ON BOUNDS ON THE NUMBER OF STEPS TO COMPUTE FUNCTIONS†

by

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Abstract

Let f be a partial recursive function defined in terms of other functions $g_1, \ldots, g_n$ such that f converges if and only if some well defined assertions about the convergency of $g_1, \ldots, g_n$ hold: then we can find a total function (depending on the number of steps required to compute $g_1, \ldots, g_n$) that bounds the step counting function of f almost everywhere f is defined. It is also shown that, in that case, to compute the bound to the step counting function is not much harder than computing the step counting function itself.

1. Introduction

Notation [1]: in what follows we assume the reader to be familiar with the notion of standard indexing $\lambda i \phi_i$ of all partial recursive functions (p.r.f.). Given any function ϕ_i, we call a p.r.f. Φ_i the step counting function (or the complexity measure) of ϕ_i if:

1) $\phi_i(x)$ converges iff $\Phi_i(x)$ converges
2) the relation $\Phi_i(x)=y$ is recursive in i,x,y.

Besides, when we write $(\overset{\infty}{\forall} x)(f(x) \leq g(x))$, we mean that $f(x) \leq g(x)$ for all but a finite number of values of the argument x.

Given a partial recursive function f defined in terms of partial recursive functions $g_1, g_2, \ldots, g_n$, it is sometimes possible to find a (total) recursive function r that bounds the number of steps required to compute f in terms of the number of steps required to compute $g_1, \ldots, g_n$, almost everywhere that f is defined. For example we can prove the following facts:

Fact 1.1 Let σ be a recursive function of two variables with the property that for all i,j,x, $\phi_{\sigma(i,j)}(x)$ converges iff $\phi_i(x)$ and $\phi_j(x)$ converge. Then there exists a recursive function r such that, for all i and j and almost all x:

$$\phi_i(x) \text{ defined } \& \ \phi_j(x) \text{ defined} \rightarrow \Phi_{\sigma(i,j)}(x) \leq r(x, \max\{\Phi_i(x), \Phi_j(x)\})$$

Proof. Let us define:

$$t(x,i,j,u) = \begin{cases} \Phi_{\sigma(i,j)}(x) & \text{if } \Phi_i(x) \leq u \text{ and } \Phi_j(x) \leq u \\ 0 & \text{otherwise} \end{cases}$$

$$r(x,u) = \max\{t(x,i,j,u) \mid i,j \leq x\}$$

r is a recursive function since t is a recursive function. Furthermore, $\phi_i(x)$ defined & $\phi_j(x)$ defined $\rightarrow \max\{\Phi_i(x), \Phi_j(x)\}$ defined $\rightarrow t(x,i,j,\max\{\Phi_i(x),\Phi_j(x)\}) = \Phi_{\sigma(i,j)}(x) \rightarrow r(x, \max\{\Phi_i(x), \Phi_j(x)\}) \geq \Phi_{\sigma(i,j)}(x)$, provided $x \geq \max\{i,j\}$. Hence our result follows for all $x \geq \max\{i,j\}$.

Fact 1.2 Let σ be a recursive function of one variable with the property that for all i,x and y, $\phi_{\sigma(i)}(x,y)$ converges iff $\phi_i(0,y), \phi_i(1,y), \ldots, \phi_i(x,y)$ converge. Then there exists a recursive function r such that for all i and almost all x

$$(\forall z \leq x)(\phi_i(z,y) \text{defined}) \rightarrow \Phi_{\sigma(i)}(x,y) \leq r(x, \max\{\Phi_i(z,y) \mid z \leq x\})$$

Proof. Let us define:

$$t(x,y,i,u) = \begin{cases} \Phi_{\sigma(i)}(x,y) & \text{if} (\forall z \leq x)(\Phi_i(z,y) \leq u) \\ 0 & \text{otherwise} \end{cases}$$

$$r(x,y,u) = \max\{t(x,y,i,u) \mid i \leq x,y\}$$

Then $(\forall z \leq x)(\phi_i(z,y)\text{defined}) \rightarrow \Phi_{\sigma(i)}(x,y) \leq r(x,y,\max\{\Phi_i(z,y) \mid z \leq x\})$ for all i and all $x,y \geq i$

Fact 1.3. Let σ be a recursive function of two variables with the property that for all i,k and all x $\phi_{\sigma(i,k)}(x)$ is defined iff the number of steps required by the computation of $\phi_i(x)$ is greater than k. Then there exists a recursive function r such that for all i and k and almost all x

$$\Phi_i(x) > k \rightarrow \Phi_{\sigma(i,k)}(x) \leq r(x)$$

Proof. Let us define:

$$t(x,i,k) = \begin{cases} \Phi_{\sigma(i,k)}(x) & \text{if } \Phi_i(x) > k \\ 0 & \text{otherwise} \end{cases}$$

$$r(x) = \max\{t(x,i,k) \mid i,k \leq x\}$$

Then $\Phi_i(x) > k \rightarrow \Phi_{\sigma(i,k)}(x) \leq r(x)$ for all i and k and all $x \geq \max\{i,k\}$

Instances and proofs of these types of facts occur frequently in the literature. For example we have:

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Blum[2]: Extended Recursion Theorem

$\exists \eta, r \in \mathcal{R}_2$, r monotonically increasing in its second variable, such that $\forall i, e \in N$

1) $\phi_{\eta(i,e)} = \phi_{\phi_e(i,\eta(i,e))}$

2) $(\forall x)(\phi_{\eta(i,e)}(x)$ defined $\rightarrow \Phi_{\eta(i,e)}(x) \geq \Phi_{\phi_e(i,\eta(i,e))}(x))$

3) $(\overset{\infty}{\forall} x)(\phi_{\eta(i,e)}(x)$ defined $\rightarrow \Phi_{\eta(i,e)}(x) \leq r(x, \Phi_{\phi_e(i,\eta(i,e))}(x)))$

where 3) holds for all $x \geq \max\{i, e, \Phi_e(i, \eta(i,e))\}$.

Borodin[3]: Proposition 5

Let $\sigma \in \mathcal{R}_n$ be such that, given a set $\underline{i} = \{i_1, i_2, \ldots, i_n\}$ of indices, for all $i_1, \ldots, i_n \in N$

$(\forall i \in \underline{i})(\phi_i(x)$ is defined$) \rightarrow \phi_{\sigma(i_1,\ldots,i_n)}(x)$ is defined.

Then there exists a total function $r(x, u_1, \ldots, u_n)$ such that for all $x \geq \max\{i_1, \ldots, i_n\}$

$\Phi_{\sigma(i_1,\ldots,i_n)}(x) \leq r(x, \Phi_{i_1}(x), \ldots, \Phi_{i_n}(x))$.

2. A General Framework

The tight similarity among the facts proved in the preceding paragraph, as much as the similarity among the proofs themselves, suggests the construction of a general framework in which to prove the existence of a recursive function that bounds almost everywhere the number of steps required to compute functions.

In fact, we are interested in producing a class $\mathcal{A}$ of "assertions about partial recursive functions," and then showing that, for every assertion A of the class $\mathcal{A}$, the following is true: "let σ be a recursive function such that, for every tuple of indices $\underline{i}$ (of a given enumeration of partial recursive functions), and every tuple of variables $\underline{x}$ $\phi_{\sigma(\underline{i})}(\underline{x})$ is defined iff $A[\underline{i},\underline{x}]$ holds. Then we can find a recursive function r and a partial recursive function u_Φ such that for all values of $\underline{i}$ and all but a finite number of values of $\underline{x}$, if $A[\underline{i},\underline{x}]$ holds then $u_\Phi[\underline{i},\underline{x}]$ is defined and

$$\Phi_{\sigma(\underline{i})}(\underline{x}) \leq r(\underline{x}, u_\Phi[\underline{i},\underline{x}])".$$

In this rather informal statement of the problem the square brackets are to specify that $\underline{i}$ and $\underline{x}$ are not arguments of A and u_Φ but both A and u_Φ "depend in some way" on functions (or multiply composed functions) whose indices and/or arguments range in $\underline{i}$ and/or $\underline{x}$.

From the preceding paragraph we can take some example of assertions A and functions r and u_Φ.

a) from Fact 1.1: $\underline{i} = \{i,j\}$, $\underline{x} = \{x\}$
$A[\underline{i},\underline{x}]$ = "$\phi_i(x)$ and $\phi_j(x)$ defined".

r is the function defined in the proof of fact 1.1

$u_\Phi[\underline{i},\underline{x}] = \max\{\Phi_i(x), \Phi_j(x)\}$

b) from Fact 1.3: $\underline{i} = \{i,k\}$, $\underline{x} = \{x\}$
$A[\underline{i},\underline{x}]$ = "$\Phi_i(x) > k$"

r is as defined in fact 1.3

$u_\Phi[\underline{i},\underline{x}] = 0$ (r depends only on x so we could have chosen any other value for u_Φ)

c) from Blum's Extended Recursion Theorem $\underline{i} = \{i,e\}$, $\underline{x} = \{x\}$
$A[\underline{i},\underline{x}]$ = "$\phi_{\phi_e(i,\eta(i,e))}$defined"

r is as defined in E.R.T.

$u_\Phi[i,x] = \Phi_{\phi_e(i,\eta(i,e))}(x)$

Before giving a formal definition of the problem and considering the class for which we will prove the theorem, we need to introduce a notation to simplify the symbol manipulation job, and some definitions for objects that we frequently deal with.

a. Notation: we will generally use a prefix notation so that we can distinguish, by position, between the operator and an unspecified number of operands. The basic operators that will appear in our expressions will be:
ϕ: $(\phi a\ b\ c)$ is interpreted as $\phi_a(b,c)$: a-th function in the enumeration, computed with b and c as arguments;
D: (D a b c) is interpreted as the predicate "$\phi_a(b,c)$ is defined"; Φ: $(\Phi\ a\ b\ c)$ is interpreted as $\Phi_a(b,c)$: step counting function for $\phi_a(b,c)$.
Other operators such as LE (less or equal), $\wedge$ (and), V(or), etc. will be used and their interpretation is clear.

b. Definitions: the terms out of which we define assertions will represent functions and step counting functions whose indices and arguments are variables or functions. There are two kinds of terms, viz. <term>, representing functions, and <s.c. term> representing step counting functions. Formally:

<i-var>::= i,j,k,...... (infinite set)

<x-var>::= x,y,z,...... (infinite set)

<var>::= <i-var>|<x-var>

<term>::= (ϕ<arg-string>)
Example: $(\phi(\phi i(\phi x j))z)$, i.e. $\phi_{\phi_i(\phi_x(j))}(z)$

<arg-string>::= <arg><arg>|<arg><arg-string>

<arg>::= <var>|<term>

<s.c. term>::= (Φ<arg-string>)
Example: $(\Phi x(\phi i\ j\ y)z)$, i.e. $\Phi_x(\phi_i(j,y),z)$

We will use the metasymbols $\underline{i}$ and $\underline{x}$ to represent finite (possibly empty) set of i-variables and x-variables. The metasymbols t and

$\underline{t}$ will represent terms and finite (possibly empty) set of terms, respectively.

Let $t \epsilon \underline{t}$: We say t is a <u>term in $\underline{i}$ and $\underline{x}$</u> and we write $(t[\underline{i},\underline{x}])$ iff any variable occurring in t belongs to $\underline{i}$ or to $\underline{x}$. In the same case if $\underline{T}$ is a finite set of s.c. terms and $T \epsilon \underline{T}$ we say T is a <u>s.c. term in $\underline{i}$ and $\underline{x}$</u> and we write $T[\underline{i},\underline{x}]$.

Our goal is to define <u>a class of assertions</u> as a formal language, $\mathcal{A}$. If $A \epsilon \mathcal{A}$ is an assertion we say A is an <u>assertion about $\underline{i}$ and $\underline{x}$</u>, written $A[\underline{i},\underline{x}]$, iff any variable occurring free in A belongs to $\underline{i}$ or to $\underline{x}$.

An assertion $A[\underline{i},\underline{x}]$ will be interpreted as a statement about the convergency of functions (or about the values of step counting functions) whose indices and/or arguments range in i and/or x. The truth-valuation will be such that $A[\underline{i},\underline{x}]$ is true if and only if the corresponding statement holds.

Now we are able to give a more formal statement of

<u>Aim 2.1</u>: we want to define classes of assertions such that we can prove the following:

"Let $A[\underline{i},\underline{x}] \epsilon \mathcal{A}$ be an assertion about $\underline{i}$ and $\underline{x}$ where $\underline{i}$ is an n-tuple of i-variables and $\underline{x}$ an m-tuple of x-variables;
let $\sigma \epsilon \mathcal{R}_n$ be a total function such that, for any $\underline{i}$ (i.e. for any n-tuple of values of variables of $\underline{i}$) and any $\underline{x}$

$$\phi_{\sigma(\underline{i})}(\underline{x}) \text{ is defined} \leftrightarrow A[\underline{i},\underline{x}]$$

then we can find a total recursive function $r \epsilon \mathcal{R}_{m+1}$ and a partial recursive function u_Φ such that

$$(\forall \underline{i})(\overset{\infty}{\forall} \underline{x})(A[\underline{i},\underline{x}] \to \Phi_{\sigma(\underline{i})}(\underline{x}) \leq r(\underline{x},u_\Phi(\underline{i},\underline{x})))$$"

3. Bounds Relative to Assertions in Disjunctive Normal Form.

The class of assertions we want to consider are assertions about the convergency of terms, closed under conjunction, disjunction and bounded quantifiers.

The associative and distributive properties of conjunctions and disjunctions allow us to put any such assertion in a sort of disjunctive normal form so that we can give an easy formal definition of $\mathcal{A}$ (see also definitions on page 2).

```
𝒜 = {<assertions>}        where

<assertion> :: = <D - pred>|<conj>|<b-univ>|
                 <disj>|<b-exist>

<D-pred> :: = (D<arg-string>)

<conj> :: = (∧<D-string>)

<D-string> :: = <D-pred><D-pred>|<D-pred>
                <D-string>

<b-univ> :: = (∀ <var><bound><body> )

<body> :: = <D-pred>|<conj>|<b-univ>

<bound> :: = <var>|<integer>

<disj> :: = (V<body-string> )

<body-string> :: = <body><body>|<body>
                   <body-string>

<b-exist> :: = (∃ <var><bound><assertion> )
```

The interpretation of an assertion of $\mathcal{A}$ is immediately derived as soon as we interpret a term as a function and a <D-pred> as the predicate asserting that a function is defined, and the usual interpretation of statements in Statement Calculus is applied.

Let us give some example of assertions of $\mathcal{A}$ whose interpretation is given in the first paragraph.

From Fact 1.1: $(\wedge\ (D\ i\ x)\ (D\ j\ x))$

From Fact 1.2: $(\forall\ z\ x\ (D\ i\ z\ y\ ))$

From Blum's E.R.T.: $(D(\phi\ e\ i\ (\phi\ j\ i\ e\ ))\ x)$
where j is such that $\eta = \phi_j$

From Borodin's Prop. 5: $(\forall\ j\ n\ (D\ (\phi\ e\ j)\ x))$
where e is such that $\phi_e(0) = i_0,\ldots,$
$\phi_e(n) = i_n$

Before proving that $\mathcal{A}$ is a class of assertions that satisfies Aim 2.1, we introduce another language $\mathcal{A}^\Phi$ whose words have to be interpreted as statements about step counting functions and their relation with the value of one free variable (u).

```
𝒜^Φ = {<Φ -assertions>}   where

<Φ -assertion> :: = <Φ -relation>|<Φ-conj>|
    <Φ-b-univ>|<Φ -disj>| |<Φ-b-exist>

<Φ -relation> :: = (LE <s.c.term> u)

<Φ -conj> :: = ( ∧ <Φ -string> )

<Φ -string> :: = <Φ -relation> <Φ -relation>|
                 <Φ -relation><Φ -string>

<Φ-b-univ> :: = (∀ <var><bound><Φ -body> )

<Φ -body> :: = <Φ -relation>|<Φ -conj>|<Φ-b-univ>

<Φ -disj> :: = ( V <Φ -body-string> )

<Φ-body-string> :: = <Φ -body><Φ -body>|<Φ-body>
                     <Φ -body-string>

<Φ-b-exist> :: = (∃<var><bound><Φ-assertion> )
```

Now we are ready to prove:

<u>Lemma 3.1</u>: Let $A[\underline{i},\underline{x}] \epsilon \mathcal{A}$ be an assertion about $\underline{i}$ and $\underline{x}$; we have an effective procedure to generate a Φ -assertion $P[\underline{i},\underline{x},u]$ such that

$$(\forall\ \underline{i},\underline{x})\ (A[\underline{i},\underline{x}] \to (\exists u)\ P\ [\underline{i},\underline{x},u])$$

<u>Proof</u>: to generate P from A we have to perform two steps of symbol manipulation;

Step 1:

1. Look for the first <D-pred> in A, from the left to the right, and call it d;
2. call the first <arg>, of the <arg-string> d, a;

3. if a is a <var> go to 6, otherwise (it is a <term>) make a copy of it, where ϕ is replaced by D, and call it d';

4. if d is in the scope of a symbol $\wedge$ go to 5, otherwise substitute d in A with ($\wedge$ d) ;

5. put d' immediately after d ;

6. if a was not the last <arg> in the <arg-string> call the next <arg> a and go to 3, otherwise go to 7;

7. if there is, still one <D-pred> to the left of d, call it d and go to 2; otherwise you are through step 1. Call the result A' and go to step 2.

Step 2:

1. Look for the first <D-pred> in A' from the left to the right, and call it d;

2. Let <arg-string> be such that d = (D<arg-string>): substitute d in A' with

   (LE (Φ <arg-string>) u)

3. look for the next <D-pred> in A', to the left of d: if there is one, call it d and go to 2; otherwise you are through step 2. Call the result $P[\underline{i},\underline{x},u]$.

Through the first step we make explicit that if a function is defined, then its index and arguments are defined. Then, if the assertion A was claiming that, for example, the functions t_1 and t_2 are defined, through step 2 we have that P is claiming that the number of steps to compute those functions is less than or equal to u: this is clearly true as soon as u is equal to the maximum of the number of steps required to compute t_1 and required to compute t_2.

Now we will prove that, in general, for every $\underline{i}$ and $\underline{x}$ such that $A[\underline{i},\underline{x}]$ holds, there exists a value of u such that $P[\underline{i},\underline{x},u]$ holds. For this we will induct on the structure of $A[\underline{i},\underline{x}]$. Since step 1 converts a word in $\mathcal{A}$ in another word in $\mathcal{A}$, the actual translation between $\mathcal{A}$ and $\mathcal{A}^\Phi$ is performed by step 2. Let $\underline{i}$ and $\underline{x}$ be such that $A[\underline{i},\underline{x}]$ holds.

- If A is a<D-pred> then there is a value of u such that A→P. In fact in that case A' is a <D-pred> or is the conjunction of two or more <D-pred>. So, let $\mathcal{U}_\Phi$ be the set of all step counting terms obtained from every <D-pred> by changing D with Φ. Let us take $\overline{u} = \max \mathcal{U}_\Phi$(*). Since $A[\underline{i},\underline{x}]$ holds max $\mathcal{U}_\Phi$ is defined and $\overline{u} = \max \mathcal{U}_\Phi$ is a value of u that satisfies every <Φ-relation> in the assertion $P[\underline{i},\underline{x},u]$.

- If A is a conjunction of <D-pred> , suppose we have found $\overline{u}_1,\dots,\overline{u}_n$ such that for each <D-pred> p_i if p_i holds then the corresponding conjunction of <Φ-relation> holds with $u = \overline{u}_i$: then we choose $\overline{u} = \max \{\overline{u}_i\}$ we have that $P[\underline{i},\underline{x},\overline{u}]$ holds.
- If A is an assertion with bounded universal quantifier and <body>(**) is such that for each value i of the running variable, from 0 to the bound, if <body> holds then <Φ-body> holds with $u = \overline{u}_i$, then if we choose $\overline{u} = \max \{\overline{u}_i\}$, we have that $P[\underline{i},\underline{x},\overline{u}]$ holds.
- If A is a disjuction and if for each <body> of <body-string> the inductive hypothesis holds with $u = \overline{u}_i$, then if we choose $\overline{u} = \min \{\overline{u}_i\}$, we have that $P[i,x,\overline{u}]$ holds.
- If A is an assertion with bounded existential quantifier and the body (that in this case may be a D-predicate or a conjunction or a <b-univ> or a disjunction or even a <b-exist>)(**), is such that, for each value i of the running variable, from 0 to the bound, if it holds then the body of the <Φ-assertion> holds with $\overline{u} = \overline{u}_i$, then if we choose $\overline{u} = \min \{u_i\}$ we have that $P[i,x,\overline{u}]$ holds.

<u>Lemma 3.2</u>: Let $A[\underline{i},\underline{x}] \in \mathcal{A}$ and $P[\underline{i},\underline{x},u] \in \mathcal{A}^\Phi$ be as in Lm 3.1. We can define a partial function u_Φ with the property that, for every $\underline{i}$ and $\underline{x}$ such that $A[\underline{i},\underline{x}]$ holds, $u_\Phi(\underline{i},\underline{x})$ is defined and its value $\overline{u}$ is such that $P[\underline{i},\underline{x},\overline{u}]$ holds.

<u>Proof</u>: we have simply to reverse the inductive argument that we have used in the existence proof and we get a recursive procedure to produce a function u_Φ, whose arguments are step counting functions and whose value satisfies the theorem.

Let us call $\mathcal{F}$ the procedure that, when applied to the assertion $A' \in \mathcal{A}$ gives the function u_Φ corresponding to A'.

- If A' is of the type ($\exists$ <var><bound><assertion>)
  $u_\Phi = \mathcal{F}A' = \min \{ \mathcal{F}$ <assertion>$|$<var> $\le$ <bound>$\}$
- If A' is of the type (V <body>,...,<body>)
  $\mathcal{F}A' = \min \{\mathcal{F}$ <body>$,\dots,$ $\mathcal{F}$<body>$\}$
- If A' is of the type ($\forall$ <var><bound><body>)
  $\mathcal{F}A' = \max \{\mathcal{F}$ <body>$|$<var> $\le$ <bound>$\}$
- If A' is of the type ($\wedge$ <D-pred> ,..., <D-pred>)
  $\mathcal{F}A' = \max \{\mathcal{F}$ <D-pred>$,\dots,$ $\mathcal{F}$<D-pred>$\}$
- If A' is of the type (D <arg-string>)
  $\mathcal{F}A'$ = (Φ<arg-string>)

By the same argument of the existence proof, it is clear that $\mathcal{F}A' = u_\Phi(\underline{i},\underline{x})$ is a function whose arguments are step counting terms on $\underline{i}$ and $\underline{x}$ and such that for every $\underline{i}$ and $\underline{x}$ u_Φ is defined if $A[\underline{i},\underline{x}]$ holds and its value $\overline{u}$ is such that $P[\underline{i},\underline{x},\overline{u}]$ holds.

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*For every i and x, max(min) $\mathcal{U}_\Phi$consists in computing the steps counting functions corresponding to the step counting terms in $\mathcal{U}_\Phi$and taking their maximum (minimum). If $\mathcal{U}_\Phi = \phi$, max $\mathcal{U}_\Phi$ = min $\mathcal{U}_\Phi$ = 0.

**If there are multiple bounded quantifiers of the same type then induct on the number of quantifiers.

Examples:

1) From Fact 1.2:

$A[\underline{i},\underline{x}] = (\forall\ z\ x\ (D\ i\ z\ y))$

$\underline{i} = \{i\}\ ,\ \underline{x} = \{x,y\}$

$A'[\underline{i},\underline{x}] = (\forall\ z\ x\ (D\ i\ z\ y))$

$P[\underline{i},\underline{x},u] = (\forall\ z\ x\ (LE\ (\Phi\ i\ z\ y)\ u))$

$u_\Phi(\underline{i},\underline{x}) = \max\ \{(\Phi\ i\ z\ y) | z \leq x\}$

2) Let us suppose:

$A[\underline{i},\underline{x}] = (V(D(\phi\ i\ x)x)\ (D\ j\ y))$

$\underline{i} = \{i,j\}\ ,\ \underline{x} = \{x,y\}$

$A'[\underline{i},\underline{x}] = (V(\wedge(D(\phi\ i\ x)x)\ (D\ i\ x))\ (D\ j\ y))$

$P[\underline{i},\underline{x},u] = (V(\wedge\ (LE(\Phi(\phi\ i\ x)x)u)(LE(\Phi\ i\ x)\ u))\ (LE(\Phi\ j\ y)u))$

$u_\Phi(\underline{i},\underline{x}) = \min\{\max\{(\Phi(\phi i\ x)x),\ (\Phi\ i\ x)\}\ ,\ (\Phi\ j\ y)\}$

Now we can prove

Theorem 3.1: Let $A[\underline{i},\underline{x}]$ be an assertion of the class $\mathcal{A}$ where $\underline{i}$ is an n-tuple of i-variables and $\underline{x}$ an m-tuple of x-variables; let σ be a recursive function in $\mathcal{R}_n$ such that

$$\phi_{\sigma(\underline{i})}(\underline{x}) \text{ is defined} \leftrightarrow A[\underline{i},\underline{x}]$$

then there is a total function $r(\underline{x},u)\varepsilon_\infty \mathcal{R}_{m+1}$ and a partial function u_Φ such that $(\forall\underline{i})\ (\forall\underline{x})(A[\underline{i},\underline{x}] \rightarrow \Phi_{\sigma(\underline{i})}(\underline{x}) \leq r(\underline{x},u_\Phi(\underline{i},\underline{x})))$

Proof. Let $P[\underline{i},\underline{x},u]$ be as in Lm 3.1 and 3.2. Let us define

$$t\ (\underline{i},\underline{x},u) = \begin{cases} \Phi_{\sigma(\underline{i})}(\underline{x}) & \text{if } P[\underline{i},\underline{x},u] \\ 0 & \text{otherwise} \end{cases}$$

we claim that

$$r(\underline{x},u) = \max\ \{t(\underline{i},\underline{x},u) | \underline{i} \leq \underline{x}\ ^{(+)}\}$$

satisfies the theorem. In fact $r(\underline{x},u)$ is total since $P[\underline{i},\underline{x},u]$ is recursive and furthermore, by Lemma 3.2, $A[\underline{i},\underline{x}] \rightarrow u_\Phi$ defined $\rightarrow t(\underline{i},\underline{x},u_\Phi(\underline{i},\underline{x})) = \Phi_{\sigma(\underline{i})}(\underline{x}) \rightarrow r(\underline{x},u_\Phi(\underline{i},\underline{x})) \geq \Phi_{\sigma(\underline{i})}(\underline{x})$ provided $\underline{x} \geq \underline{i}$.

4. Validity of the Theorem.

To clarify the meaning of theorem 3.1, let us suppose we have a function f defined in terms of other functions $g_1 \ldots g_n$ in such a way that f converges if and only if a certain statement A about the convergency of $g_1 \ldots g_n$ holds. If this statement is a certain combination of disjunctions and conjunctions (even with bounded quantifiers) of such elementary statements as "g_1 converges", ..., "g_n converges", we know we can write it in a normal form so that it is expressed by an assertion of $\mathcal{A}$. Then theorem 3.1 tells us that:

1) the number of steps required to compute f is bounded (almost everywhere f is defined) by a certain function r of the number of steps required to compute $g_1,\ldots,g_n$,
2) the function r is total,
3) there is an effective way to know how to compute r, given the statement A.

The class of assertions $\mathcal{A}$ is not so wide to include, for example, all the Facts of § 1: in Fact 1.3 we have an assertion that is not representable by words of $\mathcal{A}$. To widen the class in order to include more general types of assertions may be an interesting exercise but we tend to consider $\mathcal{A}$ large enough to be meaningful and to allow us to discuss the validity of the main result.

4.1 Reduction of the number of arguments of r: in theorem 3.1 the step counting function $\Phi_{\sigma(i)}(x)$ is bounded by $r(x,\overline{u})$ where the value $\overline{u}$ depends on the number of steps required to compute the functions that occur in A. In other words it depends on all the s.c. terms whose corresponding term occurs in A. Since in some term (like $(\phi\ j\ k)$ where $j,k\varepsilon\underline{i}$, for example) sometimes only i-variables occur, once we have fixed the values of all $i\varepsilon\underline{i}$, these terms behave like constants and so, in some cases, we can get rid of them while computing r. This happens, for example, for the subclass of $\mathcal{A}$ where only conjunctions and bounded universal quantifier occur.
Let us call this class $\mathcal{A}_o$.

We can prove:

Theorem 4.1. Let $A[\underline{i},\underline{x}]$ be an assertion of the class $\mathcal{A}_o$; let σ be a recursive function in $\mathcal{R}_n$ such that

$$\phi_{\sigma(\underline{i})}(\underline{x}) \text{ is defined} \leftrightarrow A[\underline{i},\underline{x}]$$

then there are a total function $r(\underline{x},u)\varepsilon\ \mathcal{R}_{m+1}$ and two partial functions u_Φ, v_Φ such that

$$(\forall\ \underline{i})\ \ (\forall\ \underline{x}{\geq}\underline{i},v_\Phi\ (\underline{i}))$$
$$(A[\underline{i},\underline{x}] \rightarrow \Phi_{\sigma(\underline{i})}(\underline{x}) \leq r(\underline{x},u_\Phi(\underline{i},\underline{x})))$$

Proof. Let us define

<Φ -relation> :: = (LE(Φ <arg-string>)u) | (LE(Φ <arg-string>)v).

Then let us execute step 1 as in Lemma 3.1 and change step 2 in the following way:

Step 2:

1. as in Lemma 3.1
2. Let <arg-string> be such that d = (D <arg-string>) : if some x-variables occur among the variables or in the terms of <arg-string>, then substitute d in A' with (LE(Φ <arg-string>) u), otherwise, substitute d in A' with (LE(Φ <arg-string>) v)

(+) $\underline{i} \leq \underline{x} \leftrightarrow (\forall\ i\varepsilon\underline{i})\ (\forall\ x\varepsilon\underline{x})\ (i \leq x)$

3. as in Lemma 3.1

Call the result $P[\underline{i},\underline{x},u,v]$

Now we can prove, that for every $\underline{i}$ and $\underline{x}$ such that $A[\underline{i},\underline{x}]$ holds there exist values of u and v such that $P[\underline{i},\underline{x},u,v]$ holds.

- If A is a <D-pred> or the conjunction of two or more <D-pred>, then P is a <Φ -relation> or the conjunction of two or more <Φ -relation>. Let us put their <s.c. term> in two classes according to the alternatives of instruction 1 of step 2. In other words, let us call U_Φ the set of all the step counting terms that we compare with u, and V_Φ the set of all the step counting terms that we compare with v. Since A holds, max U_Φ and max V_Φ are defined and $\overline{u} = \max U_\Phi$, $\overline{v} = \max V_\Phi$ are values of u and v satisfying P.
- If A is an assertion with bounded universal quantifier and the body is such that for each value i of the running variable from 0 to the bound, if <body> holds then <Φ -body> holds with $\overline{u} = u_i$ and $v = \overline{v}_i$, then if we choose $\overline{u} = \max\{\overline{u}_i\}$ and $\overline{v} = \max\{\overline{v}_i\}$ we have that $P[\underline{i},\underline{x},\overline{u},\overline{v}]$ holds.

Now, as in Lemma 3.2, we can reverse the argument and define u_Φ and v_Φ to be functions of i and x such that

- if A' is of the type (∀ <var><bound><body>)

$$u_\Phi = \mathcal{F}A' = \max\{\mathcal{F}\langle\text{body}\rangle \mid \langle\text{var}\rangle \leq \langle\text{bound}\rangle\}$$

$$v_\Phi = \mathcal{G}A' = \max\{\mathcal{G}\langle\text{body}\rangle \mid \langle\text{var}\rangle < \langle\text{bound}\rangle\}$$

- if A' is of the type (⋀<D-pred>,...,<D-pred>)

$$\mathcal{F}A' = \max\{\mathcal{F}\langle\text{D-pred}\rangle,\ldots,\mathcal{F}\langle\text{D-pred}\rangle\}$$

$$\mathcal{G}A' = \max\{\mathcal{G}\langle\text{D-pred}\rangle,\ldots,\mathcal{G}\langle\text{D-pred}\rangle\}$$

- if A' is of the type (D <arg-string>)

  if no x-variable occurs free in <arg-string>

  $\mathcal{F}A' = 0$

  $\mathcal{G}A' = (\Phi\ \langle\text{arg-string}\rangle)$

  otherwise

  $\mathcal{F}A' = (\Phi\ \langle\text{arg-string}\rangle)$

  $\mathcal{G}A' = 0$

For every $\underline{i}$ and $\underline{x}$, $A[\underline{i},\underline{x}]$ implies that u_Φ and v_Φ are defined and their values $\overline{u}$, $\overline{v}$ are such that $P[\underline{i},\underline{x},\overline{u},\overline{v}]$ holds. So let us define (as in theorem 3.1) :

$$t\ (\underline{i},\underline{x},u,v) = \begin{cases} \Phi_{\sigma(\underline{i})}(\underline{x}) & \text{of } P[\underline{i},\underline{x},u,v] \\ 0 & \text{otherwise} \end{cases}$$

and $r(\underline{x},u) = \max\{t(\underline{i},\underline{x},u,v) \mid \underline{i} \leq x, v \leq \underline{x}\}$

$r(\underline{x},u)$ is total recursive since $P[\underline{i},\underline{x},u,v]$ is a recursive predicate and furthermore $A[\underline{i},\underline{x}] \rightarrow u_\Phi$ and v_Φ defined →

$$\rightarrow t(\underline{i},\underline{x},u_\Phi(\underline{i},\underline{x}),\ v_\Phi(\underline{i})) = \Phi_{\sigma(\underline{i})}(\underline{x}) \rightarrow$$

$$\rightarrow r(\underline{x},u_\Phi(\underline{i},\underline{x})) \geq \Phi_{\sigma(\underline{i})}(\underline{x}) \text{ provided } \underline{x} \geq \underline{i} \text{ and } \underline{x} \geq v_\Phi(\underline{i})$$

<u>Example</u>:

In Blum's E.R.T. :

$$A[\underline{i},\underline{x}] = (D\ (\phi\ e\ i\ (\phi\ j\ i\ e))\ x)$$

$$\underline{i} = \{i,e,j\} \quad \underline{x} = \{x\}$$

$$A'[\underline{i},\underline{x}] = (\bigwedge (D(\phi\ e\ i\ (\phi\ j\ i\ e))x)(D\ e\ i\ (\phi\ j\ i\ e))(D\ j\ ie))$$

$$U_\Phi = \{(\Phi(\phi\ e\ i\ (\phi\ j\ i\ e))x)\}$$

$$V_\Phi = \{(\Phi\ e\ i\ (\phi\ j\ i\ e)),\ (\Phi\ j\ i\ e)\}$$

$$P[\underline{i},\underline{x},u,v] = (\bigwedge (LE(\Phi(\phi\ e\ i(\phi\ j\ i\ e))x)u)\ (LE(\Phi\ e\ i\ (\phi\ j\ i\ e))v)\ (LE(\Phi\ j\ i\ e)v))$$

$$u_\Phi(\underline{i},\underline{x}) = (\Phi\ (\phi\ e\ i(\phi\ j\ i\ e))x)$$

$$v_\Phi(\underline{i}) = \max\{(\Phi\ e\ i(\phi\ j\ i\ e)),(\Phi\ j\ i\ e)\}$$

We can see that in this way r depends only on the values of s.c. functions depending on $\underline{x}$ but we have to pay for this with an increase in the number of values of $\underline{x}$ for which $\Phi_{\sigma(\underline{i})}(\underline{x})$ may not be bounded by r.

4.2 Theorem 3.1 gives us a bound on the number N of values of the arguments $x \varepsilon \underline{x}$ for which $\phi_{\sigma(\underline{i})}(\underline{x})$ is defined but r does not provide a bound to the step counting function $\Phi_{\sigma(\underline{i})}(\underline{x})$. In fact we have proved that, given $\underline{i}$, N is greater than max $\{i \varepsilon \underline{i}\}$.

Since this is a consequence of the fact that we define r as the maximum of t $(\underline{i},\underline{x},u)$ while every $i \varepsilon \underline{i}$ is running between 0 and the least $x \varepsilon \underline{x}$, we can easily figure out how to improve our result by reducing the number N. In fact we can make $i \varepsilon \underline{i}$ run between 0 and the value of some functions (growing faster than the identity function) of a scapegoat variable $x_o \varepsilon \underline{x}$: i.e. define for example

$$r(\underline{x},u) = \max\{t\ (\underline{i},\underline{x},u) \mid \underline{i} \leq 2^{x_o},\ x_o \varepsilon \underline{x}\}$$

In this way the bound on the step counting function holds for every i, for every $x \varepsilon \underline{x}$, different from x_o and for all the values of x_o greater or equal to $\log_2$ (max $\underline{i}$).

It is clear that we could improve this result in order to reduce N to a number growing extremely slowly with $\underline{i}$, but we will see in 4.3, that this brings some disadvantages.

As far as the result of theorem 4.1 is concerned we must consider that the reason why we have to pay in N (i.e. the number of values of x where the bound on the s.c. function does not hold), for getting rid of some of the s.c. functions in the computation of r, is more deep than we can realize at a first moment.

In fact, suppose we can get rid of all of the s.c. functions not depending on an x-variable without paying in N: we can prove that this would contradict the undecidability of the "halting problem".

Fact 4.2 Let $A[\underline{i},\underline{x}]$ be an assertion of $\mathcal{A}$ and let σ be a recursive function in $\mathcal{R}_n$ such that

$$\phi_{\sigma(\underline{i})}(\underline{x}) \text{ is defined} \leftrightarrow A[\underline{i},\underline{x}]$$

then there is no total function $r(x)$ such that

$$(\forall\underline{i})(\forall\underline{x}\geq\underline{i})(A\ [\underline{i},\underline{x}] \to \Phi_{\sigma(\underline{i})}(\underline{x}) \leq r\ (\underline{x}))$$

Proof. Let us define

$$\phi_{\sigma(i)}(x) = \begin{cases} x \text{ if } \phi_i(i) \text{ is defined} \\ \\ \text{undefined otherwise} \end{cases}$$

Suppose such an $r(x)$ exist. Since $\phi_{\sigma(i)}(i)$ is defined if and only if $\phi_i(i)$ is defined, and since for $x \geq i(\phi_i(i)$ defined $\to \Phi_{\sigma(i)}(x) \leq r\ (x))$ we should have that for every $x \geq i$ $\phi_i(i)$ defined iff $\Phi_{\sigma(i)}(x) \leq r\ (x)$. So we could compute $r(i)$ and then start computing $\Phi_{\sigma(i)}(i)$: if this takes more than $r(i)$ steps $\phi_i(i)$ is not defined, otherwise it is defined. So we could solve the halting problem.

Thus it is impossible that we get rid of all of the s.c. functions not depending on x-variables, without paying on N (or better: without making N depend on such s.c. functions) otherwise we could have r dependent only on x and contradicting Fact 4.2.

4.3 The third point we are interested in, is discussing how hard it is to compute the bound r with respect to the step counting function $\Phi_{\sigma(\underline{i})}(\underline{x})$.

Let us call $R(\underline{x},u)$ the step counting function of $r(\underline{x},u)$.

We can prove:

Fact 4.3: For some "reasonable measures" the number of steps required to compute $r(\underline{x},u_\Phi(\underline{i},\underline{x}))$ is bounded, wherever theorem 3.1 holds, between a function $g_1(\underline{x},u_\Phi(\underline{i},\underline{x}), \Phi_{\sigma(\underline{i})}(\underline{x}))$ and a function $g_2(\underline{x},u_\Phi(\underline{i},\underline{x}), r(\underline{x},u_\Phi(\underline{i},\underline{x})))$

Proof. In the best case, to compute $r(\underline{x},u_\Phi(\underline{i},\underline{x}))$ we need to compute only $\Phi_{\sigma(\underline{i})}(\underline{x})$ because only for $\underline{1} = \underline{i}$ we have that $P[\underline{1},\underline{x},u_\Phi(\underline{i},\underline{x})]$ holds; in the worst case, for all $\underline{1} \leq x_o$ we have that $P[\underline{1},\underline{x}, u_\Phi(\underline{i},\underline{x})]$ holds and so we have to compute $\Phi_{\sigma(\underline{1})}(\underline{x})$ for every $\underline{1} \leq x_o$ (where x_o is as in point 4.2). So, let n be the cardinality of $\underline{i}$, since we have that for every $i\epsilon\underline{i}$, i runs from 0 to x_o, and for every $\underline{i}$ we need u steps to check if $P[\underline{i},\underline{x},u]$ holds or not, $R(\underline{x},u_\Phi(\underline{i},\underline{x})) \geq (x_o + 1)^n \cdot m \cdot u_\Phi(\underline{i},\underline{x}) + \Phi_{\sigma(\underline{i})}(\underline{x})$ where m is the number of step counting function to match with $u_\Phi(\underline{i},\underline{x})$

On the other side:

$$R(\underline{x},u_\Phi(\underline{i},\underline{x})\ ) \leq (x_o + 1)^n \cdot m \cdot u_\Phi(\underline{i},\underline{x}) + \sum_{o}^{x_o}{}_{\underline{1}} \Phi_{\sigma(\underline{1})}(\underline{x}) + M\ (x_o)$$

where $M\ (x_o)$ is the number of steps required to find the maximum in $\{t(\underline{1},\underline{x},u) | \underline{1} \leq x_o\}$ that we may consider growing with x_o^n for some "reasonable measures".

Considering that $r(x,u_\Phi(i,x)) = \max\{\Phi_{\sigma(\underline{1})}(x) | \underline{1} \leq x_o\}$ $R(x,u_\Phi[i,x]) \leq (x_o + 1)^n \cdot m \cdot u_\Phi[i,x] + (x_o + 1)^n \cdot r(x,u_\Phi(i,x)) + M\ (x_o)$

The meaning of Fact 4.3 is that to compute the bound to the step counting function is usually not much harder than computing the step counting function itself.

In order to reduce the strong dependence on x_o (remember that the values of x_o for which r is a bound to the s.c. function are all greater than $\underline{i}$, i.e. rather big values) we could make i (for all $i\epsilon\underline{i}$) run between 0 and $(\sqrt[n]{x_o} \dot{-} 1)$ (instead of x_o) so that the factor dependent on x_o should be reduced to x_o itself. The reason why we must be careful in doing so is that in point 4.2 we have seen that such a bound on the range of i would determine an increase in the number N of points where r is not a bound for $\Phi_{\sigma(\underline{i})}(\underline{x})$

For the opposite reason we cannot try to reduce N without increasing the difficulty of computing r. So we need to balance these two exigencies.

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References

1. Blum, M.: A Machine Independent Theory of the Complexity of Recursive Functions. J. ACM, 14,2 (April, 1967).
2. Blum, M.: On Effective Procedures for Speeding up Algorithms. ACM Symposium on Theory of Computing, Marina del Rey, California (May 1969).
3. Borodin, A.: Complexity Classes of Recursive Functions and the Existence of Complexity Gaps. Ibid.