Cartoons: An Initial Investigation of Animated Graphs

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1. Motivation

Many structures and processes representable as graphs are well known. There are, however, structures the internal relationships of which change (predictably or not) over time. An example in computer science would be an adaptive process. In this case the state transition graph changes in response to certain stimuli.

Another recent development presents a similar structural problem. The bubble memory is a two-dimensional matrix of bubbles which can be moved by altering the position of certain magnets. The possible bubble movements are representable as a maze of changing characteristics. If one represents such a movable maze as a graph with changing connectivity, problems of computation via bubble movements might become clearer.

This paper attempts to provide a formalism in which such problems can be framed.

2. Cartoons defined

For the purpose of this paper let a graph, G, be defined as

$$G=\langle V,E,\phi:E\to\{\{v_1,v_2\}\mid v_1,v_2\in V\}\rangle$$

where V is a finite set called the vertex set, E is a finite set called the edge set and ϕ is a function mapping the edges to unordered pairs of vertices. A di-graph, D, is

$$D=\langle V,E,\phi:E\to\{\langle v_1,v_2\rangle\mid v_1,v_2\in V\}\rangle$$

where V and E are as above but ϕ maps E into ordered pairs of vertices. (1)

A cartoon, C, is an animated graph. That is $C=\langle F,A\rangle$ where F is a set of graphs with common vertex set,

$$F=\{\langle V,E_0,\phi_0\rangle,\langle V,E_1,\phi_1\rangle,\ldots,\langle V,E_k,\phi_k\rangle,\ldots\},$$

and A is an infinite sequence of elements of F $A:N\to F$. F is called the frame set and A is called the animation function. There are further restrictions on A and F:

1) all elements of F must be directed or all elements of F must be undirected. (The former results in a directed cartoon or di-cartoon.)

2) A must be surjective.

Each element of F is called a frame. For convenience label $A(t)=\langle V,E_t,\phi_t\rangle$ and call it frame t.

Intuitively one can think of a cartoon as a graph whose connectivity relationships change over a time parameter. For the purpose of this initial investigation the change is discreet, however. Thus, it is clear that F is countable and may possibly be finite.

3. Sub-cartoons and editings

There are two ways of looking at parts of a cartoon. The first is to look at the subgraph of a subset of the vertices over the entire sequence. The second is to look at only a subsequence within the cartoon.

A cartoon $D=\langle F^*,A^*\rangle$ is a subcartoon of a cartoon $C=\langle F,A\rangle$ iff $\forall n\in N$ $A^*(n)$ is a subgraph of $A(n)$. This is similar to taking a close-up or cropping of a moving picture.

A cartoon $D=\langle F^*,A^*\rangle$ is an editing of a cartoon $C=\langle F,A\rangle$ iff $\exists\sigma:N\to N$ such that

a.) $A^*(n)=A(\sigma(n))$

and b.) $m<n\to\sigma(m)<\sigma(n)$

and c.) (i) σ is injective

or (ii) $\exists k\in N:\sigma|_{\{n|n\le k\}}$ is injective and for all $n>k$ $A^*(n)=A^*(n-1)$.

Intuitively an editing is a subsequence of the cartoon. The definition does, however, admit the possibility of a finite subsequence followed by infinitely many repetitions of the last frame (c(ii)).

From this appears the desirability of defining finite and infinite cartoons. Even though all cartoons are infinite sequences, let a finite cartoon, C, be defined as one such that

$\exists k\in N$: for all $n>k$ $C(n)=C(n-1)$.

All other cartoons are infinite, i.e.

for all $k\in N$ $n>k$ $C(n)\ne C(n-1)$.

One can further partition the cartoons into rational and irrational. A rational cartoon is one which after finitely many steps becomes periodic. That is, a cartoon, C, is rational iff

$\exists k\in N$ $\exists m\in N$ for all $n\in N$: $n>k\to A(n)=A(n+m)$.

A cartoon is irrational if it is not rational.

Thm 1. All finite cartoons are rational.

Pf. In the definition of "rational" let m=1.

Thm 2. Let C= F,A be a cartoon.

(i) C rational $\to$ F finite.

(ii) $|\{C|C \text{ rational cartoon}\}|=\aleph_0$

(iii) $|\{C|C \text{ cartoon}\}|=\aleph_1$

Pf. (i) Assume F infinite. F contains infinitely many distinct elements. Since A is surjective, the set $A(N)=F$. $\forall k\in N$ $\exists n\in N$ $n>k$ and for all $m\in N$ $m<n\to A(m)\ne A(n)$.

C not periodic.

C not rational. Contra

(ii) and (iii) as with rational and real numbers.

Note that editings for which c(ii) in the definition of editings hold are indeed finite and, therefore, can reasonably be called finite editings. If c(ii) does not hold, then the editing is infinite but may be rational or irrational.

It will be useful later to have the concept of smooth editings. An editing is smooth (using the symbols from the "editing" definition) iff

for all $n>0$ $\sigma(n-1)=\sigma(n)-1$.

4. Isomorphism defined

Two cartoons C=<F,A> and D=<F',A'> are isomorphic iff for all $n\in N$ A(n) is graph isomorphic with A'(n).

5. Edge freedom

A useful concept for study is that of the amount of change in the connectivity relationship from frame to frame. To capture this let the degree of edge freedom, μ, be defined as follows:

let $S_t=\phi_t(E_t)=\{\{v,v'\}|e\in E_t,\phi_t(e)=\{v,v'\}\}$

then $\mu=\max_{0\leq t}\{n|n=|(S_t\cup S_{t+1})\cap(\bar{S}_t\cup\bar{S}_{t+1})|\}$

$=\max_{0\leq t}\{n|n=|S_t-S_{t+1}|+|S_{t+1}-S_t|\}$

(This definition does not work for multi-cartoons).

What are the effects of different degrees of edge freedom? This question can be approached from many different view points corresponding to the question: "effects on what?" For example, consider frame-wise connectedness. (A cartoon is framewise connected if each frame is a connected graph.)

Is a connected graph classifiable by the minimum strictly positive degree of edge freedom of the cartoons generable from that graph? The following simple result shows that any such classification scheme is not very interesting.

Thm 3. For every finite connected graph, G, with more than two vertices, there exists a cartoon, C=<F,A>, such that

(i) A(0)=G and

(ii) for all $n\in N$ A(n) connected and

(iii) for all D=<F',A'>, (D editing of C and $|F'|>1)\rightarrow\mu(D)=1$ and

(iv) G simple $\rightarrow$ for all $n\in N$ A(n) simple.

Pf: Case (a) G contains a circuit $e_1 e_2 e..e_1$.

C=<{G,G−e_1}, A > where A(0)=G,A(1)=G−e_1, A(2)=G, A(3)=G−e_1,..., A(2t)=G, A(2t+1)=G−e_1,....

This satisfies all of (i) - (iv).

Case (b) G contains no circuits $\rightarrow$ an edge, $\hat{e}$, can be added maintaining simplicity:

C=<{G,G+$\hat{e}$}> where A(0)=G, A(1)=G+$\hat{e}$,..., A(2t)=G, A(2t±1)=G+$\hat{e}$,...

C meets the requirements (i) - (iv).

This shows that all graphs of greater than two vertices can generate a cartoon maintaining connectedness with not only $\mu=1$ but $\mu=1$ for all editings with at least two distinct frames. This makes this classification uninteresting.

Slightly more interesting is the maximum μ which maintains framewise connectedness and framewise simplicity and maintains a constant spanning tree. This is considered in the following theorem.

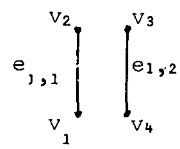
Thm 4. C=< F,A >
and A(0)=G=< V,E,ϕ> simple and connected
and ($\exists$S spanning tree of G such that
$\forall n$ S subgraph of A(n))
and C framewise simple
and C framewise connected

$\rightarrow\mu(c)\leq\left(\frac{|v|!}{(|v|-2)!\,2}-|v|+1\right)$

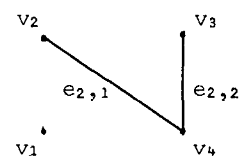
Pf: This becomes obvious when one realizes that $\frac{|v|}{(|v|-2)!\,2}$ is the cardinality of edge set of the complete graph over V and $|v|-1$ is the cardinality of the spanning tree. Equality occurs if $\exists t$ A(t) is complete and (A(t-1) or A(t+1) a tree).

6. Edge progression and connectivity

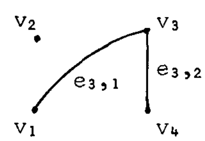
Let an edge progression in a frame be defined as in Busacker and Saaty -- a sequence of adjacent edges with an implied directionality. [1] A p,q- edge progression over k frames of a cartoon (p, q, k $\in N$) is such a sequence of edges where a minimum of p edges and a maximum of q edges appear in each frame over a sequence of k consecutive frames beginning with A(0). For a progression with no maximum, the notation is p,*-edge progression. One can also speak of a p,q-edge progression over all frames. Consider the following cartoon, C:



G_0



G_1



G_2 $\qquad G_3=G_2=G_4=G_5=$ ....

In C there exist no simple 2,2-edge progressions over three frames because G_0 will not support length 2 progressions without oscillations. There do exist, however, simple 1,1- and 1,2-edge progressions over three frames. For example, starting at v_1: $e_{1,1}$ $e_{2,1}$ $e_{2,2}$ $e_{3,1}$ is a simple 1,2-edge progression. In fact, it is a simple 1,2-circuit over three frames.

This last example is even more. It is (following standard terminology for graphs) a 1,2-Euler circuit and a 1,2-Hamiltonian circuit over three frames.

Two vertices are weakly p,q-connected iff $\exists$ k$\geq$ 1 such that there exists a simple p,q-edge progression over k frames between those two vertices (in an indirect cartoon, in either direction). They are strongly p,q-connected if ∃ k>1 such that simple p,q progressions over less than k frames exist in both directions.

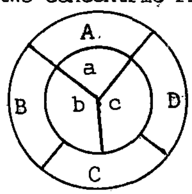
For example in the last figure, v_4 and v_2 are strongly 0,1-connected but they are not 1,*-connected even weakly. v_1 is strongly 1,1-connected with v_2 and weakly 1,1-connected with all vertices.

Clearly, the weakest connectivity is weak 0,*-connectively. Vertices which are not weakly 0,*-connected are said to be mutually unreachable. Two vertices, v_1 and v_2 are totally disconnected iff for all smooth editings of the cartoon v_1 and v_2 are mutually unreachable. Two vertices are time independent connected iff they are not totally disconnected. That is iff ∃ an editing in which they are weakly 0,*-connected.

A cartoon is (weakly,strongly) p,q-connected iff all pairs of vertices are (weakly, strongly) p-q-connected. From this one could define many other connectivity properties and structures, e.g. p,q-components. This is left for future papers.

7. Two problems.

a. Here are two concentric rings:

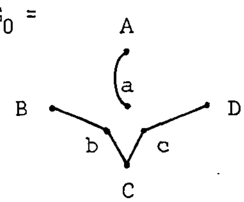


A,B,C, and D are 90° sectors in the outer ring; b, c, are 135° sectors in the inner ring, and a is a 90° sector in the inner ring. The outer ring discreetly shifts clockwise 90° per time period, t. The inner ring shifts 90° counter clockwise per time, t.

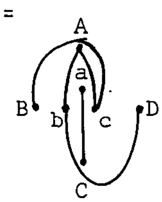
Find all the methods, starting in the outer ring, to cover all sectors exactly once, one sector every time period. The divisions between sectors on a given ring are impassible.

Solution: Find all 1,1-Hamiltonian progressions in this cartoon starting at A,B,C,D:

G_0 =



G_1 =

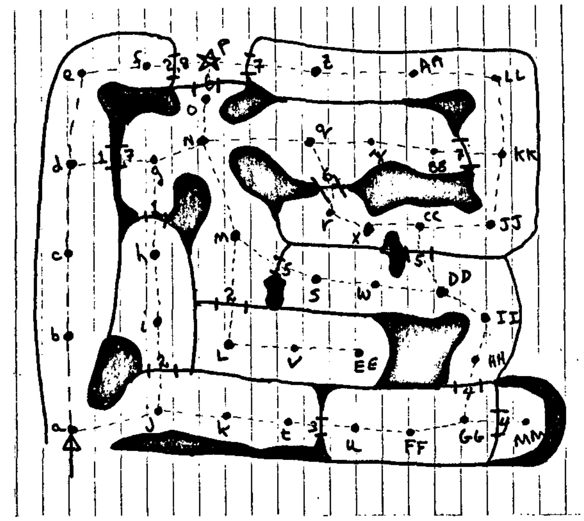


AaCcBbD
AaCbDcB

BbDcAaC
DcBdAaC

None from C

b. In the illustration below is a maze.



The only safe paths are on the dashed lines. The lettered dots signify the distance coverable in five seconds. You must go from one dot to another every five seconds and you may not retreat to a dot you were on five seconds previously. There are fourteen gates in the walls. Each has some digits associated with it. At time 15t all gates numbered with t open for 14.999+ seconds and then close. For example, the gate between "d" and "g" opens at times 15-29 and 105-119 seconds and remains closed at other times. Starting at point "a" get to point "p". All solutions?

Solution: The cartoon of maze has been deleted for space purposes. The problem is to show that a and p are weakly 1,1-connected starting at a.

Possible progressions:

1) a b c d, g h i, j k t, u FF GG, HH II DD, wsm, nop
2) a → DD as 1), CC x r, q n g, d e f, p
3) a → DD as 1), CC JJ KK, LL AA z, p

8. A useful theorem

From the first problem one might realize that all rational cartoons are reducible to a single finite directed graph preserving 1,1-edge progressions.

Pf: Let us assume the cartoon C is rational. Then it looks like the following sequence:

$$G_0\ G_1\ \cdots\ G_k\ G_{k+1}\cdots G_{k+m} G_{k+1}\cdots G_{k+m}\cdots$$

Let $G_0 = \langle V, E_o, \phi_o \rangle \quad V = \{v_1, v_2, \ldots v_n\}$

$E_t = \{e_{t,1}, e_{t,2}, \ldots, e_{t,j_t}\}$

Define $\hat{C} = \langle \hat{V}, \hat{E}, \hat{\phi} \rangle$ where

$\hat{V} = \{v_{0,1}, v_{0,2}, \ldots, v_{0,n}, v_{1,1}, \ldots v_{1,n}, \ldots, v_{k+m,n}\}$

$$\hat{E} = \bigcup_{t=0}^{k+m} E_t \cup \bigcup_{t=0}^{k+m} \{e'_{t,1}, e'_{t,2}, e'_{t,3}, \ldots, e'_{t,j_t}\}$$

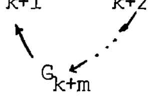
$\hat{\phi}$ such that

$$\hat{\phi}(e_{t,i}) = \begin{cases} \langle v_{t,w}, v_{t+1,x} \rangle & \text{for } t < k+m \\ & \text{if } \phi_t(e_{t,i}) = \{v_w, v_x\} \\ \langle v_{t,w}, v_{k+1,x} \rangle & \text{for } t = k+m \\ & \text{if } \phi_t(e_{t,i}) = \{v_w, v_x\} \end{cases}$$

$$\hat{\phi}(e'_{t,i}) = \langle v_{t,y}, v_{t',y'} \rangle \quad \text{where} \quad \hat{\phi}(e_{t,i}) = \langle v_{t,y'}, v_{t',y} \rangle$$

C is merely

$$G_0 \to G_1 \to G_2 \to \ldots \to G_{k+1} \to G_{k+2}$$



If a 1,1-edge progression exists over r frames between v_y and v_z in a smooth editing of cartoon C which begins at frame t and includes frame t+l-1 then and only then is there an arc progression from $v_{t,y}$ to $v_{t+r-1,z}$ in C. Thus we have constructed a <u>composite of C</u> relative to 1,1-edge progressions.

9. Conclusions

Many problems and processes find a paradigm in cartoons. Cartoon theory may provide a sensible of understanding these processes. However, many more concepts must be defined and investigated more deeply than done herein. Examples might be movement of cut points and the maintaining of planarity. Also there exist useful generalizations. For example, changing vertex sets might be allowed, or the animation map might be A: R → F and, therefore, continuous. This introductory investigation seems to lead to many possibilities.

Acknowledgements: Thanks to Dr. William Grosky who taught me graph theory and Dr. Miroslav Valach who showed me that there really might be some application after all.

REFERENCE

1. Busacker, R.G., and T.L. Saaty, Finite Graphs and Networks (New York:1965)