Computer-Aided Design of Mechanical Linkages

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An interactive computer-aided design system for kinematic analysis of mechanical linkages in three-dimensional space is described. The design system affords the designer a wide range of capabilities including the conceptual definition of a linkage by its linkage bodies, constraints, and loops, the visualization of a linkage by means of wire frame representations of linkage components, and the performance of position analyses of a linkage's motions. The system makes use of computer graphics in an interactive environment to provide the output and visual displays which contribute to the comprehension and analyzation of a linkage. The system has been adapted to assist in the design of mechanical linkages with constraints and configurations particularly suitable for analysis under the present system's capabilities. This has proven to be a significant technological advancement over previous methods of designing a linkage in a production environment in which many manual drafting layouts were required to check out the operation and proper functioning of a given design.

KEY WORDS AND PHRASES: computer-aided design, interactive graphics, mechanical linkage, kinematic analysis, engineering design, geometrical analysis, solution algorithm, loop, parallel loop, constraint, graphical input/output, display manipulation
CR CATEGORIES: 3.26, 4.9

INTRODUCTION

Mechanical linkage design is readily adaptable to the use of interactive computer graphics for several reasons. The linkage is an assembled set of mechanical devices such as levers, cams, rollers, gears, pistons which are attached to each other and constrained to a defined course of motion as a result of their combined geometries. Therefore, the design and analysis of a linkage require visualization of positions and placements of its various component devices, and being a mechanical system, it requires analysis of the movements of the linkage "bodies" and evaluation of the entire system for proper operation and functioning. The design of a linkage involves many calculations which are very lengthy when performed manually.

By using a graphic display device, the designer may request a representation of the linkage to be displayed on the display screen, view the linkage from any orientation, exercise a linkage, and request results of position computation to be displayed on the screen. Should modifications to the design be desired, the user can alter the linkage data base directly on-line at the CRT and evaluate the new design immediately.

Kinematic analysis of mechanical linkages is one engineering application for which computer-aided design techniques provide considerably more powerful methods of linkage design than other modes of operation. The approach taken in implementing the design capabilities specified above has fully utilized all of the available features of interactive graphics.

MECHANICAL LINKAGE DESIGN SYSTEM

The linkage program which has been developed is specifically intended to serve as a computer design aid in the position analysis of mechanical linkages. The program is a graphical interactive system employing an IBM 2250. Features of the IBM 2250 include a cathode-ray-tube, graphical display capability, programmable function keys, an alphanumeric keyboard, and a light pen. The user communicates with the central processor of an IBM 360 by sending interpretable interrupts from the IBM 2250. A schematic drawing of the program's architecture is given in Figure 1. The control program decides which segments of the system

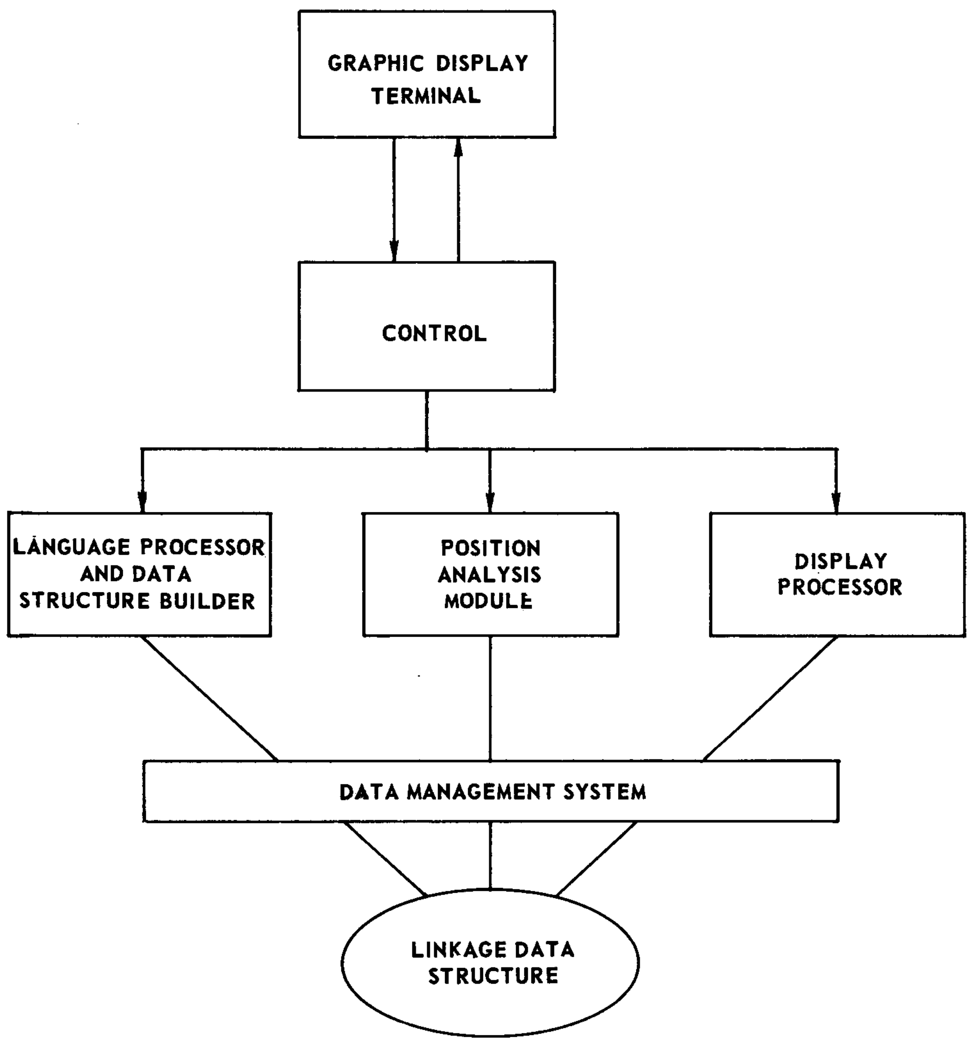

Figure 1. Mechanical Linkage Design System

should currently be processed. The primary segments of the system include a language processor (for building the data structure representing the linkage), a position analysis module (for computation of displacements of linkage bodies due to input displacement), and a display processor (for creating display images and outputting to a display device).

GENERAL SOLUTION FOR POSITION ANALYSIS

Because of many variations of complex linkages, it is impractical to develop unique solution procedures for each possible linkage. The problem of position solution of linkages may be defined in terms of a general system. A characteristic of a general system is that a general algorithmic method is defined for handling any particular input case.

It has been found that the position solution problem of linkages can be simplified to a step-by-step solution of each integral part of the whole linkage. According to the physical concepts of linkages, a given input at one end of the linkage will cause a predictable output at another end of the linkage. By solving for the positions of the given input links to the linkage, the input positions of the next links in sequence may be determined by knowing how those links are constrained to move within the linkage. Each link of the linkage is solved from the information supplied by its surrounding links. The problem of a complex linkage can, therefore, be reduced to the problem of solving individual "loops" of a series of links connected to one another. This basic "loop" concept has been previously identified in IBM's Kinematic Analysis Method (KAM) (1,2). The analytical position solution approach employed in that system was that its "loops" could be mathematically reduced to a series of simultaneous equations which could be uniquely solved. The "loops", in effect, describe the flows of motion through the linkage. This method of solution has given rise to the definition of a linkage in terms of the "loops" of the linkage, where a "loop" is defined as a series of constraints of motion.

By limiting the types of loops and types of constraints which are commonly encountered in practical design applications, it was determined that a generalized step-by-step solution algorithm could be developed. It was also determined that any meaningful system for linkage design ought to include the following types of constraints. (In its precise definition a constraint refers to a reduction of the degrees of freedom of a linkage pair or joint. However, in this system "constraint" has been used in some instances to refer to the physical pair itself. Although the terms are not truly interchangeable, it is felt that its meaning will be readily understandable by the context in which it appears.) These constraints are: pin (rotation in a plane), pivot (ball joint), roller on a flat surface, piston in a cylinder, cam with ball follower, and gear (and rack) (Figures 2-8).

By considering only physically workable linkages, it has been determined that in general, all linkages can be broken down into loops of several basic categories. In this paper, the term "loop" has been extended beyond its common usage in kinematic literature to mean a series of constraints and attached links, regardless of closure of the loop, for which the positions of one or more of the constraints are uniquely determinable from the positions and geometries of the remaining constraints and links of the loop. The word "loop" has been preserved because it is employed as the basic element in the position solution algorithm which is similar to its usage in the algorithm described in the KAM system. However, whereas a loop in the KAM system was solved by a vector algebra approach for a set of vector equations which each loop could be reduced to, a loop in this system is solved by calculating geometrical intersections involving planes, circles, spheres, and lines inherently defined by the linkage configurations. In the present system configuration, three simple "loop" types have been required for the analysis of a wide class of linkages. These loops are directly solvable by methods of analytical geometry. One loop is made up of a pin, roller on a flat, and a pin (Figure 9). Its position is determined by solving for the intersection of a circle and a plane. A second loop contains a pin (or pivot) and a piston in a cylinder (Figure 10). This loop is solved by

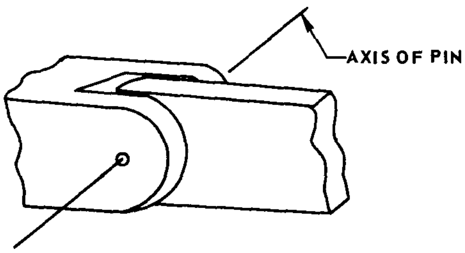

Figure 2. Pin Constraint

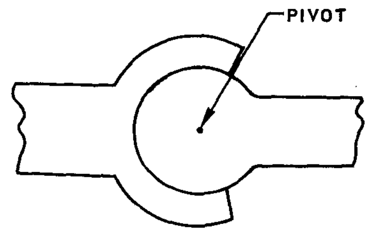

Figure 3. Pivot Constraint

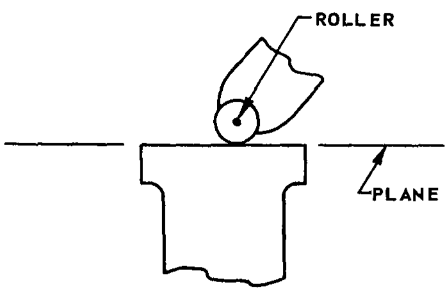

Figure 4. Roller/Flat Constraint

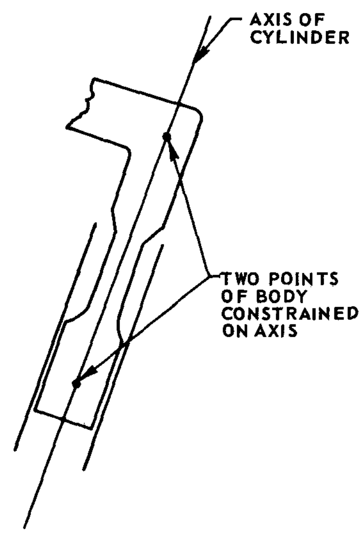

Figure 5. Piston/Cylinder Constraint

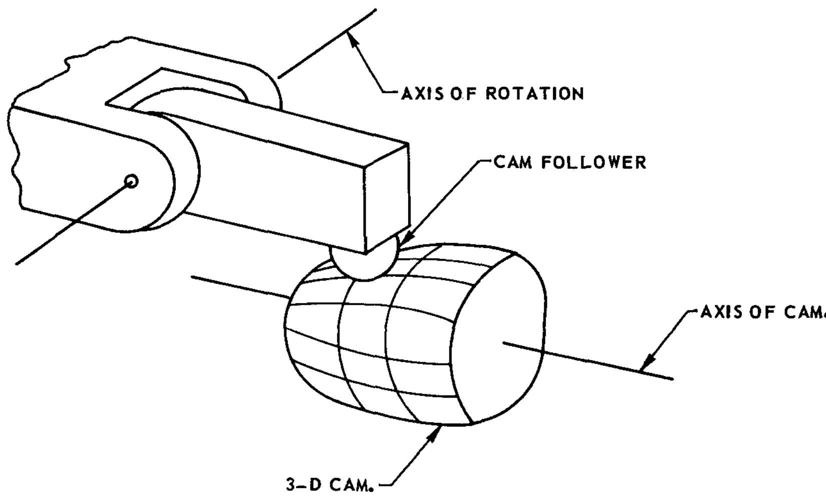

Figure 6. Cam/Cam Follower Constraint

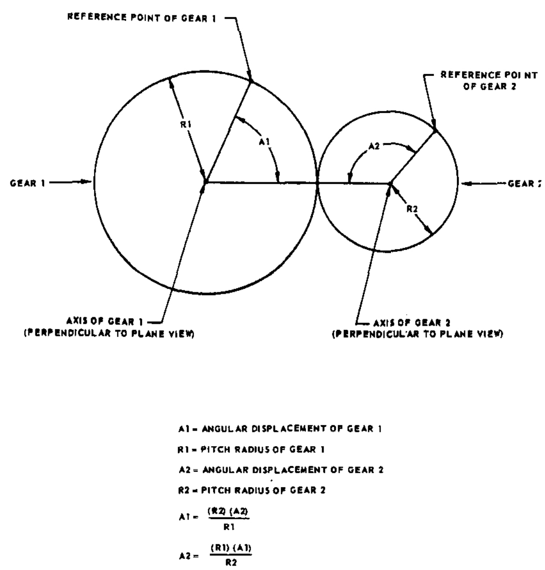

Figure 7. Gear/Gear Constraint

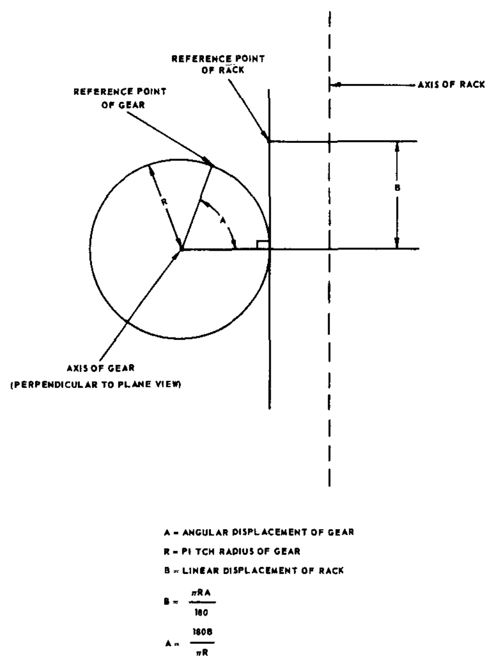

Figure 8. Gear/Rack Constraint

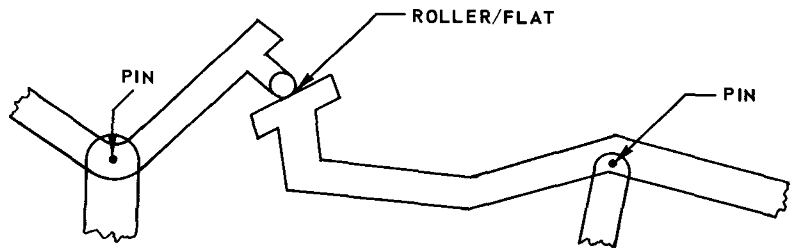

Figure 9. Pin-Roller/Flat-Pin

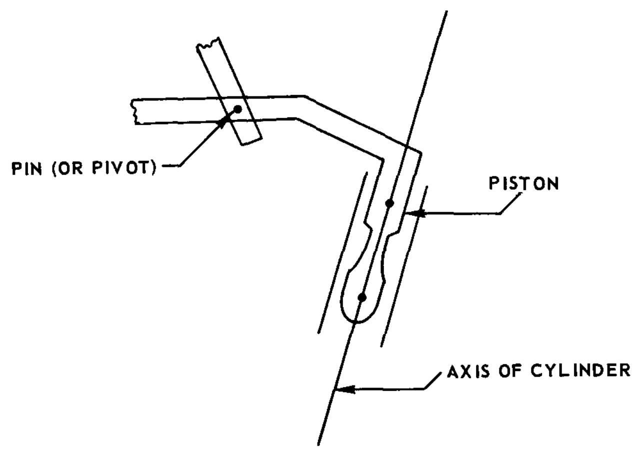

Figure 10. Pin (or Pivot) - Piston/Cylinder

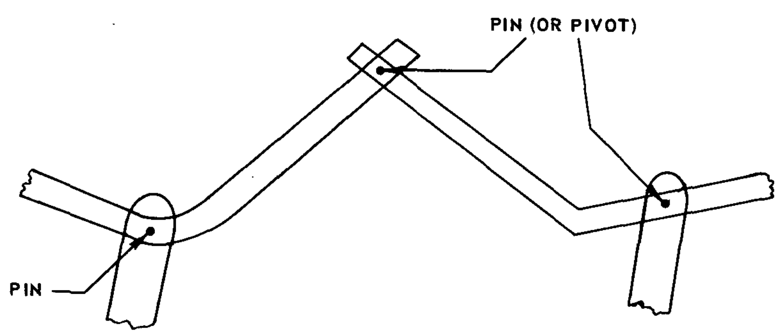

Figure 11. Pin-Pin (or Pivot) - Pin (or Pivot)

using the intersection of a circle (or sphere) and a straight line. The third loop involves a pin, pin (or pivot), and pin (or pivot) (Figure 11) and its solution involves the intersection of a circle and sphere.

There are certain complex loops which cannot be classified according to the types described above. Peculiar to these loops is the fact that there does not exist a simple analytical geometry solution for the calculation of their positions but they can be solved by an iterative technique to converge to the correct result. Mathematical methods have been developed to facilitate the solution of these loops. A particular instance of this type loop is defined by the series of constraints roller on a flat, roller on a flat, pin, piston in a cylinder (Figure 12). In the illustration, A is the position of the pin, B is the position of one roller, and C is the position of the second roller. The positions of both flat surfaces are known. It is required to find the positions of A, B and C. By taking an initial guess for A, the position of B (dependent on the choice for A) may be determined from the requirement that the roller must be in contact with the flat surface at B. (All positions of A lie on line parallel to the axis of the cylinder.) The position of C is then determinable by its known relative positions to A and B. The distance from this position of C to the flat surface is calculated. A "Secant/False Position" Method is used until the distance from the roller at C to the flat at C converges within a prescribed tolerance.

The general method of position solution which has been developed may be referred to as a "stick" solution. It is called a "stick" solution because the linkages involved may be modeled from a set of "sticks". The "sticks" are straight line segments between the center points of the constraints. The analysis of the linkage has been done without consideration to the surfaces of the constraints. Pins, pivots, and rollers are considered as point centers of zero radius. To detect possible interferences between the actual bodies of the linkage once the position has been determined by the "stick" solution, the display scope is used. A line drawing of the linkage is displayed on the screen. In addition to lines defining the "sticks", the user may define lines to represent the contours of the bodies (or links) of the linkage. By manipulating the display to view the linkage from different orientations, the user can visually determine if interferences occur.

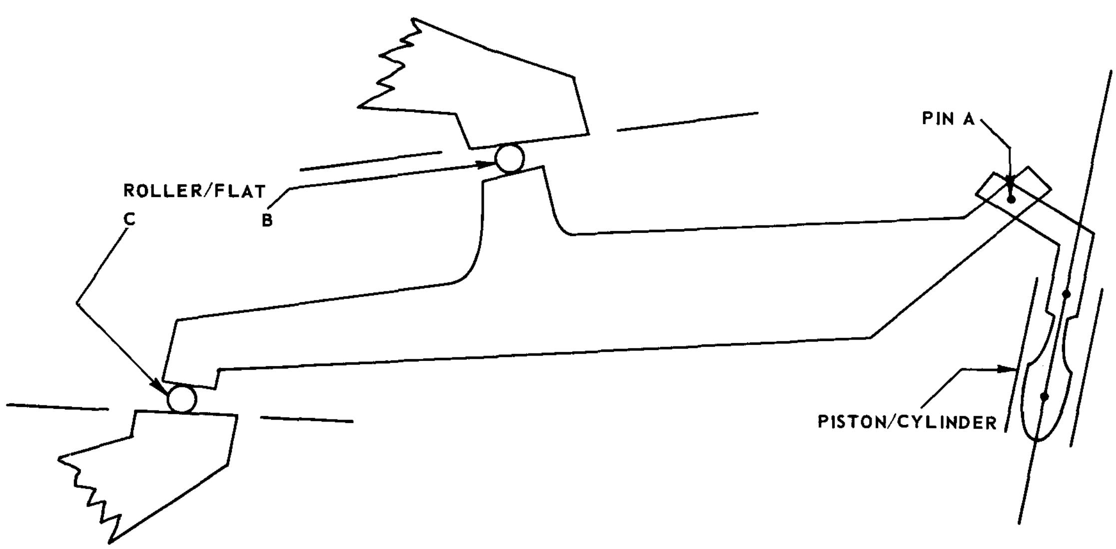

Figure 12. An Example of A Complex Loop in Equilibrium (Roller/Flat-Roller/Flat-Pin-Piston/Cylinder)

A basic concept used in the position solution algorithm is that of a "body". A "body" is a set of 3 or more points fixed relative to each other. Throughout the solution process, whenever a point has been determined, a check is made to find out if 3 noncollinear points have been solved for the body containing the particular point. If 3 such points are found, all remaining points of the body are solved for because of their constant relationship to the "known" points.

The position solution algorithm that is used in this system is based on the approach described in the KAM system but with several modifications in addition to the different method of loop solution discussed earlier. The position solution algorithm is a search and solve procedure driven by a list of constraints whose positions have been determined. Initially, a list of input constraints, those constraints whose positions are determinable from the given input motion to the linkage is created. Also included in this list are all constraints which are fixed to some external mounting or frame (which, incidentally defines the reference system of the linkage). For each constraint on the list, each loop referencing this constraint is investigated to see if there is enough information at the current stage in the search procedure by which to solve the loop. If a loop is solved, all the constraints of the loop are inserted at the end of the list of the constraints (if they do not already exist on the list). As each constraint becomes known (its position is solved), both bodies containing this constraint are searched (for 3 noncollinear known points) in an effort to fix the positions of the bodies. When a body's position is determined,the coordinates of all points on the body are calculated and all constraints involving this body are inserted on the list of constraints. Control passes back to processing the next constraint on the list. If at the end of the list any constraint's position has been determined as a result of the search down the list, the process is repeated from the beginning of the list. Otherwise, the solution procedure for the position of the linkage is completed. The linkage has been determined as much as possible and appropriate messages are printed out if any unknown positions exist.

GRAPHICAL DISPLAY INTERFACE

The graphical interface represents the entire range of options available to the user of the system during any session at the display scope. The program provides the following major capabilities: definition of a linkage, display linkage, manipulate linkage display, and exercise linkage.

The user defines a linkage by means of linkage modelling statements which may be punched on cards and entered by a batch run or typed in from the CRT keyboard. The statements contain the logical information necessary to describe a model of the linkage to the computer. A linkage is defined by bodies (links), constraints, loops, bodylines, points and initial positioning statements. Bodies define the sets of points which remain fixed relative to each other. Constraints describe the physical connections between two bodies or links. Loops define the ordered sets of constraints required for the position analysis and initial placement of the linkage's bodies and constraints. Bodylines describe the line segments to be displayed on the bodies of the linkage. Different priority lines may be defined permitting different levels of complex displays for the linkage. The simplest pictures might correspond to "stick" diagrams. Point statements define the coordinate values of points of the linkage. The coordinates are defined within the reference systems of the bodies they lie on. Initial positioning statements define the coordinates of points within the housing reference system which represent the inputs to the linkage. In addition, approximate position statements are used to resolve ambiguous solutions when loops are solved for the first time. The ambiguity is resolved by choosing the solution closest to the coordinate values of the point determined by the approximate position statement.

The modelling statements for both batch run and CRT entry contain the same data in slightly different formats. The advantage of the CRT mode is that the user is informed immediately of any errors in syntax and logical content that he may have made for each statement as he enters it. The disadvantage of the CRT mode rather than batch is that it becomes tedious and very time consuming for large cases involving hundreds of modelling statements. An example of the input for a linkage containing the complex loop depicted in Figure 12 is given by the modelling statements shown in Figure 13. Each statement has a corresponding integer code entered in the first two columns by which the input processor recognizes it. In the illustration, codes 1,2,

```
1ACCPS8F3
2BA       1.  2.  AA      2.  AA1     AA2             AB      2.  AB8
0AB9
2BB       0.  4.  AC      1.  AC15                    AB      3.  AB11
0AB12     AB13
2BC       0.  4.  AD      1.  AD23                    AC      3.  AC20
0AC21     AC22
2BD       1.  2.  AA      2.  AA3     AA4             AD      2.  AD28
0AD29
2BE       0.  2.  AC      2.  AC16    AC17            AE      2.  AE34
0AE33
2BF       1.  7.  AE      3.  AE30    AE31    AE32    AA      3.  AA5
0AA6      AA7
3LP1      1.  3.  BA      BB      BE
3LP2      8.  4.  BF      BE      BB      BC
4AA       1.
5AB       L01     1.  6.  AB14    AB10    AB11    AB12    AB13    AB11
5AB       L02     1.  2.  AB8     AB9
5AC       L03     1.  6.  AC18    AC19    AC20    AC21    AC22    AC20
5AC       L04     1.  2.  AC16    AC17
5AD       L05     1.  5.  AD23    AD24    AD25    AD26    AD27
5AD       L06     1.  2.  AD28    AD29
6AA1        4.0000  0.6700  0.5800
6AA2        4.0000  0.6700  1.7800
6AA3        0.7300  2.5000  0.5800
6AA4        0.7300  2.5000  1.7800
6AA5        3.4800  2.0000  1.7800
6AA6        3.4800  1.0000  1.7800
6AA7        3.4800  1.0000  0.5800
6AB8        0.0000  0.0000 -0.6000
6AB9        0.0000  0.0000  0.6000
6AB10      -1.3200  0.0000  0.0000
6AB11      -1.3200  0.3300  0.0000
6AB12      -1.5000  0.3300  0.1800
6AB13      -1.1400  0.3300  0.1800
6AB14       0.0000  0.0000  0.0000
6AC15      -0.8000  0.0000  0.0000
6AC16       0.0000  0.0000  0.6000
6AC17       0.0000  0.0000 -0.6000
6AC18       0.0000  0.0000  0.0000
6AC19      -1.5600  0.0000  0.0000
6AC20      -1.5600  0.2500  0.0000
6AC21      -1.7400  0.2500 -0.1800
6AC22      -1.3800  0.2500 -0.1800
6AD23       1.1900 -1.2500  0.0000
6AD24       1.1900 -0.8000  0.0000
6AD25       0.5000 -0.8000  0.0000
6AD26       0.5000  0.0000  0.0000
6AD27       0.0000  0.0000  0.0000
6AD28       0.0000  0.0000 -0.6000
6AD29       0.0000  0.0000  0.6000
6AE30       0.0000  1.0000  0.0000
6AE31       0.0000  0.0000  0.0000
6AE32       0.0000  0.0000 -1.2000
6AE33       0.0000  0.0000 -1.2000
6AE34       0.0000  0.0000  0.0000
 7  BB      AB      3.
 0  AB11      2.6800  1.0000  1.1800
 0  AB12      2.5000  1.0000  1.3600
 0  AB13      2.8600  1.0000  1.3600
 7  BC      AD      1.
 0  AD23      1.9200  1.2500  1.1800
 7  BE      AE      2.
 0  AE33      3.4800  1.0000  0.5800
 0  AE34      3.4800  1.0000  1.7800
 7  BF      AE      1.
 0  AE30      3.4800  2.0000  1.7800
10  BB      AC      1.
 0  AC15      2.6250  1.0000  1.1875
```

Figure 13. Example of Input Modelling Statements

3,4,5,6,7, and 10 represent linkage, constraint, loop, body, bodyline, point, exact position and approximate position statements, respectively, and code 0 designates a continuation of the previous statement. Codes 8 and 9 are used to input rotational and sliding motions and are presently input only at the CRT keyboard. This description has been given to provide the reader with a general idea of how a linkage is input and for the purpose of this discussion it is not necessary to describe the exact formats required.

Once the linkage model has been input to the computer, the linkage may be modified on-line from the CRT by typing in altered modelling statements. Several linkages may be stored in the library of linkage cases within the computer and each may be analyzed one at a time from the CRT terminal. The library capability permits updating and addition of linkage cases both on-line and off-line.

The display linkage capability allows for displaying different complexity pictures of a linkage. To illustrate this capability, an example is given in Figure 14. A detailed display of two particular bodies connected by a roller on flat constraint is shown along with a "stick" diagram of the entire linkage.

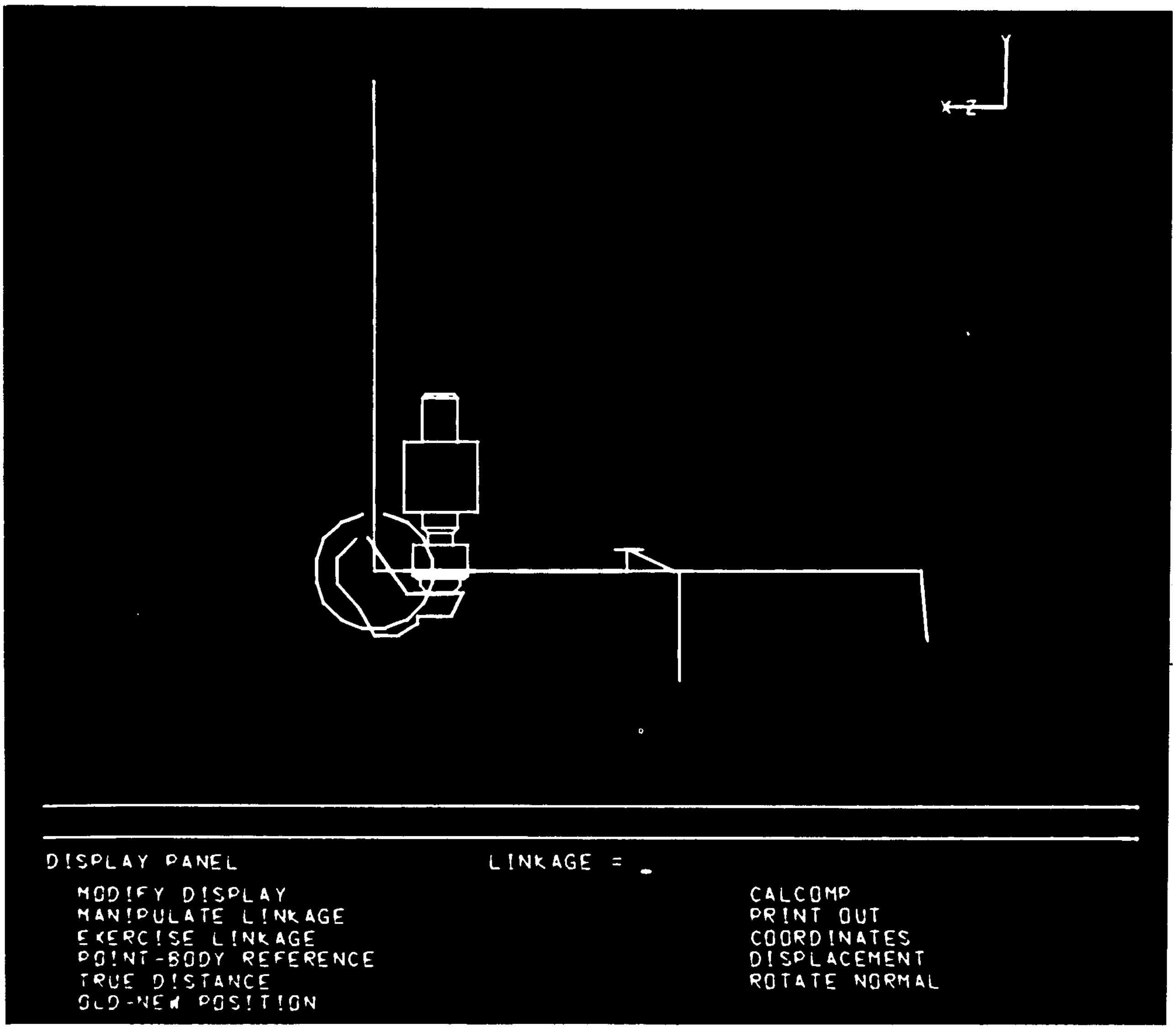

Figure 14. "Stick" Diagram and Roller/Flat Constraint

This display corresponds to the initial position (zero displacement) of the particular linkage. Different complexity diagrams of particular bodies may be intermixed throughout the display. "Stick" diagrams would most likely be modelled initially containing the minimum number of functional points. More detailed pictures would be defined should it be desirable to visualize approximate surface definitions of the linkage. Individual lines or bodies or levels of priority display may be removed from the display of any linkage. Any line contained in the display of a linkage may be detected on with the light pen. Upon detection, the linkage name, body name, line name, and priority level of display for that line will be displayed on the screen. In addition, the names of points and coordinate values of the points may be requested to be shown on the screen. The capability for hard copy output is also provided. CalComp plots may be obtained for any linkage display. A printout listing may be requested for the coordinate values of any and all points for any exercised position of the linkage.

A split screen feature exists to enable the simultaneous display of a linkage from two different orientations. This feature contributes to the improved visualization of the linkage as can be seen in Figure 15. The displayed linkage may be rotated

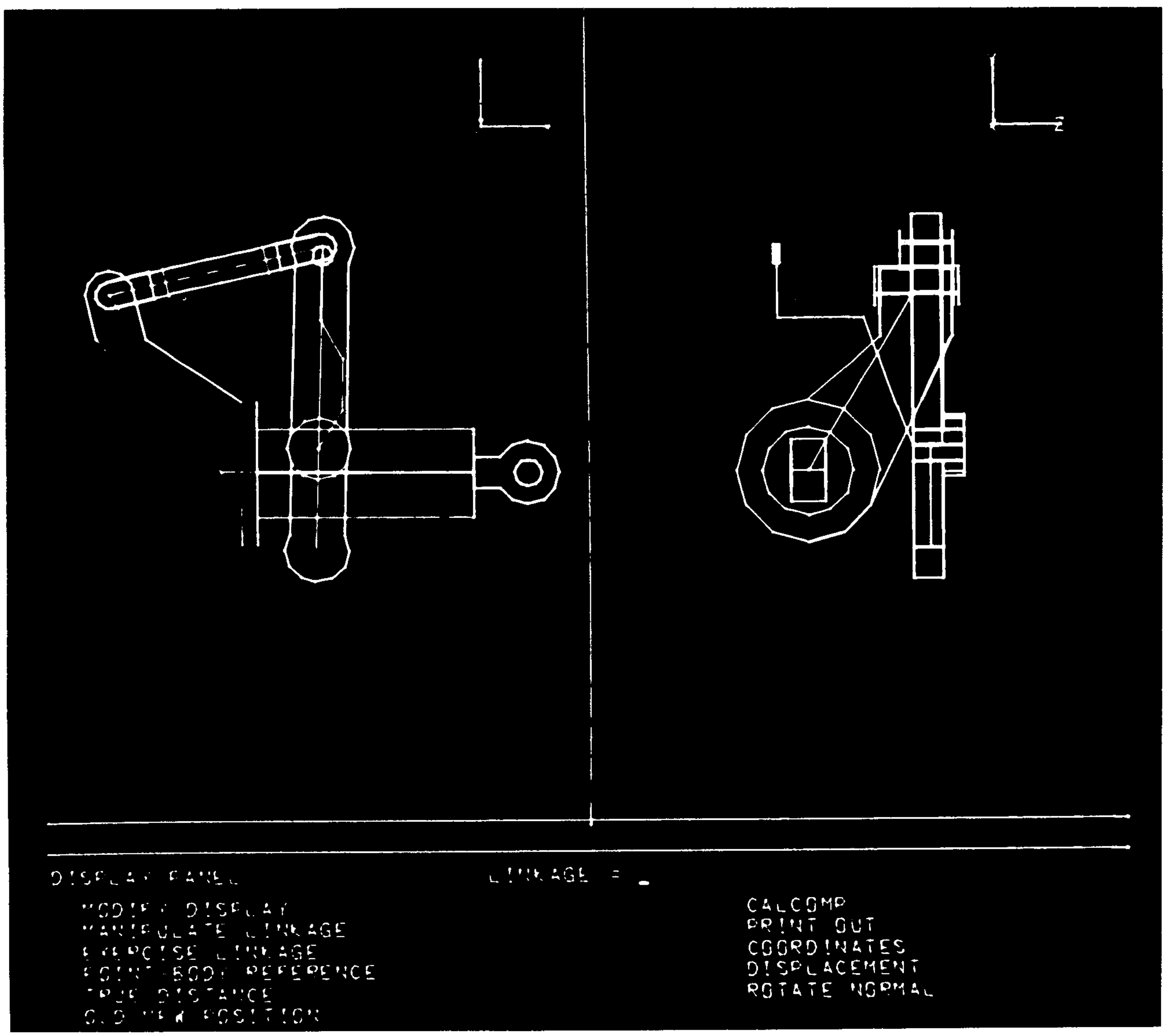

Figure 15. Split-screen Showing Two Orthogonal Views of a Linkage

by any specified angle about the horizontal, vertical, diagonal, or normal axes of the display screen. Translation of the linkage display may be done in any amount up, down, left, right, or in a diagonal direction. The linkage display may be magnified by any amount. All manipulation capabilities, rotation, translation and magnification, may be applied separately to the left and right sides, or to both sides together.

The exercise capability permits the user to enter input displacements and request the resultant position of the linkage to be calculated and displayed on the screen. Input motions may be specified by angular or linear displacement of a constraint position or by actual coordinate positions of transformed points. To aid in the visualization of the displacements of the linkage' components, two successive positions of a linkage may be displayed together on the screen. This feature is considered desirable because displacements are readily recognizable when viewed by comparing superimposed pictures containing only slightly altered positions. Several examples illustrating this feature and showing computer solutions using the mathematical approach described above for different linkage configurations are given in Figures 16 through 18. The figures show a linkage involving a simple

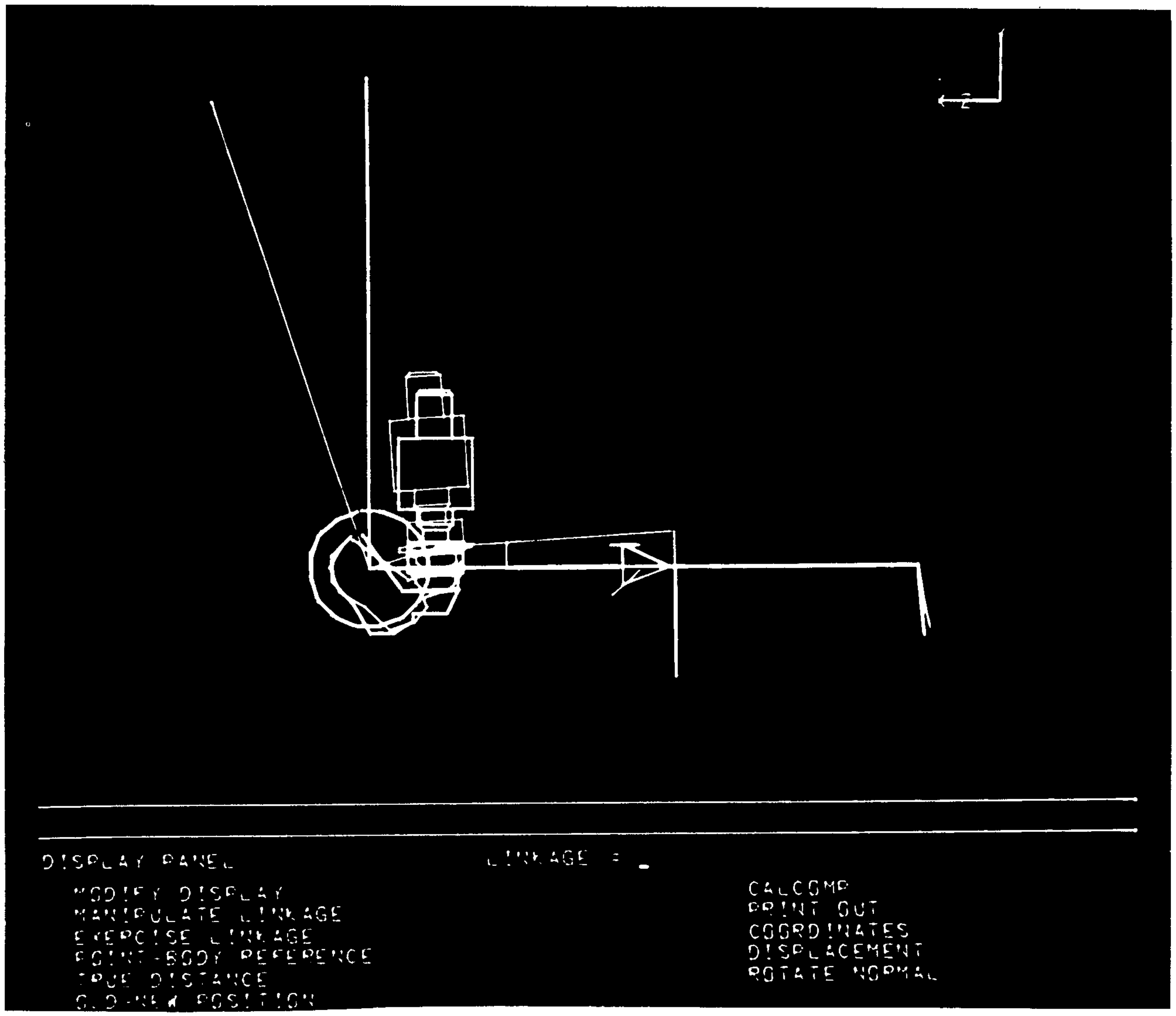

Figure 16. Two Superimposed Exercised Positions of a Linkage with Detailed View of Roller/Flat Constraint

roller/flat constraint, a linkage with a complex loop of the type described in Figure 12, and a complex "droop reset" linkage containing ten links and twenty-two constraints. This third linkage has five inputs and includes a four-bar loop and three complex parallel loop types, one of which requires a double iteration procedure for its position solution. A sample printout of the exercised position for the linkage illustrated in Figure 17 is given in Figure 19. The coordinates of each input point on every body have been printed out in this case. Several calculations may be performed at the user's request with the results displayed on the screen. These include the following: calculate linear or angular displacement of a point or lever from its previous position, obtain coordinates of any specified point in dimensioning system of any other body, and calculate true distance between any two points on the screen.

CONCLUSION

As it currently exists, the Mechanical Linkage Design System can be used initially to exercise the preliminary design of linkages (represented by "stick" diagrams) and to determine their correct functional operation. During this phase, the system would be used to check lever

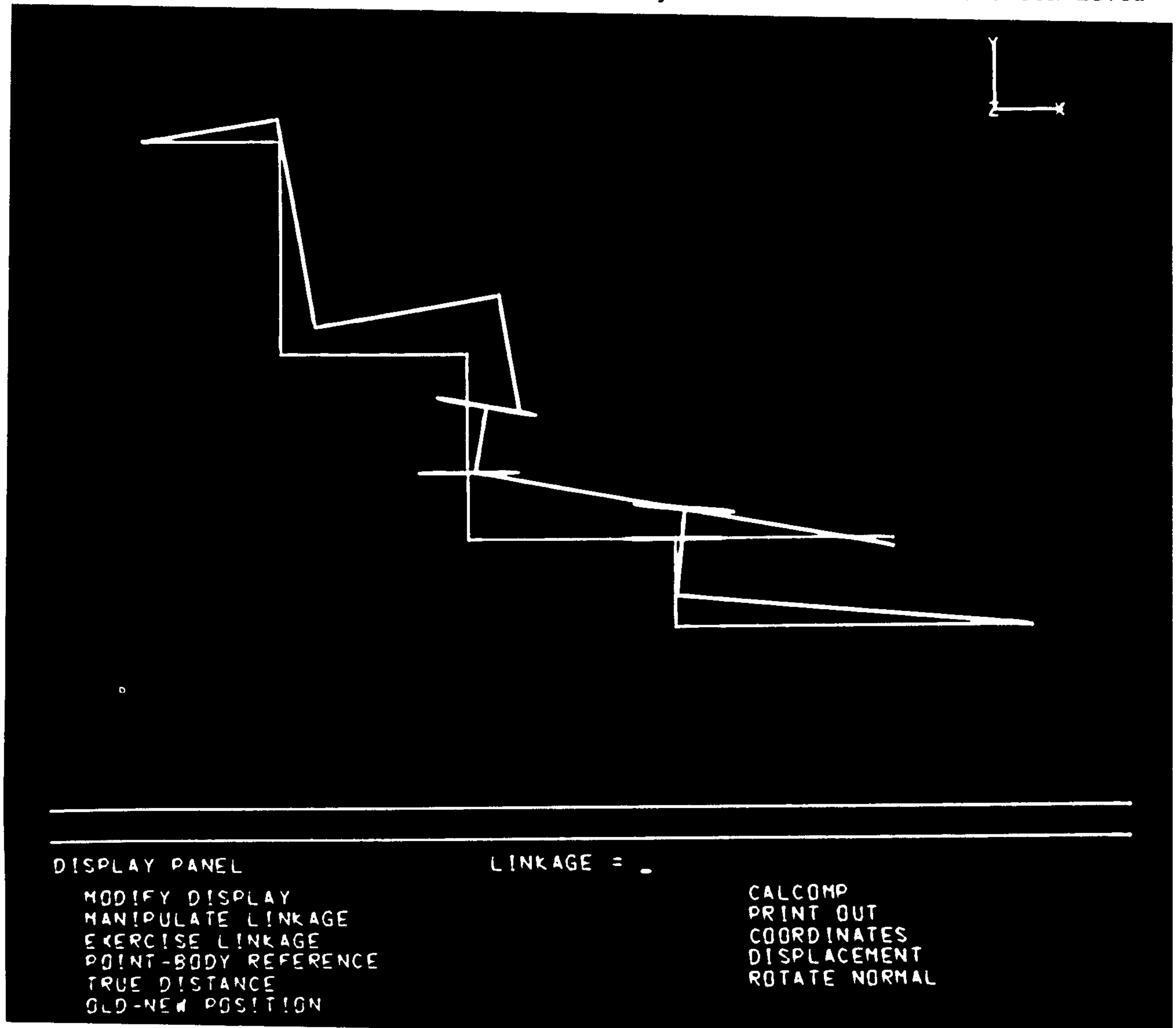

Figure 17. Two Superimposed Position Solutions of a Linkage Containing a Complex Parallel Loop

ratios of the linkage, that is, lengths of "sticks" necessary to meet the desired requirements, and to modify the lengths of "sticks" until a satisfactory operation is obtained. As the design progresses, body surface definitions (implicitly defined by "bodylines") can be added and interferences can be checked out at gross levels by rotating the linkage and viewing the linkage from various directions.

The system does not have a built-in design synthesis functional capability. That is, the program cannot determine which types of constraints and which series of constraints should be used to achieve a desired output from a specified input. The user must define the particular constraints and designate (by loops) the correct sequences of constraints defining the motion of the linkage. He may then test the linkage by exercising it to see if its performance is suitable. Should changes be necessary in the design of the linkage, modifications may be made at the graphics terminal. Thus, the system provides the tools by which a trial and error procedure may be used for designing a linkage.

The Mechanical Linkage Design System described in this paper fully utilizes all available interactive graphics features as a part of a computer-aided design system for linkage

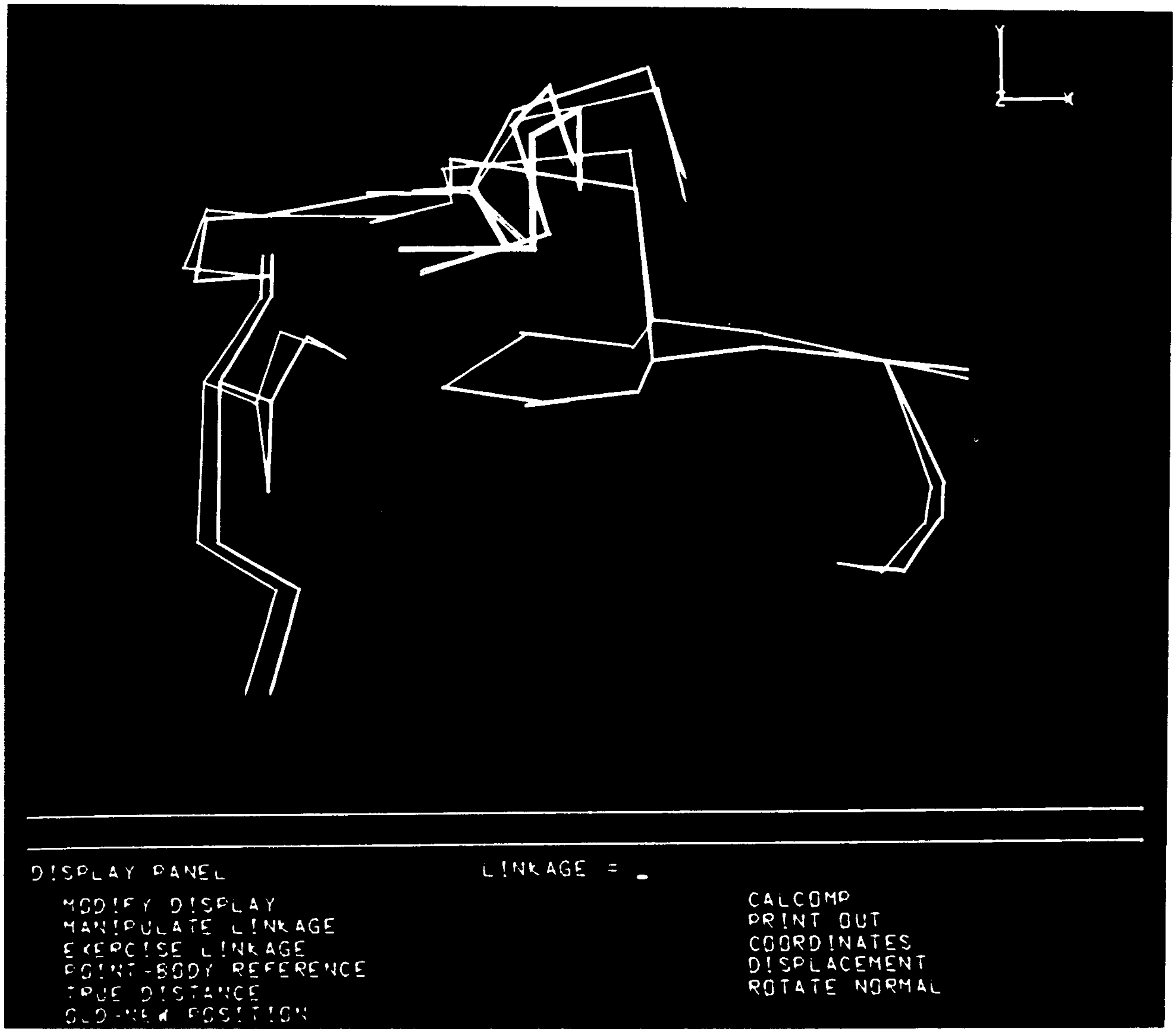

Figure 18. Two positions of a complex linkage with ten links and twenty-two constraints are shown. The linkage configuration consists of a four-bar loop and three complex parallel loops.

```
   POSITION OF POINTS
 OPTION 1 CHOSEN - PRINT ALL POINTS OF LINKAGE
LINKAGE ACCPS8F3
 PRINTED CARD OUTPUT
ROTATE BODY AD        ABOUT CONSTRAINT BD       BY 10.      DEGREES
ROTATE BODY AB        ABOUT CONSTRAINT BA       BY -5.      DEGREES
*************************************************************
BODY AA
*************************************************************
BODY AB
BODYLINE L01          PLEVEL =    1   NUMBER OF POINTS =    6
   POINT        X        Y        Z
AB14         4.0000   0.6700   1.1800
AB10         2.6850   0.7850   1.1800
AB11         2.7138   1.1138   1.1800
AB12         2.5345   1.1295   1.3600
AB13         2.8931   1.0981   1.3600
AB11         2.7138   1.1138   1.1800
BODYLINE L02          PLEVEL =    1   NUMBER OF POINTS =    2
   POINT        X        Y        Z
AB8          4.0000   0.6700   0.5800
AB9          4.0000   0.6700   1.7800
*************************************************************
BODY AC
BODYLINE L03          PLEVEL =    1   NUMBER OF POINTS =    6
   POINT        X        Y        Z
AC18         3.4800   0.9706   1.1800
AC19         1.9458   1.2533   1.1800
AC20         1.9911   1.4992   1.1800
AC21         1.8141   1.5318   1.0000
AC22         2.1682   1.4666   1.0000
AC20         1.9911   1.4992   1.1800
BODYLINE L04          PLEVEL =    1   NUMBER OF POINTS =    2
   POINT        X        Y        Z
AC16         3.4800   0.9706   1.7800
AC17         3.4800   0.9706   0.5800
*************************************************************
BODY AD
BODYLINE L05          PLEVEL =    1   NUMBER OF POINTS =    5
   POINT        X        Y        Z
AD23         2.1190   1.4756   1.1800
AD24         2.0408   1.9188   1.1800
AD25         1.3613   1.7990   1.1800
AD26         1.2224   2.5868   1.1800
AD27         0.7300   2.5000   1.1800
BODYLINE L06          PLEVEL =    1   NUMBER OF POINTS =    2
   POINT        X        Y        Z
AD28         0.7300   2.5000   0.5800
AD29         0.7300   2.5000   1.7800
*************************************************************
BODY AE
```

Figure 19. Example of Printout for Position of Linkage

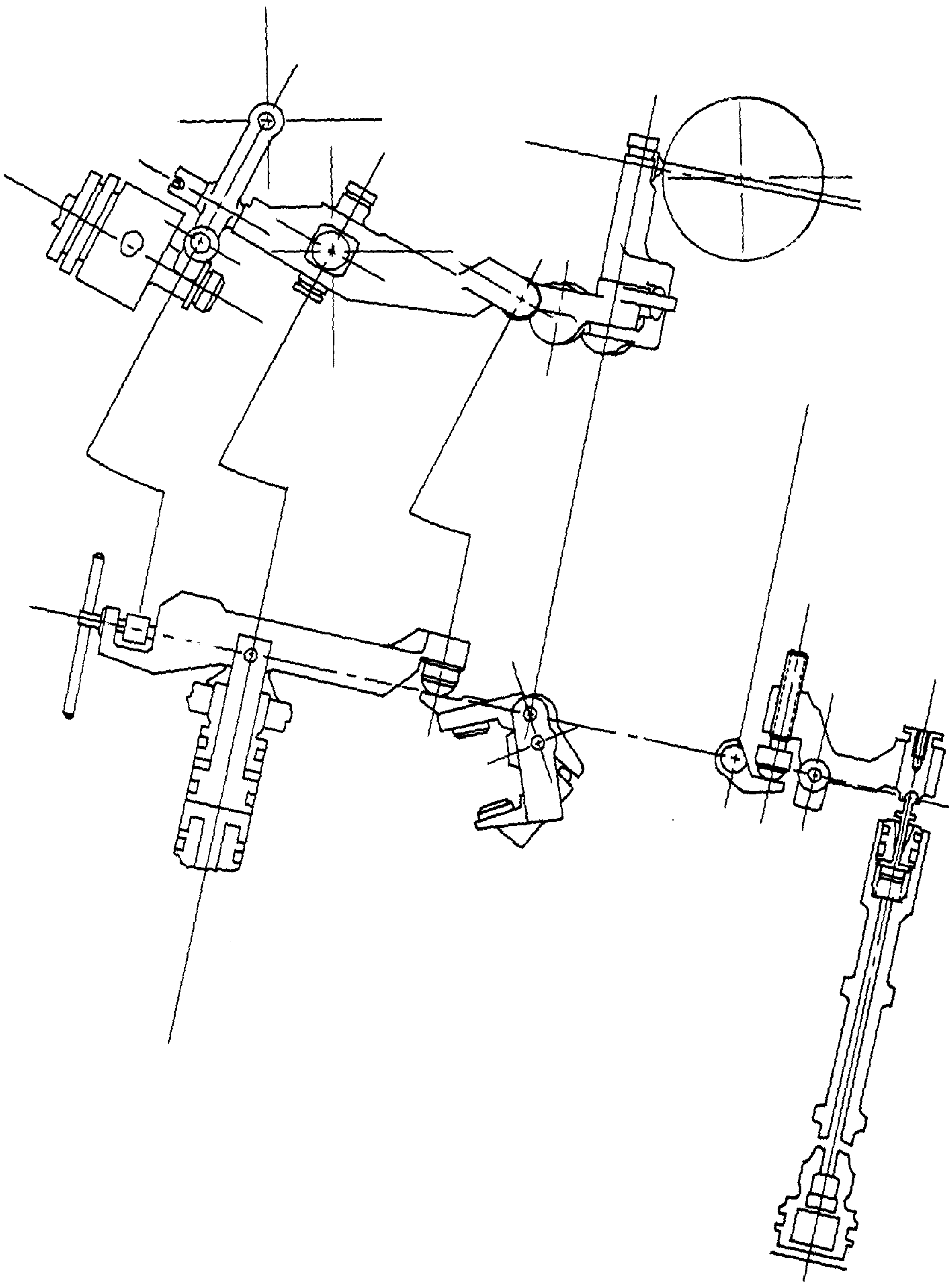

Figure 20. Mechanical Linkage, Design Layout

design. The capabilities of the system are being expanded to broaden the types of linkages which can be analyzed by the position solution algorithm and to investigate the addition of advanced computer aids. A hidden line program based on Loutrel's solution to the hidden line problem (3) has been developed and is being tested for its application to linkage design. In addition, the investigation of a tablet as both an input device and a digitizer to facilitate the description of surface definitions of bodies of the linkage has been undertaken. A basic capability incorporating the position analysis of a linkage with regard to the forces propagated and the consideration of deflections due to the elasticity of the linkage components is also being investigated.

The system's performance has been evaluated by using it to analyze several mechanical linkages representative of a typical linkage system. An example of a particular subset of such a linkage is illustrated in the design layout in Figure 20. Estimates of the man-hours and computer time required were made to compare the previous method of design and drafting analysis with a method considering an environment in which the interactive graphics linkage design system is fully utilized. Based on the use of a digitizer to reduce the manual effort involved in generating the input required by the system, it was determined that the Mechanical Linkage Design System would result in a savings in man-hours of approximately 30%. Deducting the costs due to the increased computer time, a total cost savings of 25% was predicted. Additional estimates based on anticipated results after the implementation of major analytical capabilities, especially regarding forces and deflections, indicated an overall savings in man-hours of 45% and a cost savings of 35%. These statistics serve to illustrate the fact that the system which has been developed is a significant technological advancement in the field of computer-aided design over previous methods of designing mechanical linkages which relied heavily on manual drafting operations for the various check out procedures of a linkage design.

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