preliminary version of this paper appeared in [20].

## References

- [1] O. Aichholzer, F. Aurenhammer, H. Krasser and B. Speckmann. *Convexity Minimizes Pseudo-Triangulations*, Proc. 14th Canad. Conf. Comp. Geom., pp. 158-161, 2002.
- [2] O. Aichholzer, D. Orden, F. Santos, B. Speckman. *On the number of pseudo-triangulations of certain point sets*, to appear in *Proceedings of the 15th Canadian Conference on Computational Geometry*, Dalhousie University, 2003.
- [3] O. Aichholzer, G. Rote, B. Speckmann and I. Streinu. *The Zigzag Path of a Pseudo-Triangulation*, Proc. WADS, Aug. 2003.
- [4] P. Agarwal, J. Basch, L.J. Guibas, J. Herschberger and L. Zhang. *Deformable free space tilings for kinetic collision detection*, in *Algorithmic and Computational Robotics: New Directions*, Proc. 5th Workshop Algor. Found. Robotics (WAFR), 2001, pp. 83-96.
- [5] A. Berg and T. Jordán. *A proof of Connelly's conjecture on 3-connected circuits of the rigidity matroid*. to appear in J. Combinatorial Theory, Ser. B, 2002.
- [6] S. Bespamyatnikh. *Enumerating Pseudo-Triangulations in the Plane*, Proc. 14th Canad. Conf. Comp. Geom., 2002, 162-166.
- [7] H. Brönnimann, L. Kettner, M. Pocchiola and J. Snoeyink. *Counting and enumerating pseudo-triangulations with the greedy flip algorithm*, manuscript, 2001.
- [8] M. Chrobak, M. Goodrich and R. Tamassia. *Convex Drawings of Graphs in Two and Three Dimensions*, Proc. 12th Ann. Sympos. Comput. Geom., 1996, 319-328.
- [9] E. Colin de Verdière, M. Pocchiola and G. Vegter. *Tutte's Barycenter Method applied to Isotopies*. In Proc. 13th Canad. Conf. Comput. Geom., August 2001
- [10] R. Connelly, E. Demaine and G. Rote. *Straightening Polygonal Arcs and Convexifying Polygonal Cycles*, to appear, Discrete and Comp. Geometry, 2002. Preliminary version in Proc. 41st FOCS, 2000, pp. 432-452.
- [11] H. Crapo and W. Whiteley. *Plane self-stresses and projected polyhedra I: The basic pattern* Structural Topology 20, 1993, pp. 55-77.

- [12] H. Crapo and W. Whiteley. *Spaces of stresses, projections, and parallel drawings for spherical polyhedra*, Beitrage zur Algebra und Geometrie / Contributions to Algebra and Geometry 35 (1994), 259-281.
- [13] G. Di Battista, P. Eades, R. Tamassia and I. Tollis. *Graph Drawing*, Prentice Hall 1999.
- [14] T. Eren, W. Whiteley, A. Stephen Morse and P. Belhummeur. *Sensor and Network Topologies of Formations with Distance-Direction Angle Constraints*, submitted, March 2003.
- [15] S. Felsner. *Convex drawings of Planar Graphs and the Order Dimension of 3-Polytopes*. Order 18 (2001), 19-37.
- [16] I. Fary. *On straight lines representation of planar graphs*, Acta Sci. Math. Szeged, 11 (1948), pp. 229-233.
- [17] H. de Fraysseix, J. Pach and R. Pollack. *How to Draw a Planar Graph on a Grid*, Combinatorica vol. 10, no. 1, 1990, pp. 41-51.
- [18] J. Graver, B. Servatius and H. Servatius. *Combinatorial Rigidity*, Amer. Math. Soc., Graduate Studies in Mathematics vol.2, 1993.
- [19] L. Guibas and J. Stolfi. *Primitives for the manipulation of general subdivisions and the computation of Voronoi diagrams*. ACM Trans. Graph. 4:2, 1985, pp. 74-123.
- [20] R. Haas, D. Orden, G. Rote, F. Santos, B. Servatius, H. Servatius, D. Souvaine, I. Streinu, and W. Whiteley. *Planar minimally rigid graphs and pseudo-triangulations*. Proc. 19th ACM Symp. Comp. Geom, 2003.
- [21] L. Henneberg. *Die graphische Statik der starren Systeme*, Leipzig 1911, Johnson Reprint 1968.
- [22] J. E. Hopcroft and R. E. Tarjan, *Efficient Planarity Testing*, JACM, vol. 21, No. 4, pp. 549-568, 1974.
- [23] D. Jacobs, A.J. Rader, L. Kuhn, and M. Thorpe. *Protein flexibility predictions using graph theory*. Proteins 44 (2001), 150-165.
- [24] L. Kettner, D. Kirkpatrick, A. Mantler, J. Snoeyink, B. Speckmann and F. Takeuchi. *Tight Degree Bounds for Pseudo-Triangulations of Points*, Comp. Geom. Theory. Applications, 2002, to appear.
- [25] G. Laman. *On Graphs and rigidity of plane skeletal structures*, J. Engrg. Math. 4, pp. 331-340.
- [26] R. J. Lipton, D. J. Rose and R. E. Tarjan. *Generalized nested dissection*. SIAM J. Numer. Anal. 16, 1979, 346-358
- [27] R. J. Lipton and R. E. Tarjan. *Applications of a planar separator theorem*. SIAM J. Comput. 9, 1980, 615-627
- [28] N. Mněv. *On manifolds of combinatorial types of projective configurations and convex polyhedra*, Soviet Math. Doklady, 32:335-337, 1985.
- [29] D. Orden and F. Santos. *The polytope of non-crossing graphs on a planar point set*. Preprint, February 2003. Available at <http://arxiv.org/abs/math.CO/0302126>
- [30] D. Orden, G. Rote, F. Santos, B. Servatius, H. Servatius, W. Whiteley. *Reciprocal Duals of self-stressed Pseudo-Triangulations*. in preparation, 2003.
- [31] J. O'Rourke and I. Streinu. *Vertex-Edge Pseudo-Visibility Graphs: Characterization and Recognition*. Proc. ACM Symp. Comp. Geom, 1997, pp. 119-128.
- [32] J. Pach and G. Tóth. *Monotone drawings of planar graphs*. in: Algorithms and Computation ( P. Bose, P. Morin, eds.) LNCS 2518, Springer Verlag, 2002, 647-653
- [33] M. Pocchiola and G. Vegter. *Pseudo-Triangulations: Theory and Applications*, Proc. 12th ACM Symp. Comput. Geometry, Philadelphia, PE 1996, pp. 291-300.
- [34] M. Pocchiola and G. Vegter. *Topologically Sweeping Visibility Complexes via Pseudo-Triangulations*, Discrete Comput. Geom. 16 (1996), pp. 419-453.



- [35] D. Randall, G. Rote, F. Santos and J. Snoeyink *Counting triangulations and pseudo-triangulations of wheels*. Proc. 13th Canad. Conf. Comput. Geom., 2001, pp. 149-152.  
<http://compgeo.math.uwaterloo.ca/~cccg01/proceedings>
- [36] J. Richter-Gebert, *Realization spaces of polytopes*. Springer-Verlag, 1996.
- [37] G. Rote *Two solvable cases of the traveling salesman problem*. PhD Thesis, Technische Universität Graz, Institut für Mathematik, 1988.
- [38] G. Rote, F. Santos and I. Streinu *Expansive Motions and the Polytope of Pointed Pseudo Triangulations*, in B. Aronov, S. Basu, J. Pach and M. Sharir (eds.), *Discr. Comput. Geom. - The Goodman-Pollack Festschrift*, Springer Verlag, to appear 2002.
- [39] B. Speckmann and C. Tóth. *Allocating Vertex  $\pi$ -Guards in Simple Polygons via Pseudo-Triangulations*, to appear in Proc. 14th ACM-SIAM Symposium on Discrete Algorithms, 2003.
- [40] I. Streinu. *Stretchability of Star-like Pseudo-Visibility Graphs*. Proc. ACM Symp. Comput. Geometry , 1999, pp. 274-280.
- [41] I. Streinu. *A Combinatorial Approach to Planar Non-Colliding Robot Arm Motion Planning*, Proc. 41st Symp. Found. of Comp. science (FOCS), Redondo Beach, California, 2000, pp. 443-453.
- [42] I. Streinu. *Combinatorial Roadmaps in Configuration Spaces of Simple Planar Polygons*, to appear in S. Basu and L. Gonzalez-Vega (eds.), Proc. DIMACS Wkshp. *Algorithmic and Quantitative Aspects of Real Algebraic Geometry in Mathematics and Computer Science*.
- [43] W. Schnyder. *Embedding Planar Graphs on the Grid*, Proc. ACM-SIAM Symp. Discrete Algorithms (SODA), 1990, pp. 138-148.
- [44] R. E. Tarjan. *Data Structures and Network Algorithms* CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 44, SIAM 1983.
- [45] T.S. Tay and W. Whiteley. *Generating isostatic graphs*, Structural Topology 11, 1985, pp. 21-68.
- [46] M. Thorpe and Duxbury (eds.) *Rigidity Theory and Applications*, Kluwer Academic, 1999.
- [47] W. T. Tutte, *How to Draw a Graph*. Proc. London Math. Soc., 13(52), 743-768, 1963.
- [48] W. T. Tutte, *Convex Representations of Graphs*. Proc. London Math. Soc., 10(38), 304-320, 1960.
- [49] W. T. Tutte, *Connectivity in Graphs*, University of Toronto Press (1966).
- [50] W. T. Tutte, *A Theory of 3-connected graphs*, Kominsk. Nederl. Akad. van W. Proc. 64 (1961), 441-455. Also in *Selected Papers of W.T. Tutte*, Winnipeg, Canada, 1979.
- [51] W. Whiteley. *Rigidity of Molecular Structures: Generic and Geometric Analysis*, in *Rigidity Theory and Applications* (Thorpe and Duxbury, eds.), Kluwer Academic, 1999, pp. 21-46.
- [52] W. Whiteley. *Some Matroids from Discrete Applied Geometry*, Contemporary Mathematics, vol. 197, pp. 171-311, Amer. Math. Soc. 1996.
- [53] W. Whiteley. *Rigidity and Scene Analysis*, in *Handbook of Discrete and Computational Geometry*, (J.E. Goodman, J. O'Rourke, eds.), 1997, pp. 893-916.
- [54] H. Whitney, *2-isomorphic graphs*, Amer. J. Math. 55, 1933, 245-254.