# APPLICATIONS OF FINITE FIELDS TO DYNAMICAL SYSTEMS AND REVERSE ENGINEERING PROBLEMS 

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#### Abstract

We present a mathematical model: dynamical systems over finite sets (DSF), and we show that Boolean and discrete genetic models are special cases of DFS. In this paper, we prove that a function defined over finite sets with different number of elements can be represented as a polynomial function over a finite field. Given the data of a function defined over different finite sets, we describe an algorithm to obtain all the polynomial functions associated to this data. As a consequence, all the functions defined in a regulatory network can be represented as a polynomial function in one variable or in several variables over a finite field. We apply these results to study the reverse engineering problem.


## 1. Introduction

In this paper we introduce the definition of dynamical systems over different finite sets (DSF) and we develop its applications to regulatory networks and the Reverse Engineering Problem. We consider variables over sets with different numbers of elements and we change that to variables over a finite field.

The justification for considering dynamical systems over different finite sets is related with the method Generalized Logical Networks developed by Thomas and colleagues, 11, 13, 14, 15, 16. The generalized logical networks has a mean consideration: a variable can have more that two possibilities but always the number of possibilities is finite. In addition, the network is described by a function which acts over several variables and for each variable there are different number of values. These considerations are very important for biologists because it is known that in a regulatory network all the variables do not have the same number of states. Here, we prove that all of these functions can be considered over a finite field and as a consequence of that we can represent them by polynomial functions. In section 2 we present an algorithm which changes a function over different set of values to a function over a finite field.

In Section 2 we introduce the method to construct functions over a finite field using functions defined over finite sets with different number of elements. In Section 3, we apply the partially defined functions to Reverse Engineering Problem. In Section 5, we introduce the definition of Dynamical Systems over different finite sets.

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## 2. Partially DEFined functions

Now, in this section we introduce the mathematical background which will permit the application of modelling methods such as generalized logical networks.

Let $X_{j}=\{0,1, \ldots, j-1\}$ and let $\mathbb{Z}_{p}$ be the set of integers modulo $p$, with $p$ a prime number. Suppose that $p \geq j$, and we consider a canonical map from $X_{j}$ to the field $\mathbb{Z}_{p}$ given by $a \rightarrow a(\bmod p)$. In the following we consider $X_{j} \subset \mathbb{Z}_{p}$. Let $\boldsymbol{x}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{Z}_{p}{ }^{n}$. We denote the polynomial ring in $n$ variables over $\mathbb{Z}_{p}$ by $\mathbb{Z}_{p}\left[x_{1}, \ldots x_{n}\right]$. We begin with some definitions. Let $D \subseteq \mathbb{Z}_{p}$.

Definition 2.1. Let $S \subsetneq \mathbb{Z}_{p}{ }^{n}$. Let $f: S \rightarrow D$ be a function. We will call $f a$ partially defined function over $\mathbb{Z}_{p}$.
Example 2.2. Now, let $g: X_{2} \times \mathbb{Z}_{3} \rightarrow X_{2}$, given by the following table:

| $g\left(x_{1}, x_{2}\right)$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 |

Then the table of values of the partially defined function $\hat{g}$ is the following:

| $\hat{g}\left(x_{1}, x_{2}\right)$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 |
| 2 | $*$ | $*$ | $*$ |

Let $S=X_{m_{1}} \times \cdots \times X_{m_{n}} \subsetneq \mathbb{Z}_{p}{ }^{n}$, and let $D \subseteq \mathbb{Z}_{p}$. Since a partially defined function $\hat{f}: S \rightarrow D$ is not a function from $\mathbb{Z}_{p}{ }^{n}$ to $\mathbb{Z}_{p}$, we are interested in solving the following problem:
[DF $\left.\left(\mathbb{Z}_{p}\right):\right]$ Let $S \subsetneq \mathbb{Z}_{p}{ }^{n}$ and let $f: S \rightarrow D$ be a function. We want a polynomial function $P: \mathbb{Z}_{p}{ }^{n} \rightarrow \mathbb{Z}_{p}$ such that $P(\boldsymbol{x})=f(\boldsymbol{x})$, for all $\mathbf{x} \in S \subsetneq \mathbb{Z}_{p}{ }^{n}$.

A function $P$ associated to $f$ will be called a polynomial function for $f$. Now, we prove that the problem $\mathrm{DF}\left(\mathbb{Z}_{p}\right)$ can have more than one solution.
Proposition 2.3. For each function $f: S \rightarrow D$ there is a polynomial $P(\boldsymbol{x}) \in$ $\mathbb{Z}_{p}\left[x_{1}, \ldots, x_{n}\right]$, such that $f(\boldsymbol{x})=P(\boldsymbol{x})$ for all $\boldsymbol{x} \in S$. The polynomial $P$ can be chosen with degree less than or equal to $n(p-1)$ but in general, it is not unique.
Proof. If $k$ is a finite field and $f: k^{n} \rightarrow k$ is a function then there exists a polynomial $P$ in the variables $x_{1}, \ldots, x_{n}$, with coefficient in $k$, such that $f(\boldsymbol{x})=P\left(x_{1}, \ldots, x_{n}\right)$ for all $\left(x_{1}, \ldots, x_{n}\right) \in k^{n}$, 7]. But, in our case we do not have a function from $k^{n} \rightarrow k$, so we will prove that the polynomial function exists associated to the partially defined function but it is not unique.

We will show the idea using the example 2.2. In the table of $\hat{f}$ we can complete the table in some way. Then there exists a unique polynomial for this table. But we can complete the table in many ways, so the polynomial function exists but, it is not unique.

As a consequence of Proposition 2.3 we have an algorithm which solves the problem $\operatorname{DF}\left(\mathbb{Z}_{p}\right)$. Let $f: S \rightarrow \mathbb{Z}_{p}$ be a function. Let $m=|S|$ be the cardinality of $S\left(m=\sum_{i=1}^{n} m_{i}\right.$ when $\left.f: X_{m_{1}} \times \cdots \times X_{m_{n}} \rightarrow \mathbb{Z}_{p}\right)$. Now, we write a polynomial $P$ in $n$ variables $x_{1}, \ldots, x_{n}$. $P$ has degree less than or equal to $p-1$ in each variable, so has degree less than or equal to $n(p-1)$. We denote $P$ in the following form: $P\left(x_{1}, \ldots, x_{n}\right)=\sum_{\alpha \in \mathbb{Z}_{p}{ }^{n}} b_{\alpha} x^{\alpha}$, where $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right), x^{\alpha}=x^{\alpha_{1}} \ldots x_{n}^{\alpha_{n}}$. Now, we
evaluate $P$ for all $\mathbf{a} \in S$ and we obtain a system of $m$ linear equations in the $p^{n}$ unknowns $b_{\alpha}$ which always has solutions. The system is the following:

$$
\text { (I) } \sum_{\alpha \in \mathbb{Z}_{p} n}(\mathbf{a})^{\alpha} b_{\alpha}=f(\mathbf{a}) \text { for all } \mathbf{a} \in S
$$

Solving the system using elementary row operations, we finally obtain all the solutions. In [3], it is proved that the rank of this system is $m$. Then, there are $b_{\beta_{1}}, \ldots, b_{\beta_{m}}$ coefficients of the polynomial $P$ whose are determined in term of the free coefficients denoted by $b_{\gamma_{1}}, \ldots, b_{\gamma_{p^{n}-m}}$. Now, let $\mathbb{Z}_{p}{ }^{(p-1)}\left[x_{1}, \ldots, x_{n}\right]$ be the subspace of $\mathbb{Z}_{p}\left[x_{1}, \ldots, x_{n}\right]$ of all polynomials with maximum degree $p-1$ in each variable and coefficients in $\mathbb{Z}_{p}$. So, we have the following theorem.

Theorem 2.4. All the polynomial solutions with degree $\leq n(p-1)$ of the problem $D F\left(\mathbb{Z}_{p}\right)$ are given by a particular solution $f_{0}(\boldsymbol{x})$ of $(I)$ plus the subspace

$$
U=\left\{g \in \mathbb{Z}_{p}^{(p-1)}\left[x_{1}, \ldots, x_{n}\right] \mid g(\boldsymbol{a})=0, \forall \boldsymbol{a} \in S\right\}
$$

of dimension $p^{n}-m$.
Proof. We know by linear algebra that all the solutions of (I) are given by

$$
f_{0}(\boldsymbol{x})+b_{\gamma_{1}} g_{1}(\boldsymbol{x})+\cdots+b_{\gamma_{p^{n}-m}} g_{p^{n}-m}(\boldsymbol{x})
$$

where $b_{\gamma_{1}}, \ldots, b_{\gamma_{p^{n}-m}} \in \mathbb{Z}_{p}$ and $g_{1}, \ldots, g_{p^{n}-m} \in U$. Let $h_{1}$ and $h_{2}$ be two polynomial solutions of $(I)$. Then $h_{1}-h_{2} \in U$, so the theorem holds.

## 3. Reverse Engineering Problem over finite sets

Now, we connect the problem $\operatorname{DF}\left(\mathbb{Z}_{p}\right)$ with the Reverse Engineering Problem over $\mathbb{Z}_{p}$. The problem for partially defined functions is equivalent to the following.
$\left[P\left(\mathbb{Z}_{p}\right):\right]$ Given $\mathbf{a}_{1}, \ldots, \mathbf{a}_{m} \in \mathbb{Z}_{p}{ }^{n}, \mathbf{b}=\left(b_{1}, \ldots, b_{m}\right) \in \mathbb{Z}_{p}{ }^{m}$, with $m<p^{n}$. Find a polynomial $P \in \mathbb{Z}_{p}\left[x_{1}, \ldots, x_{n}\right]$ such that $P\left(\mathbf{a}_{j}\right)=b_{j}$ for $j=1, \ldots, m$.

The problems $P\left(\mathbb{Z}_{p}\right)$ and $\operatorname{DF}\left(\mathbb{Z}_{p}\right)$ are equivalent. In fact, we only need to take $S=\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{m}\right\}$ and $\mathbf{b}=\left(f\left(\mathbf{a}_{1}\right), \ldots, f\left(\mathbf{a}_{m}\right)\right)$.

The problem $P\left(\mathbb{Z}_{p}\right)$ was solved by E . Green in [2]. He called the problem $P\left(\mathbb{Z}_{p}\right)$ for $\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{m} ; \mathbf{b}\right)$ and he proved that if $P\left(\mathbb{Z}_{p}\right)$ has solutions then the Reverse Engineering Problem over $\mathbb{Z}_{p}$ has solutions.

Now, we define the Reverse Engineering Problem over sets with different number of elements. Let $\left\{k_{j}\right\}$ be a family of $n$ finite sets where $\left|k_{j}\right|=m_{j}$. We denote by $k=k_{1} \times \cdots \times k_{n}$. Let $\mathbf{r}_{1}, \ldots, \mathbf{r}_{m+1} \in k$. We assume that the vectors $\mathbf{r}_{j}=\left(r_{j 1}, \ldots, r_{j n}\right)$ are obtained by experiments (like microarray) and we assume that $\mathbf{r}_{j}$ determines $\mathbf{r}_{j+1}$. Then the Reverse Engineering Problem over $k$ is to find a function $F=\left(f_{1}, \ldots, f_{n}\right): k \rightarrow k$ such that $F\left(\mathbf{r}_{j}\right)=\mathbf{r}_{j+1}$ for $j=1, \ldots, m$. But, we rewrite:
[(REP)] The Reverse Engineering Problem over $k$ is to find polynomial functions $f_{s}: k \rightarrow k_{s}$ such that

$$
f_{s}\left(\mathbf{r}_{j}\right)=r_{s, j+1} \text { for } j=1, \ldots, m \text { and } s=1, \ldots, n
$$

Now, we prove that if we can solve $\operatorname{DF}\left(\mathbb{Z}_{p}\right)$, we can solve (REP) and use the same algorithm. In fact, if $\mathbb{Z}_{p}$ is the field such that $p \geq m_{j}$ for all $j$, we take the partially defined functions $\hat{f}_{s}$ over $\mathbb{Z}_{p}$ considering $k_{s} \subseteq \mathbb{Z}_{p}$. Let $\mathbf{r}=\left(r_{1}, \ldots, r_{n}\right) \in k$. So, we consider $S \subsetneq \mathbb{Z}_{p}{ }^{n}$ and $\hat{f}_{s}\left(\mathbf{r}_{j}\right)=r_{s, j+1}$ for $j=1, \ldots, m$ and $s=1, \ldots, n$.

We have proved the following proposition.
Proposition 3.1. The reverse engineering problem over set with different number of elements has polynomial solutions by Proposition 2.3, and Theorem 2.4.

Definition 3.2. The matrix $A=\left(\boldsymbol{r}_{j}\right)_{m \times n}$ will be called the matrix of the problem REP.

Example 3.3. Suppose we have the following data: $\boldsymbol{r}_{1}=(1,2,0), \boldsymbol{r}_{2}=(2,2,1)$, $\boldsymbol{r}_{3}=(1,0,1), \boldsymbol{r}_{4}=(0,1,1)$, and $\boldsymbol{r}_{5}=(1,1,0)$. And, we have the additional information:
(a) the variables $\{x, y, z\}$ are defined over different finite sets, but we take finite fields: $x, y \in \mathbb{Z}_{3}=\{0,1,2\}$ and $z \in \mathbb{Z}_{2}=\{0,1\}$.
(b)the variable $x$ depends of $x$ and $z, y$ depends of $x$ and $y$, and $z$ depends of $y$ and $z$.

The matrix $A$ of the problem is the following:

$$
A=\left(\begin{array}{lll}
1 & 2 & 0 \\
2 & 2 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{array}\right)
$$

We want polynomial functions $f_{1}, f_{2}$, and $f_{3}$, such that $F=\left(f_{1}, f_{2}, f_{3}\right)$ and $F\left(\mathbf{r}_{j}\right)=$ $\mathbf{r}_{j+1}$ for $j=1,2,3,4$.

The additional information (b) means that the functions that we are looking for are as follows:

$$
\begin{array}{ccc}
f_{1}(x, z)= & a_{0}+a_{1} x+a_{2} z+a_{3} x z+a_{4} x^{2}+a_{5} z^{2} \\
& +a_{6} x^{2} z+a_{7} x z^{2}+a_{8} x^{2} z^{2} \\
f_{2}(x, y)= & b_{0}+b_{1} x+b_{2} y+b_{3} x y+b_{4} x^{2}+b_{5} y^{2} \\
& +b_{6} x^{2} y+b_{7} x y^{2}+b_{8} x^{2} y^{2} \\
f_{3}(y, z)= & c_{0}+c_{1} z+c_{2} y+c_{3} y z+c_{4} y^{2}+c_{5} z^{2} \\
& +c_{6} y^{2} z+c_{7} y z^{2}+c_{8} y^{2} z^{2}
\end{array}
$$

Using the data we have the table of $f_{1}$.

| $f_{1}(x, z)$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $*$ | 1 | $*$ |
| 1 | 2 | 0 | $*$ |
| 2 | $*$ | 1 | $*$ |

Using the above table and the algorithm for problem $\mathrm{DF}\left(\mathbb{Z}_{p}\right)$, we obtain a system of 4 linear equation with 9 unknown. The matrix of the system of linear equation is:

$$
\overline{A_{1}}=\left(\begin{array}{lllllllll|l}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\
1 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Using elementary row operations, we have the following:

$$
\begin{gathered}
a_{2}=1+2 a_{3}+2 a_{5}+2 a_{6}+2 a_{7}+2 a_{8}, a_{4}=1+2 a_{6}+2 a_{8} \\
a_{1}=1+2 a_{3}+2 a_{7}, a_{0}=a_{3}+a_{7}+a_{6}+a_{8}
\end{gathered}
$$

A particular solution of the system is $f_{1}=x+z+x^{2}$. And all the solutions are given by $f_{1}+a_{3} g_{1}+a_{5} g_{2}+a_{6} g_{3}+a_{7} g_{4}+a_{8} g_{5}$, where

$$
\begin{gathered}
g_{1}=1+2 x+2 z+x z, g_{2}=2 z+z^{2}, g_{3}=1+2 z+2 x^{2}+x^{2} z \\
g_{4}=1+2 x+2 z+x z^{2}, \text { and } g_{5}=1+2 z+2 x^{2}+x^{2} z^{2}
\end{gathered}
$$

If we denote by $U_{1}$ the subspace of $\mathbb{Z}_{3}[x, y, z]$ generated by $\left\{g_{1}, g_{2}, g_{3}, g_{4}, g_{5}\right\}$, then all the solutions with degree $\leq 2$ in each variable, are $f_{1}+U_{1}$.

Similarly we obtain:
(1) $f_{2}=x+y^{2}$ and all the solutions are $f_{2}+U_{2}$, where $U_{2}$ is the subspace generated by the polynomials

$$
\begin{gathered}
h_{1}=2+x+x y+y^{2}, h_{2}=2+2 y+x^{2}+2 y^{2} \\
h_{3}=1+2 x+2 y^{2}+x y^{2}, h_{4}=y+2 y^{2}+x^{2} y, h_{5}=2 y+y^{2}+x^{2} y^{2}
\end{gathered}
$$

(2) $f_{3}=1+y+y^{2}$ and all the solutions are $f_{3}+U_{3}$, where $U_{3}$ is the subspace generated by the polynomials

$$
\begin{gathered}
v_{1}=2+z+2 y+y z, v_{2}=1+2 z+2 y^{2}+y^{2} z \\
v_{3}=2+z+2 y+y z^{2}, v_{4}=1+2 z+2 y^{2}+y^{2} z^{2}, v_{5}=2 z+z^{2}
\end{gathered}
$$

Finally one of the functions that can describe the genetic network is the following

$$
f(x, y, z)=\left(x+z+x^{2}, x+y^{2}, 1+y+y^{2}\right)
$$

## 4. Solution over the finite field $\operatorname{GF}\left(p^{n}\right)$

We can solve the problems $\mathrm{DF}\left(\mathbb{Z}_{p}\right)$ and (REP) using Lagrange interpolation over the field $\operatorname{GF}\left(p^{n}\right)=K$ [8, 9]. Let $f: S \rightarrow \mathbb{Z}_{p}$ be a function with $S \subsetneq \mathbb{Z}_{p}{ }^{n}$. Let $|S|=m$ be the cardinality of $S$. let $\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right\}$ be a fixed basis of $K$. There is a natural one to one correspondence between the sets $\mathbb{Z}_{p}{ }^{n}$ and $K$, namely

$$
\lambda:\left(a_{1}, \ldots, a_{n}\right) \mapsto a_{1} \alpha_{1}+\cdots+a_{n} \alpha_{n}
$$

Let $\bar{S}=\lambda(S) \subsetneq K$. Now we have the partially defined function $\hat{f}=\lambda \circ f \circ \lambda^{-1}$ : $\bar{S} \rightarrow \mathbb{Z}_{p}$. We denote the elements of $\bar{S}$ by $\overline{\boldsymbol{a}}$.

Now, using the Lagrange interpolation formula we have the following: $\overline{\boldsymbol{a}}_{1}, \ldots$, $\overline{\boldsymbol{a}}_{m}$ are $m$ distinct elements of the finite field $K$ and $\hat{f}\left(\overline{\boldsymbol{a}}_{1}\right)=b_{1}, \ldots, \hat{f}\left(\overline{\boldsymbol{a}}_{m}\right)=b_{m}$, with $b_{1}, \ldots, b_{m}$ elements in $\mathbb{Z}_{p}$. We know that $\mathbb{Z}_{p} \subset K$. We rewrite the problem $\mathrm{DF}\left(\mathbb{Z}_{p}\right)$ as follows:
[DF $\left.\left(p^{n}\right):\right]$ Let $\bar{S} \subsetneq K$ and let $f: \bar{S} \rightarrow K$ be a function. We want a polynomial $P(x) \in K[x]$ such that

$$
P(\overline{\boldsymbol{x}})=f(\overline{\boldsymbol{x}}), \text { for all } \overline{\boldsymbol{x}} \in \bar{S} \subsetneq K
$$

We can observe that this is the same problem (REP) if we consider $\bar{S}=\left\{\mathbf{r}_{1}, \ldots, \mathbf{r}_{m+1}\right\}$ and $f\left(\mathbf{r}_{j}\right)=\mathbf{r}_{j+1} \in K$. So in the following we denote both problem by $\operatorname{DF}\left(p^{n}\right)$.

Using Lagrange Interpolation, we know that: there exists a polynomial $\bar{P} \in K[x]$ of degree $d \leq m-1$ such that $\bar{P}\left(\overline{\boldsymbol{a}}_{i}\right)=\boldsymbol{b}_{i} \in K$ for $i=1, \ldots, m$. The polynomial is given by

$$
\bar{P}(x)=\sum_{i=1}^{m-1} \boldsymbol{b}_{i} \prod_{k=1, k \neq i}^{m-1}\left(\overline{\boldsymbol{a}}_{i}-\overline{\boldsymbol{a}}_{k}\right)^{-1}\left(x-\overline{\boldsymbol{a}}_{k}\right)
$$

Then, we have proved the following theorem.

Theorem 4.1. The problem $D F\left(p^{n}\right)$ has solutions over the field $K=G F\left(p^{n}\right)$ using Lagrange Interpolation. That is, there exists a polynomial $\bar{P}_{0} \in K[x]$, such that $\bar{P}_{0}\left(\overline{\boldsymbol{a}}_{i}\right)=\boldsymbol{b}_{i} \in K$, for $i=1, \ldots, m$. The degree of the polynomial $\bar{P}_{0}$ is less than or equal to $m-1$. If $I$ is the ideal of $k[x]$ generated by $\bar{P}(x)=\left(x-\overline{\boldsymbol{a}}_{1}\right) \cdots\left(x-\overline{\boldsymbol{a}}_{m}\right)$, then all the solutions are given by $\bar{P}_{0}(x)+G(x)$, where $G(x) \in I$.

Now, we have a new algorithm to solve the problem. We know by the Theorem 4.1 that the solution is a polynomial of the following form: $\bar{P}(x)=\sum_{k=0}^{m-1} B_{k} x^{k}$. We evaluate $\bar{P}(x)$ in all the elements of $\bar{S}$. Then we obtain a system of linear equations with one solution

$$
\text { (II) } \sum_{k=0}^{m-1} B_{k} \overline{\boldsymbol{a}}_{i}^{k}=b_{i} \text { for } i=1, \ldots, m
$$

We want to remark that the system $(I I)$ has rank $m$, since $\overline{\boldsymbol{a}}_{i} \neq \overline{\boldsymbol{a}}_{j}$ for $i \neq j$. Finally we have an output a polynomial in one variable with degree less than or equal to $m-1$.

Example 4.2. Let $S=\{(0,1),(1,0),(1,1),(2,1)\} \subsetneq \mathbb{Z}_{3}{ }^{2}$, and $f: S \rightarrow \mathbb{Z}_{3}$. The function $f$ has the following table of values.

| $f\left(x_{1}, x_{2}\right)$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | $*$ | 1 | $*$ |
| 1 | 2 | 0 | $*$ |
| 2 | $*$ | 1 | $*$ |

Let $\alpha$ be a root of the polynomial $X^{2}+X+2$ in $\mathbb{Z}_{3}$. Then $\alpha^{2}=2 \alpha+1$, and a basis for $G F\left(3^{2}\right)$ is $\{\alpha, 1\}$. A natural correspondence is $\left(x_{1}, x_{2}\right) \mapsto x_{1} \alpha+x_{2}$ We have $\mathbf{a}_{1}=(0,1), \overline{\boldsymbol{a}}_{1}=1 ; \boldsymbol{a}_{2}=(1,0), \overline{\boldsymbol{a}}_{2}=\alpha ; \boldsymbol{a}_{3}=(1,1), \overline{\boldsymbol{a}}_{3}=\alpha+1$; and $\boldsymbol{a}_{4}=(2,1), \overline{\boldsymbol{a}}_{4}=2 \alpha+1$. Then, the particular solution is given by a polynomial $P_{0}(x)=c_{0}+c_{1} x+c_{2} x^{2}+c_{3} x^{3}$. We evaluate in the four elements of $\operatorname{GF}\left(3^{2}\right)$ using the table of values. The matrix of the system is the following:

$$
M=\left(\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 1 \\
1 & \alpha & \alpha^{2} & \alpha^{3} & 2 \\
1 & \alpha+1 & 2 \alpha+1 & 2 \alpha & 0 \\
1 & 2 \alpha+1 & 2 & \alpha+2 & 1
\end{array}\right)
$$

Solving the system we obtain the polynomial $P_{0}(x)=\alpha^{3}+2 x+\alpha^{6} x^{2}+x^{3}$. Now, we change the polynomial in two polynomials with two variables $x_{1}, x_{2}$ and coefficients in $\mathbb{Z}_{3}$. We use the correspondence $\lambda^{-1}$ and obtain the following:

$$
f\left(x_{1}, x_{2}\right)=\left(2+x_{1}+2 x_{1} x_{2}+x_{2}^{2}, 2+2 x_{1}+x_{1}^{2}+2 x_{1} x_{2}+2 x_{2}^{2}\right)
$$

## 5. Dynamical Systems over finite sets with different number of ELEMENTS

In this section we present a definition of dynamical system over finite sets.
Definition 5.1. A dynamical system over finite sets is a pair $(S, f)$ such that (a) $S \subseteq \mathbb{Z}_{p}{ }^{n}$.
(b) A function, $f=\left(f_{1}, \ldots, f_{n}\right): S \rightarrow S$, and each function $f_{i}: S_{i} \rightarrow \mathbb{Z}_{p}$ is a partially defined function with $S_{i} \subseteq S$ for all $i$.

The DSF is a time discrete dynamical system, that is the dynamics is generated by iteration of the function $f$.

Definition 5.2. The state space $\mathcal{S}_{f}$ of the dynamical system $f: S \rightarrow S$ is a finite directed graph (digraph) with vertex set $S$ and arc set $A=\{(x, y) \in S: f(x)=y\}$, that is the ordered pair $(x, y)$ stands for and arrow from a vertex $x=\left(x_{1}, \ldots, x_{n}\right) \in$ $S$ to a vertex $y=\left(y_{1}, \ldots, y_{n}\right) \in S$ if and only if $f(x)=y$.

Theorem 5.3. Let $(S, f)$ be a dynamical system over finite sets. Then
(a) $f$ can be represented as a polynomial function in one variable over the finite field $G F\left(p^{n}\right)$, for some prime $p$.
(b) If $f=\left(f_{1}, \ldots, f_{n}\right)$ then each function $f_{i}$ can be represented as a polynomial function in one variable or in several variables over $\mathbb{Z}_{p}$, for some prime $p$.
(c) Part (a) and (b) hold for all prime number $p \geq \max _{i}\left|\overline{S_{i}}\right|$, where $\left|\overline{S_{i}}\right|$ is the maximal number of different elements in the coordinate $i$ of the set $S_{i}$

Proof. It is a consequence of the Sections 1 and 2.

## 6. Examples and applications

In this section we present two applications of the representation of the dynamical systems by polynomials over a finite fields. One very important things is to determine the steady states, that is the elements $\mathbf{x}$ such that $f(\mathbf{x})=\mathbf{x}$. We determine that in the first example.

In the reverse engineering problem, we have a set of solutions and here we suggest a method for biologist to determine if one of the solution is the right one.

In Fig. 1, an example of regulatory network is shown. This example of Generalized Logical Networks appear in [5]. Here, we use the usual words for biologists. Gene 1 regulates genes 2 and 3 , so that it has two thresholds (two values different 0 ) and the corresponding logical variable $x_{1}$ takes its value from $\{0,1,2\}$. Similarly, $x_{2}$ and $x_{3}$ have one and two thresholds, respectively, and hence possible values $\{0,1\}$ and $\{0,1,2\}$. The functions $f_{1}(\mathbf{x}), f_{2}(\mathbf{x})$, and $f_{3}(\mathbf{x})$ need to be specified such as to be consistent with the threshold restrictions in the graph. Examples of logical functions allowed by the generalized logical method are shown in Fig. 1(b).

Consider the case of $f_{2}(\mathbf{x})$. If $x_{1} \neq 0$ and $x_{3} \neq 0$, so that $x_{1}$ and $x_{3}$ have values above their first threshold, the inhibitory influences of genes 1 and 3 on gene 2 become operative. Figure 1(b) indicates that $x_{2}$ will tend to 0 , that is, below the first threshold of the protein produced by gene 2 . If either $x_{1}=0$ or $x_{3}=0$, that is, if only one of the inhibitory influences is operative, then gene 2 is moderately expressed. This is here represented by the value 1 for the image of $x_{2}$. In general, several logical functions will be consistent with the threshold restrictions. Exactly which logical function is chosen may be motivated by biological considerations or may be a guess reflecting uncertainty about the structure of the system being studied.

## Example 6.1.



Figure 1 (a)

| $f_{2}\left(x_{1}, x_{3}\right)$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 2 | 1 | 0 | 0 |
| 2 |  |  |  |$|$| $f_{3}\left(x_{1}, x_{3}\right)$ | 0 | 1 | 2 |
| :---: | :---: | :---: | :---: |
| 0 | 2 | 2 | 1 |
| 1 | 2 | 2 | 1 |
| 2 | 0 | 0 | 0 |

Figure 1 (b)
Now, we describe this example using a dynamical system over finite sets. We have three genes, and the regulatory network is the following:

$$
\text { (1) The digraph: } Y \quad 1 \begin{array}{ccc}
\nearrow & \downarrow \\
& 1 & \\
\hline
\end{array}
$$

(2) The variables $x_{1}$ and $x_{3}$ are in $\mathbb{Z}_{3}=\{0,1,2\}$, and the variable $x_{2}$ is in $X_{2}=$ $\{0,1\}$,
(3) There are several possibilities for functions $f_{1}$ and $f_{2}$, but $f_{3}$ is the unique function. Then the polynomial functions $f_{1}, f_{2}$, and $f_{3}$ are:

$$
\begin{aligned}
f_{1}\left(x_{2}\right)= & 2 x_{2} \\
f_{2}\left(x_{1}, x_{3}\right)= & 1+2 x_{1}^{2} x_{3}^{2} \\
f_{3}\left(x_{1}, x_{3}\right)= & 2+x_{1}+2 x_{3}+x_{1} x_{3}+2 x_{1}^{2}+x_{3}^{2}+2 x_{1}^{2} x_{3} \\
& +2 x_{1} x_{3}^{2}+x_{1}^{2} x_{3}^{2}
\end{aligned}
$$

(4) The global function $f:\left(\mathbb{Z}_{3}\right)^{3} \rightarrow\left(\mathbb{Z}_{3}\right)^{3}$,

$$
f\left(x_{1}, x_{2}, x_{3}\right)=\left(f_{1}\left(x_{2}\right), f_{2}\left(x_{1}, x_{3}\right), f_{3}\left(x_{1}, x_{3}\right)\right)
$$

The state space has vertices $V=\mathbb{Z}_{3} \times X_{2} \times \mathbb{Z}_{3}$. We want to know the steady states of the dynamical system $f$, in general it is a very difficult problem. But, in this particular case we have three equations in three variables,

$$
\begin{gathered}
2 x_{2}=x_{1}, 1+2 x_{1}^{2} x_{3}^{2}=x_{2} \\
2+x_{1}+2 x_{3}+x_{1} x_{3}+2 x_{1}^{2}+x_{3}^{2}+2 x_{1}^{2} x_{3}+2 x_{1} x_{3}^{2}+x_{1}^{2} x_{3}^{2}=x_{3}
\end{gathered}
$$

For $x_{2}$ we have only two values, if $x_{2}=0$ then $x_{1}=0$ and we obtain $0=1$ in the second equation, that is impossible. If $x_{2}=1$ then $x_{1}=2$ and $1+2 x_{3}^{2}=1$ so $x_{3}=0$. We can check in the last equation and the only solution is $(2,1,0)$.

Example 6.2. In the example that we present in Section 3, we have $3^{15}=14,348,907$ different solutions. But, we can select the particular solution $f(x, y, z)=(x+z+$ $\left.x^{2}, x+y^{2}, 1+y+y^{2}\right)$, and try to find which vectors in the state space go to the first state (1, 2, 0).

Solving over $\mathbb{Z}_{3}$ the equations $x+z+x^{2}=1, x+y^{2}=2$, and $1+y+y^{2}=0$, we obtain that: $1+y+y^{2}=0$ has one solution $y=1$. So $x=1$ and $z=2 \equiv 0$ ( $\bmod 2)$. Therefore, $f(1,1,0)=(1,2,0)$. Now, we can check that in the laboratory. Since the state space has only 18 elements, there are several functions with this property.

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[^0]:    1991 Mathematics Subject Classification. Primary:11T99; 05C20
    Key words and phrases. Finite fields, dynamical systems, partially defined functions, regulatory networks.

