Optimum Recurrence Formulas for a Fourth Order Parabolic Partial Differential Equation

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The equation

$$\frac{\partial^4\psi}{\partial x^4} + \frac{\partial^2\psi}{\partial t^2} = 0 \qquad (1)$$

in the unit strip $0 \leq x \leq 1$, $t > 0$ represents the transient motion of a finite beam according to the Euler-Bernoulli theory. Approximate solutions can be obtained by covering the domain by a rectangular network with spacing Δx and Δt and marching out a finite difference approximation to (1). An explicit recurrence formula for this process was given by Collatz [1] and an implicit formula was given by Crandall [2]. Both of these formulas had truncation errors proportional to $(\Delta x)^2$. In this note we consider a more general class of recurrence formulas which contain those previously considered but also contain formulas which have truncation errors proportional to $(\Delta x)^4$ and $(\Delta x)^6$.

<u>Description of the formulas</u>. Let the dimensions of the network be $\Delta x = h = 1/M$ and $\Delta t = r/M^2$. Let the value of ψ at the point $P_{j,k}$ with coordinates $(j\,\Delta x, k\,\Delta t)$ be denoted by $\psi_{j,k}$. We then consider approximations to (1) which make use of the 15 values in the rectangle whose corners are $P_{j-2,k-1}$, $P_{j+2,k-1}$, $P_{j+2,k+1}$, and $P_{j-2,k+1}$. In the explicit recurrence formula of Collatz [1] the fourth x-derivative in (1) is approximated along the line k and the second t-derivative is approximated along the line j. The implicit formula [2] is similar except that the average of approximations to the fourth derivative along the lines $k+1$ and $k-1$ is used.

Let us now consider an average of approximations to the x-derivative in which the $k+1$ and $k-1$ lines are weighted with the factor θ and the line k is weighted with $(1-2\theta)$. At the same time we use an average of approximations to the t-derivative in which the lines $j+1$ and $j-1$ are weighted with the factor β and the line j is weighted with $1-2\beta$. In this way we are led to the following approximation to (1)

$$r^2\left(1+\theta\delta_t^2\right)\delta_x^4\,\psi_{j,k} + \left(1+\beta\delta_x^2\right)\delta_t^2\,\psi_{j,k} = 0 \qquad (2)$$

where the partial difference operators are defined as usual; e.g.,

$$\delta_x^4 \psi_{j,k} = \psi_{j-2,k} - 4\psi_{j-1,k} + 6\psi_{j,k} - 4\psi_{j+1,k} + \psi_{j+2,k}$$

$$\delta_t^2 \psi_{j,k} = \psi_{j,k-1} - 2\psi_{j,k} + \psi_{j,k+1} \tag{3}$$

If in (2) we set $\theta = \beta = 0$ we get the formula of [1] and if we set $\theta = 1/2$, $\beta = 0$ we get the formula of [2]. All formulas are implicit except the case $\theta = \beta = 0$.

Truncation error. We assume that the solution to (1) has continuous derivatives of all orders up to 8 in x and 4 in t. Let the derivative $\partial^{m+n}\psi/\partial x^m \partial t^n$ evaluated at $P_{j,k}$ be denoted by (m,n). In this notation the governing equation (1) is simply

$$(4,0) = -(0,2) \tag{4}$$

By repeated differentiation of (4) we have the following identities.

$$\begin{aligned} (6,0) &= -(2,2) \\ (8,0) &= -(4,2) = (0,4) \end{aligned} \tag{5}$$

If the neighboring values such as $_{j+2,k}$ which appear in (2) are expressed in terms of the Taylor's series centered at $P_{j,k}$ it is a straightforward matter to obtain

$$\begin{aligned} \frac{(1+\theta\delta_t^2)\,\delta_x^4\psi_{j,k}}{(\Delta x)^4} + \frac{(1+\beta\delta_x^2)\,\delta_t^2\psi_{j,k}}{(\Delta t)^2} &= \Big[(4,0) + (0,2)\Big] + \\ &\quad h^2(6,0)\Big[\tfrac{1}{6} - \beta\Big] \\ &\quad + h^4(8,0)\Big[\tfrac{1}{80} - \tfrac{\beta}{12} + \tfrac{r^2}{12}(1-12\theta)\Big] \\ &\quad + O(h^6) \end{aligned} \tag{6}$$

by using (5) to simplify. This shows that (2) is ordinarily an approximation to (1) with $O(h^2)$ truncation error but that when $\beta = 1/6$ the truncation error is $O(h^4)$. If, moreover

$$r^2(1 - 12\theta) = \frac{1}{60} \tag{7}$$

the truncation error is $O(h^6)$.

<u>Stability</u>. If we assume a solution to (2) of the form

$$\psi_{j,k} = \sin\frac{n\pi j}{M} \quad \cos\frac{r\Omega_n k}{M^2} \tag{8}$$

and if for all n satisfying $0 < n < M$ we obtain a real value for Ω_n the recurrence formula (2) is stable (See discussion in [2]). Solving for Ω_n we find

$$\Omega_n = \frac{2M^2}{r} \quad \sin^{-1}\left\{\frac{\frac{1}{4}\lambda_n r^2}{1 + \theta\lambda_n r^2}\right\}^{\frac{1}{2}} \tag{9}$$

where

$$\lambda_n = \frac{4(1 - \cos\frac{n\pi}{M})^2}{1 - 2\beta(1 - \cos\frac{n\pi}{M})} \tag{10}$$

The stability limit occurs when the argument of the arc sin becomes unity and this occurs first for the largest λ_n. In the limit as $M \to \infty$ we see from (10) that $(\lambda_n)_{max} \to 16(1 - 4\beta)^{-1}$. Setting the arc sin equal to unity for $(\lambda_n)_{max}$ gives the stability limit for large M.

$$r^2 \leq \frac{1}{4} \quad \frac{1 - 4\beta}{1 - 4\theta} \tag{11}$$

The combinations of r and θ for stability and $O(h^6)$ truncation error are shown in Fig. I for the case $\beta = 1/6$ which gives $O(h^4)$ truncation error.

<u>Solution error</u>. The discretization error in the solution of (2) as compared with (1) depends essentially on how faithfully the lower space modes and their effective frequencies are modelled. This can be studied by returning to (8) and shifting the emphasis from large n to small n. If the beam is hinged at its ends the boundary conditions $\psi = \partial^2\psi/\partial x^2 = 0$ can be approximated by $\psi_{o,k} = \psi_{M,k} = 0$ and $\psi_{-1,k} = -\psi_{1,k}$

$\psi_{M+1,k} = -\psi_{M-1,k}$ and (8) is an exact solution. From (9) we find

$$\Omega_n = n^2\pi^2\left\{1 + \frac{n^2\pi^2}{2M^2}\left(\beta - \frac{1}{6}\right) + \frac{n^4\pi^4}{24M^4}\left[r^2(1 - 12\theta) - \frac{1}{60}\right] + O\left(\frac{1}{M^6}\right)\right\} \qquad (12)$$

whereas the corresponding solution to (1) is

$$\psi = \sin n\pi x \cos \omega_n t \qquad (13)$$

with $\omega_n = n^2\pi^2$. Note that the finite difference space modes are identical with the continuous modes. Note that the conditions for agreement of ω_n with Ω_n for large M are the same as those obtained for the truncation error. Even for small M there is considerable advantage in using optimum values for β and θ. The following table compares the frequencies of the first three modes for the case M = 8, r = 1/3 (this guarantees stability for all of the formulas).

		Finite Difference Approximations		
	Exact Solution	$\beta=0,\ \theta=0$	$\beta=\frac{1}{6},\ \theta=\frac{1}{4}$	$\beta=\frac{1}{6},\ \theta=\frac{17}{240}$
ω_1	9.8696	9.7445	9.8673	9.8696
ω_2	39.4784	37.5501	39.3278	39.4769
ω_3	88.8264	79.5850	87.1365	88.7905

The first finite difference formula is explicit. The second has $O(h^4)$ truncation error and the third $O(h^6)$ truncation error.

1 L. Collatz, Zur Stabilitat des Differenzenverfahrens bei der Stabschwingungsgleichung, Z.a.Math. Mech. 31, 392-393 (1951).

2 S. H. Crandall, Numerical Treatment of a Fourth Order Parabolic Partial Differential Equation, J. Assoc. Comp. Mach. 1, 111-118 (1954).

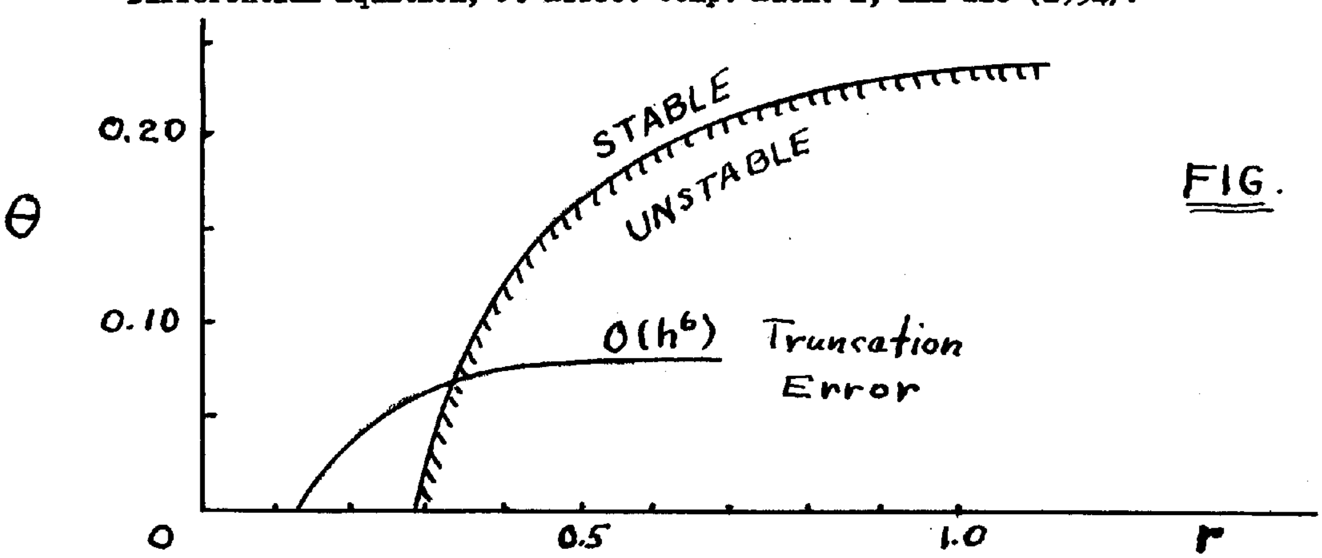

FIG. I