

Graph Based Topological Analysis of Tessellated Surfaces

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ABSTRACT

In this paper a graph-based method is presented which not only characterizes topological classification of the tessellated surfaces but also simultaneously generates the substantial circles or generators on the surface. Canonical polygons cannot always be mapped back to the original surface in terms of the edges of the given triangles. Hence, instead of applying canonical transformation to the initial "word", an associated graph is constructed using the unique vertices in the word. The graph is then decomposed into its constituent loops and paths. Based on the type of edges present, the loops are classified into three types. The number of loops of each type in the graph is then used for counting the rank or genus and classification of the given surface as being **open or closed, orientable or non-orientable**. The image of the loops and paths on the original surface give the substantial circles and arcs on the surface respectively.

Categories and Subject Descriptors

I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling – *geometric algorithms, languages and systems*.
G.2.2 [Discrete Mathematics]: Graph Theory – *graph labeling, path and circuit problems*.

General Terms

Algorithms.

Keywords

Computational topology, simplicial complex, surface classification, characterization, cut lines.

1. INTRODUCTION

A tessellated surface as a representation is simple and versatile in the sense that it can represent surfaces of arbitrary topology. However, the topological nature of the surface is generally implicit which calls for its topological verification for modeling or topology identification for surface parameterization as in reverse engineering [1,10,13,16,17,26,29,33]. Topology identification makes geometric computations more efficient and meaningful [6,7,27].

Characterization of manifold surfaces involves identification of the topological invariants associated with a class of surface. This has been carried out in literature by using sphere-handle convention [5], Betti numbers and Euler characteristics [12,28],

Morse theory, Reeb graph and polygon schema [14], substantial arc/circles [5] etc. Euler characteristic relates the geometric features (vertex, edge, face) of a closed orientable surface with its global topological nature (genus, components etc.) and is applicable mostly for polyhedral solids. Morse theory relates local differential geometric attributes of a closed smooth surface and its global topological nature. Canonical polygon based characterization in some sense is more general in that it characterizes both orientable and non-orientable closed 2-manifolds [14]. Substantial arc/circles based characterization of surfaces [5] considers the set of non-homoeomorphic curves/surfaces on the surface whose counts are used to identify the invariants associated with a class of surface. This method is suitable for characterizing even open surfaces. Application of topology in geometric modeling has been mostly done for validating a model using Euler characteristics [12,28]. Computational topology based modeling schemes have been suggested in [2,11,19,20,21,32]. Utility of topological information in different fields including robotics, computer aided drug design, genetics, chemical synthesis, flow visualization etc. has been indicated in [3,6,25]. Hilga et al. [22] employed a Reeb graph based approach to 3D shape recognition problems.

Extraction of topological features of geometric models is computationally challenging. Vegter and Yap [34] presented an $O(n \log n)$ algorithm for computation of canonical polygonal schema of a tessellated surface, where n is the total number of vertices, edges and faces on the surface. Kartasheva [23] considered a cell complex in 3D consisting of tetrahedrons which makes it applicable only for solids (oriented, closed boundary surface). Also, the surface based algorithms also can be extended to determine the cutting sections that, in any case, cannot be guaranteed to be planar. Lazarus et al. [24] gave a $O(gn)$ algorithm for the closed orientable surfaces, where g is the genus of the surface. The method marches through all the triangles in the surface for construction of a spanning tree. The map of the generators on the surface is very complex and non-intuitive. A still improved $O(n)$ algorithm for construction of non-canonical but optimal sized polygonal schema and $O(n + k \log g)$ algorithm for detection null-homotopy cycles of size k is proposed in [8]. Apart from that one can find work concerning extraction and application of topological information based on Morse theory [4, 31], knot theory [15] etc. Application of topological dissection to homeomorphically deforming (flattening) the surface is reported in [10,33] which could in turn be used for global parameterization of surfaces in reverse engineering. In [9,30] it is theoretically proved that the problem of topological equivalence could be reduced to a problem of graph isomorphism for general (non-manifold) 2-complexes. However, such objects are sparingly encountered in practice; graph isomorphism test in itself is difficult and the necessity to consider all the triangles for constructing the graph is likely to make algorithms resource

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intensive. There is a need for simpler methods for topological analysis of surfaces for their wider applicability.

Although Euler characteristic, Betti numbers and Morse numbers all characterize compact 2-manifolds, it is the polygonal schema and substantial arc based characterizations which can actually construct the homeomorphism as a step towards global parameterization, texture mapping etc. Even though substantial arcs/circles provide a geometrically simple, direct and more general way of classifying surfaces, computationally identifying the best of these characteristic curves (minimum cut [10]) on the surface is difficult (NP-hard). On the other hand, although canonical words under polygonal schema give a simple method for classifying closed 2-manifolds and generation of a word polygon is conceptually straight forward, reducing it to canonical form is difficult because it involves a great deal of pattern search and symbolic manipulation of the word. Even more difficult is the process of mapping the canonical polygon on the given surface. In this paper we propose to unify the two concepts in a single characterization by using a graphs associated with any word [14], a non-canonical polygon, for identification of generators on the surface. Edges in the graph are coloured. *The number of circuits with different kinds of edges in the graph completely characterizes the given surface because their image on the given surface correspond to the substantial or boundary circles; the paths not participating in any circuit correspond to the substantial arcs on an open surface.*

2. CONVENTIONAL SURFACE CLASSIFICATION SCHEMES

Topological classification of surfaces requires identification of homeomorphism among surfaces. Two surfaces are considered topologically equivalent (homeomorphic) if there is a continuous transformation that is continuously invertible [5]. As given in [14], *any smooth, compact, connected surface (without boundary) is homeomorphic (and diffeomorphic) either to a sphere with g handles, where $g \geq 0$, or to a sphere with k cross-caps, where $k > 0$; a surface of the first type cannot be homeomorphic to one of the second type, nor can surfaces with different values of g or k be homeomorphic.* Also, since a handle on a non-orientable surface is equivalent to two cross-caps [14], a sphere with x handles and $y > 0$ cross-caps, is homeomorphic to a sphere with $2x+y$ cross-caps.

2.1 Substantial Arcs and Circles for Surface Characterization

A *substantial arc* is a cut line that does not separate the surface into unconnected domains. On a closed surface, the corresponding entity is called a *substantial circle* [5]. The maximum number of substantial arcs/circles that can be simultaneously present on a surface, without separating it, is called the rank of the surface. *Two surfaces are homeomorphic if and only if they have the same rank, same number of boundary circles and are either both orientable or both non-orientable.* When a surface is cut along the maximum number of substantial arcs/circles, it becomes equivalent to a *disc*. The bounding circle of this disc contains pairs of segments (mates), which are parts of the substantial arcs/circles. As one traverses the edges of the disc (indicated with an arrow in Fig.1(b)), if the end vertices of an edge and its mate is found to occur in the same order, the surface is non-orientable,

otherwise it is orientable. Rank of closed orientable surfaces is even; half rank is called the *genus* of the surface, which is same as the number of handles on a sphere.

2.2 Canonical Polygon Schema for Closed Surface Characterization

Every smooth compact 2-manifold (surface) can be triangulated with finitely many *triangles*. Providing a sense to these triangles and cutting the surface along these edges we get a set of triangles with oriented edges. Two triangles with a matching edge can be glued together to get a polygon (quadrilateral); other triangles in the set can be glued to the polygon in a recursive fashion by identifying one of the edges of a triangle with a boundary edge of the polygon. Marking the edges of the polygon with a label the polygon can be coded as a "*word*". Using a sequence of symbolic transformations involving grouping, reordering, renaming and removal of edges [13,34] any word can be reduced to one of the following three canonical forms.

- (1) $W = aa^{-1}$,
- (2) $W = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$,
- (3) $W = c_1 c_1 c_2 c_2 \dots c_k c_k$

where g and k are positive numbers.

Here, (1) corresponds to a *sphere*, (2) corresponds to a *sphere with g handles* and (3) corresponds to a *sphere with k cross-caps*. A word expressed in its canonical form is also referred to as *canonical polygon* in this paper.

2.3 Relationship Between Substantial Circles and Word Polygons

The canonical polygon corresponding to the canonical word consists of even number of edges with duplication of edge labels. These labels either occur adjacent to each other, when their traversal in the polygon is in the same sense, or alternate with another edge, when their traversal in the polygon is in the opposite sense. When these edges are glued together respecting the sense, *each mated edge in the word manifest as a substantial circle on the surface* with the specialty that all substantial circles are **mapped distinctly** on the surface but for one point, which is shared by all of them. However, a general set of substantial circles on a given surface has a wide variety of topological relationship, as discussed in section-3.2 below. On the other hand, dealing with a given tessellated surface, there may not be any vertex with enough number of edges incident on them, or such a vertex may not belong to the canonical polygon. A triangulated double torus requires at least one vertex with degree eight which may not be available (Fig.4(a)). This implies that computationally mapping the canonical polygon on the surface *using the edges of the given tessellation* is difficult and sometimes impossible.

3. GRAPH BASED CHARACTERIZATION

In the following a method is described which, by using a graph of the word, obtains the substantial arcs and circles in terms of the original edges of the triangles representing the surface S , from a word in its non-canonical form and classifies the surface using the features of the graph. The graph simplification algorithm minimizes the lengths of the cut lines in terms of the number of edges in them. The methods use only the connectivity information of the constituent triangles.

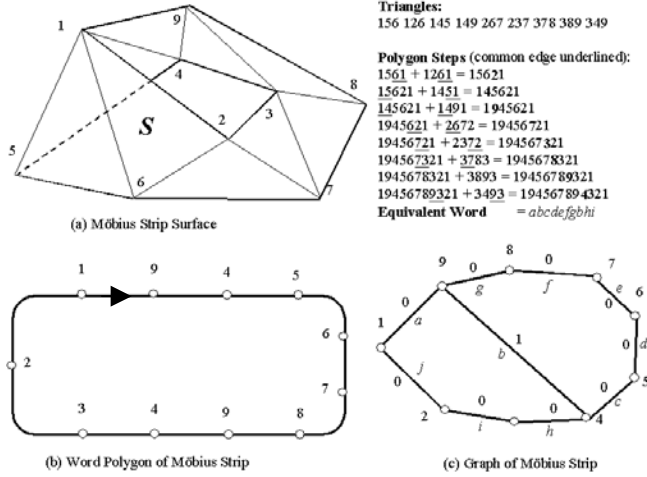


Figure 1: Graph construction for an open non-orientable surface, Möbius strip

3.1 Word and Associated Graph

Word Polygon: To construct the word, first the polygon is constructed as a sequence of vertices. Edges are identified using two consecutive vertices in a triangle or a polygon. We start from any triangle as the starting polygon and glue an edge of an adjacent triangle to it to get the updated polygon, which in effect adds a vertex in the polygon. At each step we eliminate redundant edges as identified by occurrence of the vertex pattern, ...x,y,x... in the polygon adjacent the newly added vertex to maintain the polygon of minimum length. When all the triangles have been added, we have the required polygon (Fig.1(b)).

Edge Classification: As we traverse the polygon (indicated with an arrow in Fig.1(b)), the first occurrence of an edge is considered positive. Subsequent occurrence of the edge is considered positive if the delimiting vertices appear in the same order, negative otherwise.

Graph of a Word: For a word polygon P , the graph, $G(V,E)$, is defined as follows: the set of unique vertices in P gives V and E is constructed by using pairs of vertices that are adjacent in P . Let, W be the word represented as the sequence of edges of P with appropriate classification. Thus G is completely defined by keeping only the unique edges in W . The edges in G are coloured by using a tag indicating the nature of its occurrence in W . This we call the **order of the edge**. Since we are *not* assuming the given surface to be a closed manifold, in W an edge may occur once or twice where the second occurrence may have either positive or negative classification. If an edge appears only once in W , its order is 0; if an edge appears twice with opposite sense, its order is -1 and if an edge appears twice with same sense, its order is 1. Fig.1(c) illustrates the graph construction for a triangulated Möbius strip.

3.2 Graph and the Characteristic Curves on a Surface

Mapping: There are three types of characteristic curves on a surface: substantial arcs, substantial circles and boundary circles. Since the graph has been created from the word without bringing it to the canonical form, the vertices and edges in the graph, G , can be easily identified on the original surface, S . It is easy to see in Fig.2 that homologous substantial circles on S , can be

identified with the circuits in G . In the case of open surfaces (Fig.2(h),(i)), the boundary circles also get mapped to the circuits of G and the homologous substantial arcs can be mapped to the *cut edges* [18] of G (Fig.2(i)) or to paths not participating any of the circuits identified (Fig.1(c)).

Relationship Among Characteristic Curves: When the maximum number of substantial circles/arcs or cut-lines are introduced on a surface, the surface is homologous to a disk, which in turn is homologous to the canonical polygon. However, there are many different ways the cut-lines can be introduced, each with a probably different topological relationship among the arcs of the circles. Fig. 2 shows some of the different ways the cut-lines can be introduced on a torus and a double-torus along with the graph of the cut-lines. It can be observed that, even though the graph has different number of vertices and edges (hence non-isomorphic), the *number of independent cycles in the graph is same* for the homologous objects and is equal to the

	Surface S with cut-lines	Graph of cut-lines
a		
b		
c		
d		
e		
f		
g		
h		
i		

Figure 2: Relationship among characteristic curves

number of substantial circles the surface can accommodate. When the surface has boundary, the boundary circles would also appear as cycles in the graph. Thus the type of cycle/circles is an independent information.

Graph Simplification: The purpose of graph simplification is to reduce the lengths of the circuits so as to get the shortest characteristic curves on the surface. The idea is, *if in a circuit, two consecutive edges belong to the same triangle, the graph can be modified* by introducing the third edge of the triangle and modifying the previous graph. As shown in Fig.3, the modification can be performed in two ways: by vertex deletion and edge swap. If the degree of the vertex common to the two earlier edges is two, the vertex and the two earlier edges can be deleted. In the second case, where the degree of the vertex common to the two earlier edges is more than two, the vertex cannot be deleted but one of the earlier edges can be deleted. This is called edge swap and it is performed if the operation increases the degree of some vertex or it facilitates a vertex deletion in the next step. The graph, G , in Fig.4(c) is simplified to the graph, G^* , in Fig. 4(d) with shortest substantial circles (in terms of number of edges rather than their geometric length) in the present case.

Classification of Fundamental Circuits: Since edges of S are shared at most twice, G is always planar [18] even though S is topologically non-planar [4]. Hence the number of fundamental circuits, in G is $e-v+1$, where v and e are the number of vertices and edges in G . These fundamental circuits are classified as follows: order 0 circuits are circuits with all edges of order 0; an order -1 circuit does not have order 1 edges but it has at least one order -1 edge; an order 1 circuit has at least one order 1 edge.

Extraction of Fundamental Circuits: The fundamental circuits of G can be extracted by constructing the minimum spanning tree

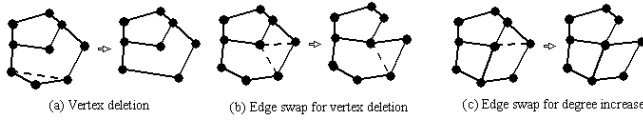


Figure 3. Graph simplification

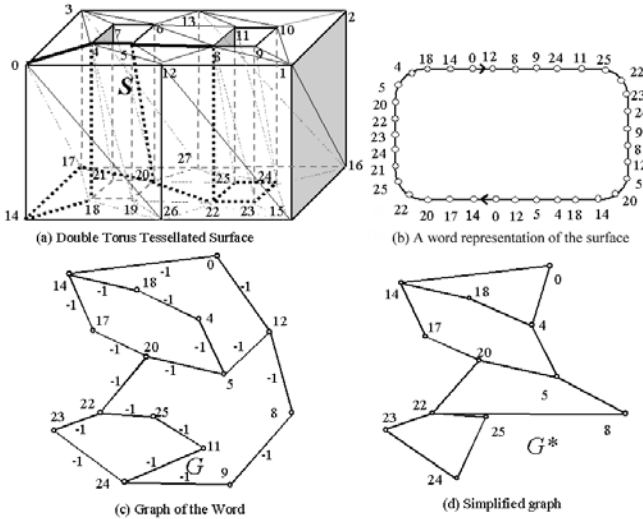


Figure 4: Graph construction and simplification for a double Torus

(MST) for G and then adding the remaining edges, RE . We needed an algorithm which extracts the circuits in the order of *0-order circuits*, *order -1 circuits* and then *order 1 circuits*. Since standard methods of cycle decomposition does not guarantee such specific decomposition we adopted a special traversal strategy.

For extraction of 0-order circuits, traversal is allowed only along 0-order edges in MST . A 0-order edge is picked up from RE and added in MST to start the process of circuit construction from its two ends. If at any point a 0-order edge is not available in the MST for traversal, we refer RE and more 0-order edges are picked up to complete the circuit. All but one of the 0-order edges used is replaced back in RE to prevent duplication of circuits extracted in subsequent stages. Used edges in RE are marked in order to prevent initiation of another 0-order circuit extraction using these edges. After all 0-order circuits are extracted; each edge in RE is used to extract a circuit in the usual manner and are classified based on the type of edges in the circuit.

3.3 Surface Characterization

Let, in G , C_0 , C_1 , C_2 be the number of circuits of orders 0, -1 and 1 respectively. When S is open, then union of C_0 , C_1 , C_2 may not give all the edges in G . The remaining edges in G are then grouped together using simple linear traversal to construct paths. Edges in a path cannot be order-0; they are either all of order 1 or all of order -1. Let L_1 and L_2 be the number of paths of order -1 and 1 respectively. Using C_0 , C_1 , C_2 , L_1 and L_2 surfaces can be classified as follows.

1. $C_0 = 0$ implies there is no boundary circle; hence the surface is closed and $L_1 = L_2 = 0$; otherwise the surface is **open**.
2. $C_2 = 0$ and $L_2 = 0$ imply there is no order 1 edge in G ; hence the surface is **orientable**.
3. $C_2 \neq 0$ or $L_2 \neq 0$ imply the surface is **non-orientable**.
4. For orientable surfaces ($C_2 = 0$ and $L_2 = 0$),
 - a. $C_0 = 0$ and $C_1 = 0$ imply the surface is homeomorphic to a **sphere**;
 - b. $C_0 \neq 0$ and $C_1 = 0$ imply the surface is homeomorphic to a **planar** surface with C_0 boundary circles and $L_1 = C_0 - 1$ substantial arcs; in particular, $C_1 = 1$ implies the surface is a **disc**;
 - c. $C_1 \neq 0$ then $C_1 = 2g$ where g is the genus of the surface; the surface is homeomorphic to a sphere with $C_1/2 = g$ handles, C_0 boundary circles and L_1 substantial arcs;
5. For non-orientable surfaces ($C_2 \neq 0$ or $L_2 \neq 0$), since adding a handle is equivalent to adding two cross-caps to a sphere [13], we consider $k = C_1 + C_2$; then
 - a. $C_0 = 1$, and either $k + L_2 = 1$, or $L_1 = 1$, imply the surface is a **Möbius strip**.
 - b. $C_0 = 0$ and $k \neq 0$ imply the surface is self-intersecting, closed and homeomorphic to a sphere with k cross-caps; in addition,
 - i. $k = 1$ implies the surface is a map of the projective plane;
 - ii. $k = 2$ implies the surface is a map of the Klein's bottle;
 - c. $C_0 \neq 0$ and $k \neq 0$ imply surface is open and homeomorphic to a sphere with k cross-caps and C_0 boundary circles and $L_1 + L_2$ substantial arcs;

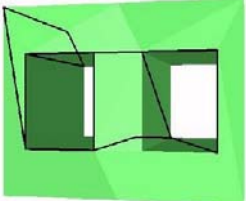
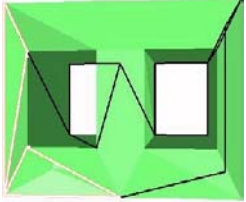
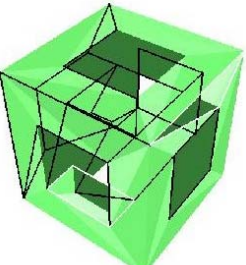
In all the cases above, the actual geometric description of the substantial arcs/circles and boundary circles is obtained by mapping C_0 , C_1 , C_2 circuits and L_1 L_2 paths on the surface. This, in turn, is easily done by using the coordinate of the points associated with the vertices in G .

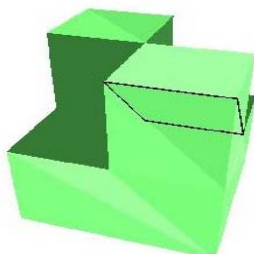
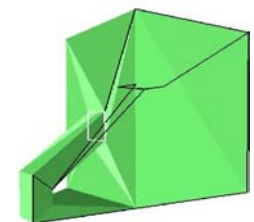
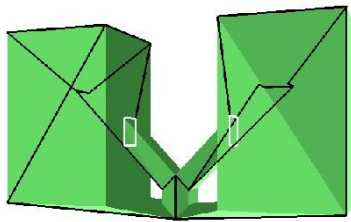
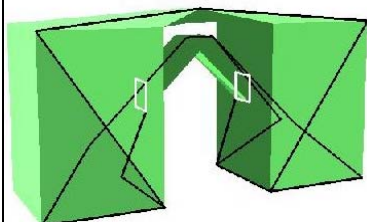
4. ILLUSTRATIVE EXAMPLES

The methodology described above has been implemented using C++ and the graphical output is generated using OpenGL. The results are shown in a tabulated form for the sake of brevity in Table 1. The dark lines in the graphics indicate the detected substantial arcs/circles and the lighter lines indicate the detected boundary circles. Two boundary circles sharing one point are interpreted as part of a single circle. Input data file is in the format of indexed triangles: first coordinates of the vertices are specified followed by a list of triplets of vertex indices. In the examples shown, data for open surfaces are generated by omitting one or more triangles from the input data file. Program can handle any single component manifold surface: closed, open, orientable, and non-orientable.

Discussion: The program has been tested for all standard surfaces such as sphere, Möbius strip, torus and their various combinations obtained by surgery. It is interesting to observe that in example 7, the surface obtained from two Klein's bottles can result in an orientable surface, which is a self-intersecting embedding of a Torus. For a torus containing about 15000 triangles, it took about 8 seconds for word formation and 1.5 seconds for classification and graph simplification on a Pentium-4 PC. Work is underway to extend the method for handling non-manifold surfaces.

Table 1 Illustrative Examples

	Object with characteristic curves	Curves count	Identification
1		$C_0=0$ $C_1=4$ $C_2=0$	Closed Orientable Genus 2 (Double Torus)
2		$C_0=2$ $C_1=4$ $C_2=0$	Open Orientable Two boundary Circles Genus 2
3		$C_0=2$ $C_1=10$ $C_2=0$	Open Orientable Sphere with 5 handles 2 boundary circles

4		$C_0=0$ $C_1=0$ $C_2=1$	Closed Non-orientable Sphere with 1 cross-cap (Projective plane)
5		$C_0=1$ $C_1=1$ $C_2=1$	Open Non-orientable Sphere with 2 cross-caps 1 boundary (Klein's Bottle)
6		$C_0=2$ $C_1=2$ $C_2=2$	Open Non-orientable Sphere with 4 cross-caps, 2 boundary Circles. (2 Klein's Bottles Joined along 1 circle)
7		$C_0=2$ $C_1=2$ $C_2=0$	Open Orientable Sphere with one Handle (two Klein's Bottles Joined along Two circles)

5. CONCLUSIONS

The paper presented a novel graph based methodology for topological analysis of any triangulated 2-manifold. The method constructs a graph out of the non-canonical word polygon obtained by sequential gluing of the triangles. The edges of the graph are classified according to their occurrence in the word and the circuits in the graph are classified based on the types of edges they contain. These circuits along with the paths that do not participate in the circuits are then used to characterize the surface. Since the edges in the word are only used for characterization, which is orders of magnitude less compared to the total number of triangle and edges in the surface, the procedure is efficient. Since the required graph decomposition can be done efficiently, the implementation is simple. Most importantly the validity of the method can be easily verified geometrically because it unifies the two traditional concepts of surface characterization based on canonical words and substantial arc/circles. The efficacy of the method has been demonstrated with a number of illustrative examples.

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