



MDELTA - A DIGITAL PROGRAM FOR CONTROL  
SYSTEM ANALYSIS

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## MDELTA - A Digital Program for Control System Analysis

### Introduction

Design and analysis of high-order control systems requires a great deal of laborious calculations. Stability analyses are usually conducted with standard techniques such as root locus or frequency response. The calculations involved in these analyses are generally too complex to do by hand. Graphical techniques and approximations may be used, but often introduce unacceptably large errors.

The advent of modern digital computers led to the development of a great many programs to perform stability analysis. Generally, these programs were designed to perform one specific analysis using a single technique such as root locus. Such programs seldom shared common input formats, making multiple analyses difficult. Program input was often in the form of polynomial coefficients of the system's transfer function. The determination of these coefficients was difficult, for all but the most simple control systems.

The inconvenience of such programs demonstrated the need for a more flexible analysis tool. To provide general usefulness a new program would require a more basic input scheme. The input language should have the flexibility to allow concise descriptions of both large and small systems. In addition, it would be desirable for more than one analysis technique to be available in a single computer run. To reduce input errors, the program should use self-explanatory control cards. A complete plotting capability would also be essential.

The MDELTA program was designed to satisfy all of the above requirements. New features are still being included after three years of successful usage in Boeing aerospace projects. The major analytical techniques used within the program are outlined below.

### Program Description

The configuration of the control system to be analyzed is defined to the program using a block diagram input language. An example for a small system is shown below.

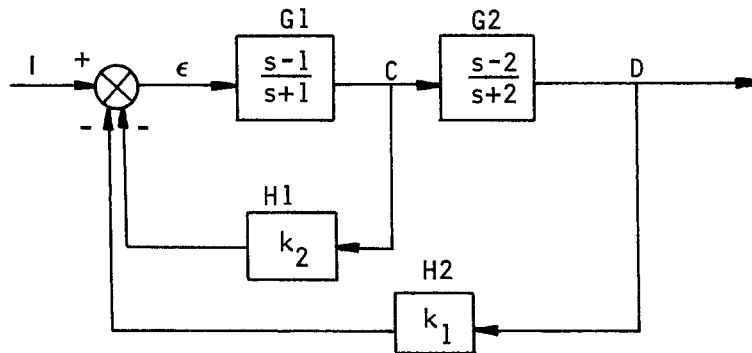


Fig. 1

# BLOCK DIAGRAM EQUATIONS

$$EPS = I - H1 * C - H2 * D$$

$$C = G1 * EPS$$

$$D = C * G2$$

# TRANSFER FUNCTIONS

$$G1 = (s-1) / (s+1)$$

$$G2 = (s-2) / (s+2)$$

$$H1 = K2$$

$$H2 = K1$$

# FORCING FUNCTIONS

$$I = -1 / s$$

END

As seen above, the user has chosen variable names for the variables in the block diagram, i.e.,

$$k_1 \sim K1$$

$$k_2 \sim K2$$

$$\epsilon \sim EPS$$

The variable name "S" is reserved to represent the Laplace transform operator. Double slashes indicate separation between numerator and denominator polynomials in s for Laplace transforms and transfer functions.

The MDELTA program accepts the above control system description and automatically generates Laplace transform equations in matrix form. The equations are written so that each matrix element is a proper polynomial function of  $s$ . A two-step derivation illustrates the process for the above example system:

$$1. \quad \begin{bmatrix} 1 & K2 & K1 \\ -\frac{s-1}{s+1} & 1 & 0 \\ 0 & -\frac{s-2}{s+2} & 1 \end{bmatrix} \begin{Bmatrix} EPS \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{s} \\ 0 \\ 0 \end{Bmatrix}$$

This initial form of the matrix equation is somewhat cumbersome to process, so the program writes the nonhomogeneous equation in the following modified form, clearing each row of denominator polynomials in  $s$ :

$$2. \quad \begin{bmatrix} s & K2*s & K1*s \\ -s+1. & s+1. & 0 \\ 0 & -s+2. & s+2. \end{bmatrix} \begin{Bmatrix} EPS \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} -1. \\ 0 \\ 0 \end{Bmatrix}$$

or

$$[A(s)] \{X(s)\} = \{F(s)\}$$

where  $[A(s)]$  is the system characteristic matrix, elements of which are functions of the control system parameters and proper polynomial functions of  $s$ .

$\{X(s)\}$  is the vector of Laplace transforms of the control system variables (displacements, voltages, torques, etc.).

$\{F(s)\}$  is the vector of Laplace transform terms independent of the control system variables.

The elements of the characteristic matrix and forcing function vector are each formed symbolically (as explicit functions of system parameters). Elements of  $[A(s)]$  are transformed into FORTRAN card images of the form:

$$A(i,j,k) = \underline{\hspace{2cm}}$$

where  $i$  and  $j$  represent the row and column indices of the element and  $k$  is an indicator of the degree in "s". Card images defining the elements of  $\{F(s)\}$  are also generated.

Before specifying the block diagram equations the user is allowed to define other parameters. This can be done with FORTRAN arithmetic statements and may use standard library functions such as the square root or sine. Single or double argument table look-ups are also available. The parameters so calculated can form coefficients of the block diagram transfer functions.

Some complex systems are not easily expressed in the block diagram input language. For these cases, the user may elect to specify the elements of  $[A(s)]$  directly. A third input option is the state variable form:

$$3. \quad [A(s)] = \{s[I] - [A']\}$$

where  $[I]$  represents the identity matrix.

To define a state variable system, only the  $A'$  matrix elements must be input. Special time response techniques discussed later are available for a system defined in this form.

The set of card images comprising the system definition forms the body of a FORTRAN subroutine. The program detects all variable names employed by the user and necessary statements such as SUBROUTINE ..., COMMON, RETURN, etc., are automatically formed. A second subroutine is also generated to define the elements in the forcing function and initial conditions vectors. The final results are subroutines which can be called to evaluate  $[A(s)]$  or  $\{F(s)\}$  and  $\{x_i(0)\}$  for various user parameter values.

After specifying the system definition the user may select any of the four currently available analysis options: Frequency Response, Root Locus, Time Response, or Gain Boundary. Each option is discussed separately in the sections to follow.

## Frequency Response

In general, any control system may be described in the following form for the purpose of analyzing the margins of stability for any specified gain parameter "k":

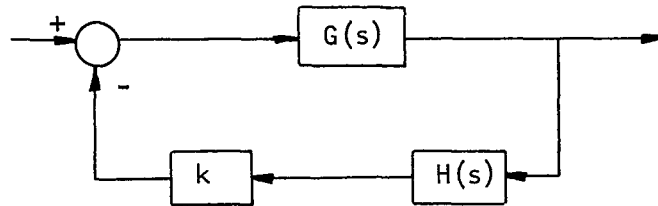


Fig. 2

The system appears to have just one loop, although many others may be implicitly contained in the transfer functions  $G(s)$  and  $H(s)$ . Stability analysis for the gain "k" may be accomplished by "opening" the "k" loop and studying the frequency response of the transfer function  $G(s)H(s)$ :

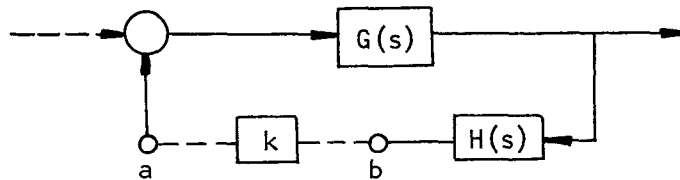


Fig. 3

The analysis performed corresponds to applying a unit sine wave signal source at point "a" and measuring the magnitude and phase of the resulting signal at point "b" as the frequency of the source is varied over a range of interest.

This analysis can be obtained directly from the characteristic matrix equation. The system loop which is to be opened is specified to the program by a control card which states the variable name, "k", for the gain parameter within that loop.

The homogeneous characteristic equation is:

$$4. \quad [A(s)] \{X(s)\} = \{0\}$$

or, equivalently:

$$5. \quad |A(s)| = Q(s) + kP(s) = 0$$

where  $Q(s)$  represents that portion of the characteristic polynomial which is independent of the gain " $k$ " and  $P(s)$  represents the characteristic polynomial coefficient of the gain " $k$ ".

The open-loop frequency response for loop " $k$ " is:

$$6. \quad F(j\omega_i) = \frac{P(s)}{Q(s)} \Big|_{s=j\omega_i} = \frac{|A(s)|_{k=1} - |A(s)|_{k=0}}{|A(s)|_{k=0}} \Big|_{s=j\omega_i}$$

where  $j$  represents the complex operator  $\sqrt{-1}$  and  $\omega_i$  the frequency in radians/sec.

Equation 6. is implemented by direct calculation of the eigenvalues for both denominator and numerator (open-loop poles and zeros for gain " $k$ "). This is accomplished by manipulation of  $[A(s)]$  to yield matrix equivalents for  $Q(s)$  and  $P(s)$ . These poles and zeros provide valuable information toward a clear understanding of frequency response results.

The user may specify only a frequency maximum and minimum and the program will generate a frequency array composed of evenly-spaced points on a logarithmic scale. Additional points are also generated around points of rapid change in the response magnitude. These can be predicted by examination of the open-loop poles and zeros for each case. Conjugate roots very near the imaginary axis in the  $s$ -plane are detected and appropriate frequency values calculated.

Plotting in the Frequency Response option consists of any combination of: 1) Bode Plots (response magnitude in db. and phase angle vs. log frequency), 2) Nichols Plots (response magnitude in db. vs. phase angle), or 3) Nyquist Plots (response magnitude vs. phase angle in polar coordinates).

#### Root Locus

Root loci for the closed loop system may be obtained by evaluating eigenvalues of the characteristic equation:

$$7. \quad |A(s)| = 0$$

as any desired system parameter (e.g., "k") is varied over a range of values. In most practical cases the characteristic equation is equivalent to the form:

$$8. \quad Q(s) + k P(s) = 0$$

where  $P(s)$  and  $Q(s)$  are independent of the parameter "k".

$Q(s)$  and  $P(s)$  represent the polynomial expansions of open-loop poles and zeros of the system. The method used to calculate open-loop eigenvalues in the Frequency Response option may also be used in Root Locus. The calculated eigenvalues are expanded to polynomial form to yield  $Q(s)$  and  $P(s)$ . The root locus equation 8. is then solved repeatedly for various values of "k" using polynomial root-finding techniques. Identical results can be obtained by finding the eigenvalues of the characteristic determinant (equation 7.) evaluated for the same values of "k". This method is available as an option but is seldom used due to run time considerations. Plots made by the Root Locus option consist of large X's denoting system poles, large O's for zeros of  $P(s)$ , and small boxes denoting points of the loci.

#### Gain Boundary

Gain Boundary is a technique for determining stability ranges for two loop gains in a multi-loop system. This technique defines a closed region of values for two gains wherein stable system operation is assured (provided all other system parameters are maintained at constant value).

For any specified operating frequency it may be possible to calculate one or more sets of values for the two selected gain parameters such that the Frequency Response for both loops is at the boundary between stable and unstable operation (phase angle =  $180^\circ$ ). The curve of such points for a specified range of frequency values defines the boundary between stable and unstable operating regions. One closed region of stable operation may be defined by this curve.

As the program is implemented, the characteristic equation is assumed to be:

$$9. \quad Q_1(s) + k_1 Q_2(s) + k_2 P_1(s) + k_1 k_2 P_2(s) = 0$$

as an explicit function of "k<sub>1</sub>" and "k<sub>2</sub>", gain parameters for the two loops of interest.



Two simultaneous equations in  $k_1$  and  $k_2$  can be found by letting  $s = j\omega$  and separating real and imaginary components. Solution of these equations for  $k_1$  and  $k_2$  is then performed for various values of  $\omega$ . A set of gain values defining a bounded area is produced. For  $P_2(s)$  non-zero the equations become quadratic and two solutions for  $k_1$  and  $k_2$  are found.

Plotted output is shown in the example problems. The four quadrants on the plot represent different configurations of sign for the loop feed-back. These plots thus signify if a different choice of sign for the feedback element would be more desirable in terms of gain margin.

### Time Response

Two methods are available to calculate the time response of variables within the control system. The first method is used for systems defined using the block diagram input language. Time histories of the variables in continuous, linear systems can be found from:

$$10. \quad [A(s)] \{X_i\} = \{F(s)\}$$

where  $[A(s)]$  represents the system characteristic matrix

$\{X_i\}$  = vector of system variables

$\{F(s)\}$  = vector of the Laplace transformations of system forcing functions.

Inverse Laplace transformation by partial fraction expansion is used. The denominator eigenvalues are calculated initially. Partial fraction expansion can be accomplished without finding the numerator eigenvalues which results in a considerable savings in machine time. The numerator is formed by a Cramer's Rule column substitution.

A second method may be used if the system is described by the state variable form input option. An advantage of this method is the capability for introducing system non-linearities. A solution to the following system of simultaneous time-domain differential equations (which correspond directly to the Transfer Function equations discussed earlier) is calculated:

$$11. \quad \{[A] - s[I]\} \{X_i\} = \{F(s)\}$$

or

$$12. \quad \{\dot{X}_i\} = [A'] \{X_i\} + \{f(t, \dot{X}_i, X_i)\}$$

and initial conditions  $\{X_i(0+)\}$  .

The  $f(t, \dot{X}_1, X_1)$  vector may contain any non-linearities desired. Numerical integration by variable-step Runge Kutta methods is used to solve the set of simultaneous equations.

### Example Problems

The control system illustrated in Fig. 1 was chosen as the first example problem.

Open-loop frequency response for only the inner loop is obtained by setting gain  $k_1$  to zero and opening the loop at  $k_2$ . This results in the transfer function  $G1*H1$  from the block diagram. Results from this analysis are a Bode Plot with a magnitude of 1. for all frequencies and the corresponding Bode phase plot shown in Fig. 4.

Root locus results for the gain  $k_1$  with  $k_2 = 0$  are shown in Fig. 5. The root locus equation solved is:

$$(s+1)(s+2) + k_1(s-1)(s-2) = 0$$

Fig. 6 shows the gain boundary option results for the two gains in the system. The figure is composed of four quadrants, each plotting the magnitude of  $k_1$  against the magnitude of  $k_2$  in db. This allows user visibility of the gain boundary results, even though wide excursions in functional value may occur. It does, however, cause plot discontinuities when  $k_1 = 0$  or  $k_2 = 0$ . The stable operating region is in the closed area enclosing the origin.

Fig. 7 shows the time response for variable "C" to a step input with gains  $k_1$  and  $k_2$  both set to unity. The stable response shown can be verified by observing that the point ( $k_1 = 1, k_2 = 1$ ) lies within the stable region in Fig. 6.

The second example problem illustrated in Figures 8 and 9 represents a basic autopilot with two bending modes and aerodynamic coupling. Figures 10-13 show the plotted Frequency Response results for the KTHETA loop. The parameters OMEGA1 and OMEGA2 represent the two lightly-damped structural bending modes which cause the spikes in Figure 10.

Figure 14 represents the root loci as a function of the gain KTHETA. The origin is expanded in Figure 15 to show break-away points and low-frequency axis crossings.

The shaded portion of Figure 16 shows the region of stability for the two gains KTHETA and KTHDOT. This region lies wholly in the first quadrant which shows that only certain positive gain values can produce a stable system.

The time response results shown in Figure 17 are for the variable PS11 with a step input for variable DELCOM.

### Summary

The MDELTA program provides the capability for comprehensive analysis of control systems. The ease and flexibility of usage have resulted in wide acceptance by control systems engineers. For one Boeing missile contract alone, it is estimated that \$250,000 was saved in the control systems design activity without including the probable savings which may result during the flight test program.

The heart of the program is the eigenvalue routine which is required in every analysis option. This routine has consistently proven successful in cases involving more than 60 complex eigenvalues.

MDELTA is coded entirely in FORTRAN IV for operation on the SRU 1108 Exec II and the CDC 6600. Current development is directed toward sampled data system analysis with options paralleling those provided for continuous analog systems. Preliminary versions of these sampled data options have been in use for the past several months.

### Acknowledgements

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### References

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2. I. Tarnove, Determination of Eigenvalues of Matrices Having Polynomial Elements, J. Soc. Indust. Appl. Math 6 (1958), pp. 163-171.

3. J. T. Tou, Modern Control Theory, McGraw-Hill, New York, 1964.
4. D. E. Muller, A Method of Solving Algebraic Equations Using an Automatic Computer, Mathematics Tables Aids Comput. 10 (1956), pp. 208-215.





# BLOCK DIAGRAM EQUATIONS

$$\text{DELTAC} = \text{DELCOM} + \text{KF1} * \text{THETAT} + \text{THETAT} * \text{KF2}$$

$$\text{DELTA} = \text{DELTAC} * \text{SERVO}$$

$$\text{PSI1} = \text{BB1} * \text{DELTA}$$

$$\text{PSI2} = \text{BB2} * \text{DELTA}$$

$$\text{THETAT} = \text{PSI1} * \text{P1} + \text{PSI2} * \text{P2} + \text{THETA} * \text{UNITY}$$

$$\text{ALPHA} = \text{A2} * \text{Z} + \text{A1} * \text{THETA}$$

$$\text{THETA} = - \text{T1} * \text{ALPHA} + \text{T2} * \text{PSI1} + \text{T3} * \text{PSI2} - \text{T4} * \text{DELTA}$$

$$\text{Z} = - \text{ALPHA} * \text{D1} + \text{THETA} * \text{D2} + \text{D3} * \text{PSI1} + \text{PSI2} * \text{D4} - \text{DELTA} * \text{D5}$$

## TRANSFER FUNCTIONS

$$\text{KF1} = \text{KTHDOT} * \text{S}$$

$$\text{KF2} = \text{KTHETA}$$

$$\text{SERVO} = \text{OMEGAS}^{**2} // \text{S}^{**2} + 2. * \text{ZETAS} * \text{OMEGAS} * \text{S} + \text{OMEGAS}^{**2}$$

$$\text{BB1} = 1. / \text{MASS1} * (\text{NBAR} * \text{MASSN} * \text{PH1E} - \text{IN} * \text{PH1PE}) * \text{S}^{**2} + \text{T} * \text{PH1E} / \text{MASS1} // \text{S}^{**2} + 2. * \text{ZETA1} * \text{OMEGA1} * \text{S} + \text{OMEGA1}^{**2}$$

$$\text{BB2} = 1. / \text{MASS2} * (\text{NBAR} * \text{MASSN} * \text{PH2E} - \text{IN} * \text{PH2PE}) * \text{S}^{**2} + \text{T} * \text{PH2E} / \text{MASS2} // \text{S}^{**2} + 2. * \text{ZETA2} * \text{OMEGA2} * \text{S} + \text{OMEGA2}^{**2}$$

$$\text{P1} = \text{PH1PK}$$

$$\text{P2} = \text{PH2PK}$$

$$\text{UNITY} = 1.$$

$$\text{A2} = \text{S} // .0175 * \text{U}$$

$$\text{A1} = \text{S} * (\text{XCG} - \text{XCP}) // .0175 * \text{U}$$

$$\text{T1} = \text{Q} * \text{AREA} * \text{CNA1} * (\text{XCG} - \text{XCP}) / \text{I} // \text{S}^{**2}$$

$$\text{T2} = \text{T} / \text{I} * (\text{PH1PE} * (\text{XCG} - \text{XE}) - \text{PH1E}) // \text{S}^{**2}$$

$$\text{T3} = \text{T} / \text{I} * (\text{PH2PE} * (\text{XCG} - \text{XE}) - \text{PH2E}) // \text{S}^{**2}$$

$$\text{T4} = 1. / \text{I} * (\text{NBAR} * \text{MASSN} * (\text{XCG} - \text{XE}) + \text{IN}) * \text{S}^{**2} + \text{T} / \text{I} * (\text{XCG} - \text{XE}) // \text{S}^{**2}$$

$$\text{D1} = \text{Q} * \text{AREA} * \text{CNA1} / \text{MASS} // \text{S}^{**2}$$

$$\text{D2} = \text{U} * \text{S} - \text{G} * \text{SIN}(\text{THETA0} / 57.3) // \text{S}^{**2}$$

$$\text{D3} = \text{T} * \text{PH1PE} / \text{MASS} // \text{S}^{**2}$$

$$\text{D4} = \text{T} * \text{PH2PE} / \text{MASS} // \text{S}^{**2}$$

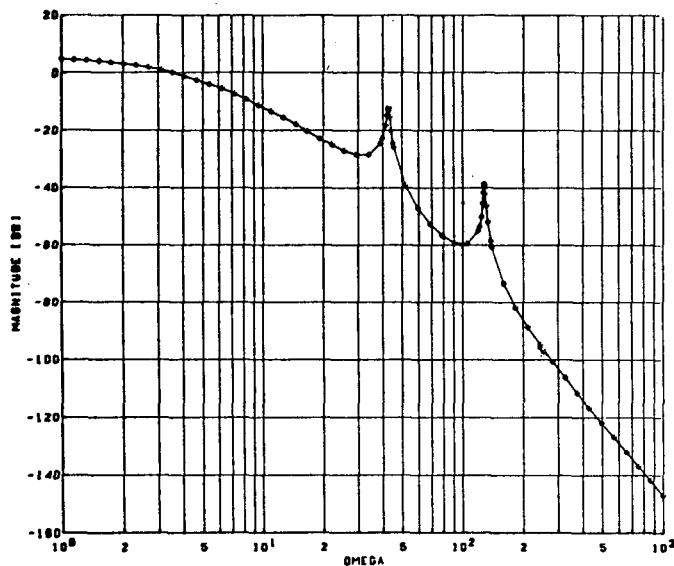
$$\text{D5} = \text{NBAR} * \text{MASSN} / \text{MASS} * \text{S}^{**2} + \text{T} / \text{MASS} // \text{S}^{**2}$$

## FORCING FUNCTIONS

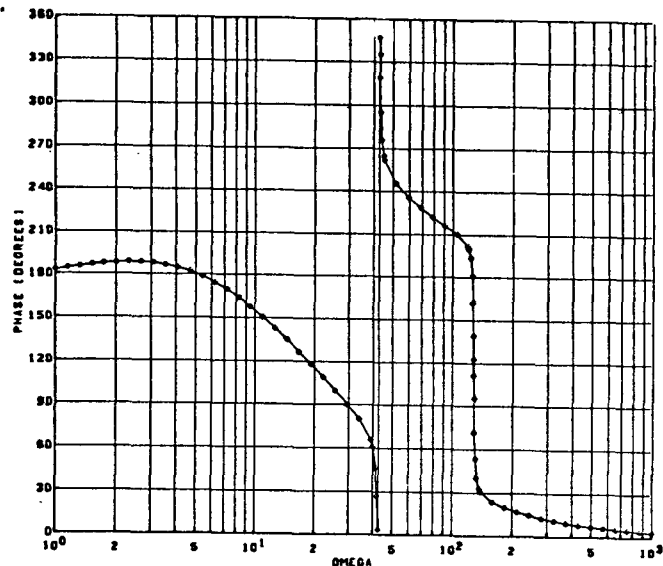
$$\text{DELCOM} = 1. // \text{S}$$

END

Figure 9



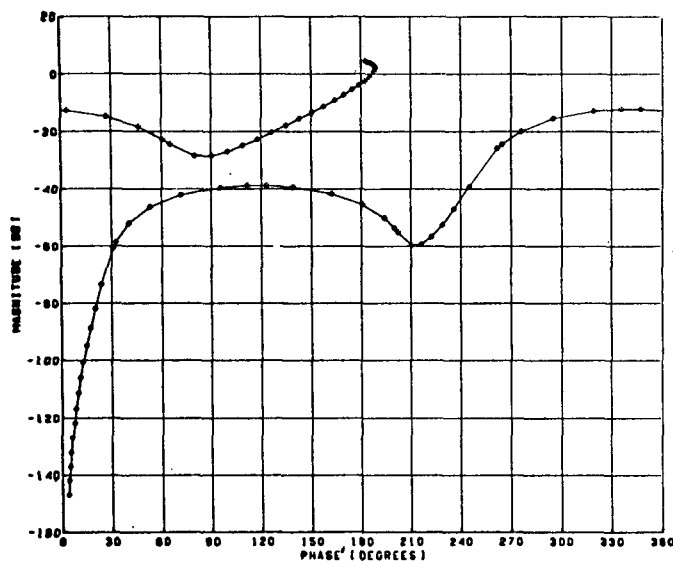
BODE PLOT, PART A. IDENTIFICATION NO. = 1  
 FREQUENCY RESPONSE FOR KTHETA LOOP  
 OMEGA1= 4.330000+01 KTHD01= 1.000000-01 OMEGA2= 1.280000+02  
 17 MAY 68



BODE PLOT, PART B. IDENTIFICATION NO. = 1  
 FREQUENCY RESPONSE FOR KTHETA LOOP  
 OMEGA1= 4.330000+01 KTHD01= 1.000000-01 OMEGA2= 1.280000+02  
 17 MAY 68

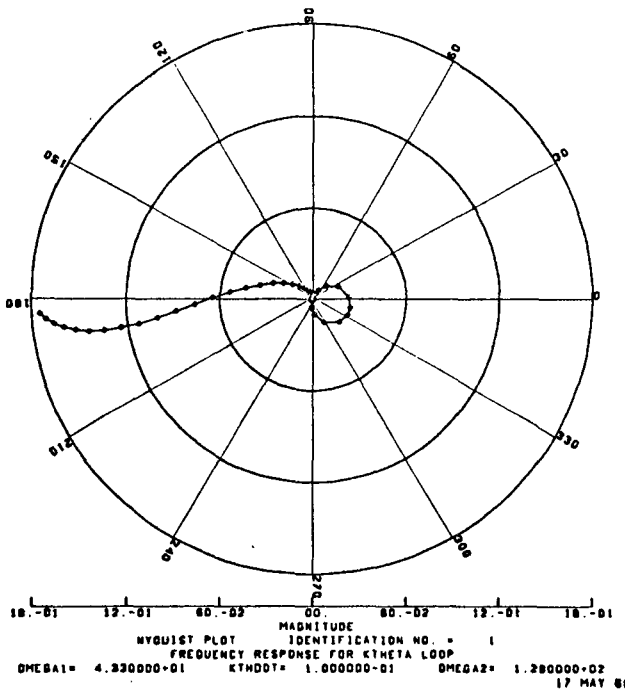
Figure 10

Figure 11



NICHOLS PLOT. IDENTIFICATION NO. = 1  
 FREQUENCY RESPONSE FOR KTHETA LOOP  
 OMEGA1= 4.330000+01 KTHD01= 1.000000-01 OMEGA2= 1.280000+02  
 17 MAY 68

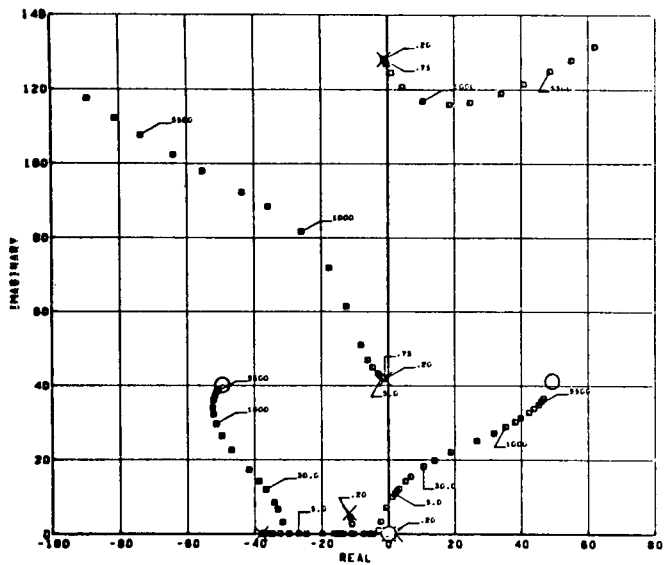
Figure 12



NYQUIST PLOT. IDENTIFICATION NO. = 1  
 FREQUENCY RESPONSE FOR KTHETA LOOP  
 OMEGA1= 4.330000+01 KTHD01= 1.000000-01 OMEGA2= 1.280000+02  
 17 MAY 68

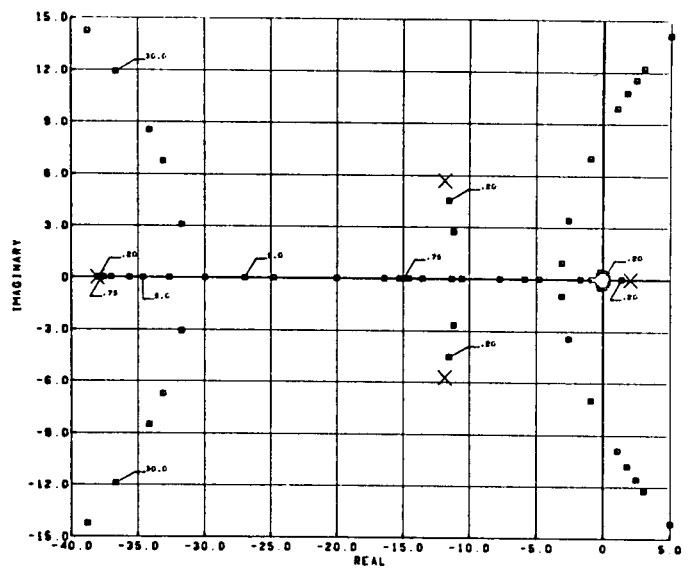
Figure 13





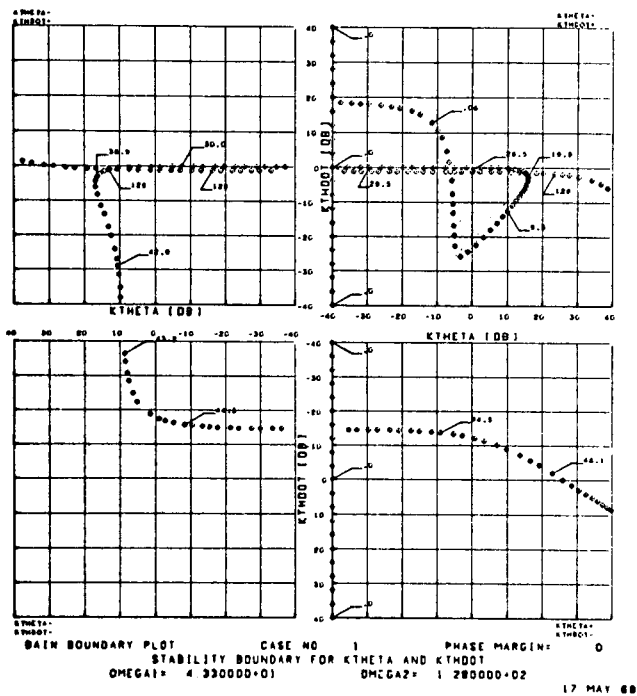
17 MAY 68

Figure 14



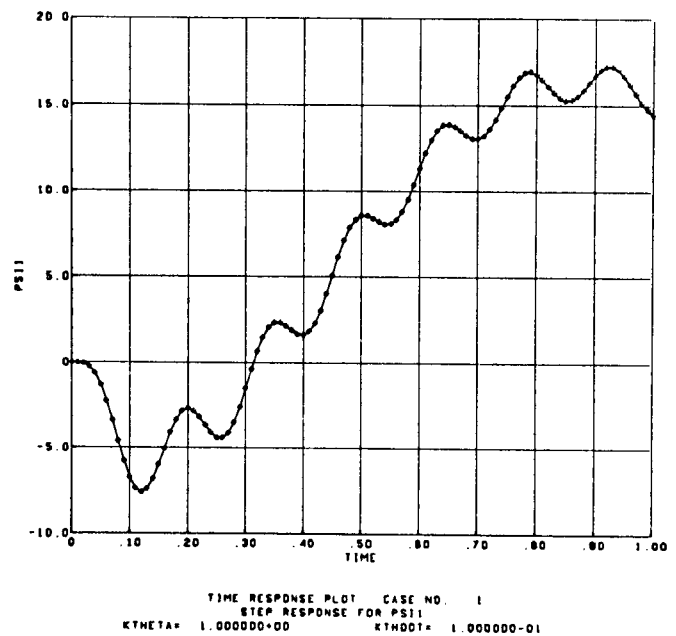
17 MAY 68

Figure 15



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Figure 16



17 MAY 68

Figure 17