EVALUATION OF INCOMPLETE ELLIPTIC INTEGRALS BY GAUSSIAN INTEGRATION

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In certain problems of applied mathematics it is necessary to obtain numerical solutions of expressions of the form:

$$\int_{\theta_1}^{\theta_2} G\left[\theta,\ F(\phi, k),\ E(\phi, k)\right] d\theta \tag{1}$$

where G is finite, real and continuous on the interval (θ_1, θ_2), and

$$F(\phi, k) \equiv \int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}\ , \quad E(\phi, k) \equiv \int_0^{\phi} \sqrt{1-k^2 \sin^2 \phi}\ d\phi$$

are the Incomplete Elliptic Integrals of the First and Second Kinds, respectively. ϕ is defined as the amplitude and k as the modulus. In equation (1), both ϕ and k are functions of θ and $\left[0 \le \phi \le \frac{\pi}{2},\ 0 \le k \le 1\right]$. This essentially a special type of double integration problem; the integration of G will be discussed later.

There are three basic computer methods of evaluating the elliptic integrals:

1. Table look-up and interpolation
2. Series expansion
3. Polynomial approximation

For the first method the classic tables of Legendre[1] are available which give functional values of both integrals, to 9 decimal digits of accuracy, by one degree increments of the amplitude and the modulus. For a very restricted range of ϕ and k, table look-up with linear interpolation is feasible. For any amount of generality, a rather large table must be stored and non-linear interpolation used. This can become quite costly in both space and time in computers with 2000 (or fewer) word capacity.

There are several series expansions available for the second method.[2, 3] Two serious faults exist; the series in general converge rather slowly, and they involve high order terms of both k and some $f(\phi)$, which necessitates rather complex and extensive programming. It is probable that this method might be feasible on a large scale computer.

Apart from the disadvantages mentioned, neither of the above methods can be utilized in the integration of G. It would be desirable to find a method which also could be applied to the entire problem. This suggests the use of polynomials.

Of the several polynomial approaches available, only Gaussian integration possesses the unique advantage that for integration on the interval (-1, 1), if the n abscissae chosen are the zeroes of the Legendre polynomial $P_n(x)$, the difference between the approximating polynomial and the function is a minimum for the corresponding n ordinates. Thus

$$\int_{-1}^{1} f(x)\,dx \simeq a_1\, f(x_1) + a_2\, f(x_2) + \cdots + a_n\, f(x_n)$$

contains 2n arbitrary constants. For all functions of degree $\leq 2n-1$, the error is zero. An exact expression for the error involves evaluating the 2nth derivative of f(x);[4] this appears to be somewhat heroic. Tables of the zeroes (x_i) and the corresponding weight coefficients (a_i) are found in reference (4), to 15 decimal digits for n=1 to 16.

A transformation on the interval (0, 1) is desirable for computer applications, hence

$$\int_{p}^{q} f(x)\,dx = (p-q)\int_{o}^{1} F(X)\,dX \simeq (p-q)\sum_{j=1}^{n} A_j\, f\left[(p-q)\,X_j + q\right] \qquad (2)$$

where

$$X_j = \frac{x_j + 1}{2}\,, \qquad A_j = \frac{a_j}{2}$$

Tables 1 and 2 give the transformed roots and coefficients for n=6 to ten decimal digits.

Table 1

$X_1 = 0.0337652429$	$X_4 = 0.6193095930$
$X_2 = 0.1693953068$	$X_5 = 0.8306046932$
$X_3 = 0.3806904070$	$X_6 = 0.9662347571$

Table 2

$A_1 = A_6 = 0.0856622462$
$A_2 = A_5 = 0.1803807865$
$A_3 = A_4 = 0.2339569673$

Applying equation (2) to the elliptic integrals gives

$$F(\phi, k) \simeq \phi \cdot \sum_{j=1}^{n} \frac{A_j}{\sqrt{1-k^2 \sin^2(\phi \cdot X_j)}}\,, \quad E(\phi, k) \simeq \phi \cdot \sum_{j=1}^{n} A_j \sqrt{1-k^2 \sin^2(\phi \cdot X_j)} \qquad (3)$$

It is seen from equations (3) that if the radicals for each X_j are computed and stored, the remaining evaluation of either elliptic integral involves only a summation of n quotients or n products. This tends to appreciably reduce both computing time and programming. However, a compromise is necessary. $F(\phi, k)$ has no upper bound and in general is of higher degree (polynomial-wise) than $E(\phi, k)$. Computation of both from the same polynomial will give greater accuracy for $E(\phi, k)$. Therefore, it is necessary to investigate only how the accuracy of $F(\phi, k)$ varies. In the particular problem which gave rise to this investigation, it was found that for n=6 (11th degree polynomial, Tables 1 and 2) six decimal digit accuracy is obtained over ϕ for $\left[0 \le k \le .5\right]$. For $\left[.5 < k \le .7\right]$, it is necessary to divide the range of ϕ into two equal parts, i.e., two iterations of the polynomial are required for at least the same accuracy. Similarly, for $\left[.7 < k \le .9\right]$, ϕ is divided into three equal intervals. The range $\left[.9 < k < .99\right]$ was not investigated; a large number of iterations would probably be necessary. (If only $E(\phi, k)$ is desired, the same polynomial with only one iteration gives an error of 2 in the 8th decimal place over ϕ for $\left[0 \le k \le 1\right]$.) For greater accuracy, either increase n, decrease the initial range of k, or both. Excluding double precision routines, n=10 is the largest practicable polynomial to use in this problem with machines of 10 decimal digit word length. All of the foregoing assumes that the necessary accuracy is available in the sine and square root subroutines used.

The elliptic integral program uses only 115 words of storage,* exclusive of the subroutines. An expression for the approximate time (t) in seconds to compute both integrals is

$$t = \left[.6 + .35n\right] i,^{*}$$

where $n(\not> 10)$ is the polynomial taken and i the number of iterations. This presupposes minimum access programming and subroutines with a fixed number of terms.[6] The simplicity of programming makes the method easily adaptable to any internally stored program machine and, with some restrictions, to card-programmed calculators.

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* Model 650 Magnetic Drum Computer (IBM)

Now to consider the integration of G. Using the same equation (2) as for the elliptic integrals, equation (1) may be written as

$$\int_{\theta_1}^{\theta_2} G \simeq (\theta_2 - \theta_1) \sum_{i=1}^{m} \sum_{j=1}^{n} A_i \left[f(X_i),\ F_i(X_j),\ E_i(X_j) \right] \tag{4}$$

where F_i and E_i are the same as equations (3) except that ϕ and k are now functions of the X_i. For m=n, the same polynomial is used, further conserving total storage space. A separate program, of course, is written for the integration of G.

It is suggested that a graph of G be drawn if its approximate behavior is not known. The range (θ_1, θ_2) can always be divided into sub-intervals if G is quite irregular.

REFERENCES

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