Some Provably Hard Crossing Number Problems

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Abstract. This paper presents a connection between the problem of drawing a graph with minimum number of edge crossings, and the theory of arrangements of pseudolines. In particular, we show that any given arrangement can be forced to occur in every minimum-crossing drawing of an appropriate graph. Using recent results of Goodman, Pollack and Sturmfels, this yields that there exists no polynomial-time algorithm for producing a straight-line drawing of a graph, with minimum number of crossings from among all such drawings. We also study the problem of drawing a graph with polygonal edges. Here we obtain a tight bound on the smallest number of breakpoints which are required in the polygonal lines, in order to achieve the (unrestricted) minimum number of crossings.

1. Introduction.

A *drawing* of a simple graph G is a subset of the plane $\mathbb{R}^2$ where each vertex is represented by a different point, and each edge is represented by a homeomorph of the closed unit interval I^1, with appropriate ends. Further, the drawings of any two edges meet at most once, and if they do, then either the two edges are incident to a common vertex, where their drawings meet, or the two drawings cross at their intersection point (the term cross is assumed to be understood). The *crossing number* problem consists of producing a drawing of G so as to achieve the least possible number of crossings (this parameter is called the *crossing number* of G and denoted cr(G)). This problem is of interest in VLSI theory and in wiring layout problems (see [Le]), and it has long been of interest in the graph theory community (see [EG1], [Th2], [Tu]). Testing whether the crossing number of a graph is at most an input number k is an NP-complete problem [GJ].

In many of the applications, further, it is desirable that the edges be drawn as straight-line segments, with no restriction to orientation. Such a drawing has classically been called *rectilinear*, and the smallest number of crossings in all such drawings is called the *rectilinear crossing number*, denoted $cr_1(G)$. We remark that practically every paper on crossing numbers has

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in fact also dealt with rectilinear crossing numbers (the latter sometimes used to approximate the former). Using the proof in [GJ], it can be shown that computing the rectilinear crossing number is NP-hard. While this problem is not yet known to be in NP, clearly the (cartesian) coordinates of the vertices in a drawing can be assumed to be rational, and thus, integral.

In this paper we study the connection between crossing number problems and the theory of arrangements of pseudolines, an area of high interest in combinatorial geometry, which has most notably been studied by Grünbaum, Goodman, Pollack and others (see [Gr], [GP1-6]). For the purposes of this paper, a *pseudoline* is a homeomorph in the plane of the real line $\mathbb{R}$. An *arrangement* of pseudolines is a collection of pseudolines, every two of which meet at exactly one point, where they cross. The pseudolines in an arrangement are usually drawn truncated, with their ends in the infinite region of the drawing]. An arrangement divides $\mathbb{R}^2$ into regions or *faces*, and two arrangements are said to be *isomorphic* (or that one is a *realization* of the other) if they have the same facial structure.

Our first result is as follows (stated in abridged form here):

Theorem 1. Let A be an arrangement of pseudolines. There is a graph G_A, so that

(i) Every drawing of G_A with $cr(G_A)$ crossings contains a realization of A.

(ii) If the members of A are straight-lines, then every rectilinear drawing of G_A with $cr_1(G_A)$ crossings contains a straight-line realization of A. ∎

Thus if we can construct arrangements all of whose realizations are "bad" in some technical sense, we will also have graphs, all of whose minimum-crossing drawings are also "bad" in the same sense. In particular, recent work of Goodman, Pollack and Sturmfels implies the following result:

Theorem [GPS]. For any n there exists an arrangement E^n of straight-lines, such that in every straight-line realization of E^n the equations of the lines require exponentially many bits. ∎

Together with Theorem 1, this yields:

Theorem 2. There exists an infinite family of graphs $\{G^n\}$, such that in every rectilinear drawing of G^n with $cr_1(G^n)$ crossings, the coordinates of the vertices require more than polynomially many bits. ∎

As a result, there does not exist a polynomial-time algorithm for producing a rectilinear drawing of a graph which achieves the rectilinear crossing number. Here we are assuming a model where either (a) A physical drawing must be produced (as in the VLSI applications), or (b) The coordinates of the vertices must be explicitly written down. Of course, the result does not imply P≠NP, for the simple reason that there may be a polynomial-time algorithm for *computing* the rectilinear crossing number, which does not rely on *drawing* the graph. In fact such an algorithm could not even write the coordinates of all vertices.

Let $t \geq 1$. Rather than drawing a graph with straight-line edges, one might ask instead for *t-polygonal* drawings, in which each edge is drawn as a t-polygonal line (a polygonal line with at most t segments. Thus t=1 yields rectilinear drawings). As shown in [BD], in fact, using t=2 instead of t=1 can dramatically decrease the number of crossings. However, we remark that for any fixed t there is a result similar to Theorem 2. On the other hand, already with t=2, in polynomial space (logarithmically many bits per vertex) one can get within a quadratic bound of the crossing number. Details are left to the full paper.

How complicated can a minimum-crossing drawing of a graph be? In other words, is there a fixed number t, so that for all graphs G, the crossing number of G can be achieved with a t-polygonal drawing? We show the answer is no, and thus, for graphs of crossing number k or less, the minimum possible such t depends on k; let us call it t(k). We prove:

Theorem 3. There exist constants c_1 and c_2, so that for every $k \geq 1$,

$$c_1 k^{1/2} \leq t(k) \leq c_2 k^{1/2}. \blacksquare$$

To obtain the lower bound, we apply a recent result of Kratochvil and Matousek [KM] on polygonal realizations of pseudoline arrangements, and our construction in Theorem 1. The upper bound is obtained by a direct construction.

2. Proof of Theorem 1.

In this section we will provide a proof of Theorem 1. The full statement of the theorem, for the non-rectilinear case, is:

Theorem 1. Let A be an arrangement of n pseudolines. There exists a graph G_A with $cr(G_A)=5n(n-1)$ and $|E(G_A)| = O(n^3)$, with a distinguished subset S_A of edges, such that in every drawing of G_A with $cr(G_A)$ crossings, S_A contains a realization of A.

Proof: The graph G_A is obtained in several steps. See Fig. 1 for an example.

1. We begin by replacing each $x \in A$ with two copies, x_1 and x_2 drawn very close to each other. Next, we obtain a plane graph L by placing a vertex at crossing and at each end of every (truncated) pseudoline. Thus, for each $x \in A$, L contains two edge disjoint paths $p_1(x)$ and $p_2(x)$, where $p_i(x)$ has ends (degree one vertices) $u_{i1}(x)$, $u_{i2}(x)$ (we assume the labeling has been done so that $u_{2k}(x)$ and $u_{1k}(x)$ are next to each other in the outer face of L, see Fig. 1 (b)). Next, we add to L a cycle C, joining all the vertices $u_{ik}(x)$ in the cyclic order, so as to form with L a plane graph. Further, C contains a vertex $v_k(x)$ between every two consecutive vertices $u_{1k}(x)$, $u_{2k}(x)$, k=1,2, $x \in A$.

2. For each $x \in A$, we add an edge e(x) with endpoints $v_1(x)$ and $v_2(x)$, and so that e(x) is drawn inside C and "between" $p_1(x)$ and $p_2(x)$. Let S be the set of all edges e(x), $x \in A$, and $H = L \cup C \cup S$. See Fig. 1 (c).

3. Take a copy $H' = C' \cup L' \cup S'$ of H, drawn outside H (and with the obvious notation), and a matching M joining the vertices of C to those in C', so that $L \cup C \cup L' \cup C' \cup M$ is plane. We replace every edge $e=\{u,v\}$ of $C \cup C' \cup M$, in this drawing, by a set b(e) of m edge-disjoint paths of length 2 (where $m = 5n(n-1)+1$) from u to v. Let W be the graph obtained from $C \cup C' \cup M$. Thus the edges of W have no crossings. The resulting graph is G_A, and we set $S_A = S \cup S'$. See Fig. 1(d)

Let D_A be the drawing we have just constructed, which has 5n(n-1) crossings.

Suppose now that D^* is a drawing of G_A with $cr(G_A) \leq 5n(n-1)$ crossings. Consider an edge t of $C \cup C' \cup M$. Since $m > cr(G_A)$, it follows that at least one of the paths in b(t) has no crossings in D^*. Thus we obtain a plane drawing of $C \cup C' \cup M$ in the obvious way. Since $C \cup C' \cup M$ is 3-connected, it has a unique plane embedding, and its drawings in D^* and D_A are homeomorphic in the sphere. So in D^* both C and C' bound faces of $C \cup C' \cup M$, which must contain the drawings of $C \cup L$ and $C' \cup L'$ (respectively). Next, let $x, y \in A$. Then, by construction, the endpoints of e(x) and e(y) alternate along C. In fact, the endpoints of e(x) and the ends of each path $p_i(y)$, i=1,2, alternate along C (and similarly, the endpoints of e(y) and the ends of each path $p_i(x)$, i=1,2, alternate along C). Since for any $z \in A$, $p_1(z)$ and $p_2(z)$ are edge-disjoint, we count 5 crossings in D^* corresponding to the pair x,y (the Jordan curve theorem). In this way we count $5n(n-1)/2$ crossings in the region bounded C. Similarly with C'. Thus $5n(n-1) = cr(G_A)$, and the $5n(n-1)/2$ crossings we have just counted are all the crossings in the region bounded by C (resp. C'). Either C or C' bound an inner face of $C \cup C' \cup M$ in D^*, say C does. Consequently, in D^*, (i) the drawing of $C \cup L$ is plane, and (ii) For each $x \in A$, e(x) does not cross any edge in $p_i(x)$, i=1,2. Then (ii) implies that for each $x \in A$, e(x) is drawn "between" $p_1(x)$ and $p_2(x)$. We conclude that S realizes A. ∎

Comment: The graph G_A has vertices of large degree, but the same result can be achieved with a graph of maximum degree 3.

The rectilinear version of Theorem 1 is obtained by subdividing the edges in M O(1) times, so that D_A is rectilinear. Using for A one of the straight-line arrangements obtained from the results in [GPS], we have:

Theorem 2. There exists an infinite family of graphs $\{G^n\}$, such that in every rectilinear drawing of G^n with $cr_1(G^n)$ crossings, the coordinates of the vertices require more than $\exp(c\, |E(G^n)|^{1/3})$ many bits, where c is a fixed constant. ∎

3. Approximating the crossing number with polygonal edges.

Let us denote with $cr_t(G)$ the t-polygonal crossing number of G. In [BD], the following results where obtained:

Theorem [BD] For every $m \geq 1$ there exists a graph G^m with $cr(G^m) = 4$, but $cr_1(G^m) \geq m$. On the other hand, for every graph G, $cr_2(G) \leq 2(cr(G))^2$. ∎

As a result, the natural question arises, is there any fixed t, so that for all G, $cr_t(G) = cr(G)$? The answer to the question turns out to be no. In fact, let us define

$$t(k) = \min \{ t : cr_t(G) = cr(G) \text{ for all } G \text{ with } cr(G) \leq k \}.$$

Then, as stated in Theorem 3 in the introduction, there are constants c_1 and c_2, so that for every $k \geq 1$,

$$c_1 k^{1/2} \leq t(k) \leq c_2 k^{1/2}.$$

In the rest of the paper we will sketch the proofs of the lower and upper bounds in (1).

3.1. The lower bound.

In order to obtain the lower bound, we will use the following theorem, recently proved by Kratochvil and Matousek.

Theorem [KM]. For each n, there exists an arrangement Z(n) of n pseudolines, which cannot be realized with t-polygonal pseudolines unless $t \geq cn$, where c is a constant. ∎

[Remark: in [KM], arrangements are allowed to have pairs of pseudolines that do not meet, but this detail is easily dealt with to obtain the above theorem].

Now given n, consider the graph $G_{Z(n)}$ produced by Theorem 1. Write $H = G_{Z(n)}$. We have that $cr(H)=O(n^2)$. On the other hand, unless $t \geq cn$, we also have $cr_t(H) > cr(H)$ by definition of Z(n). Thus the lower bound in Theorem 3 is proved.

We point out that as a corollary of the lower bound, one can prove the following (curious) result:

Corollary. For every $t > 1$ there exists a graph G, with $cr(G) = cr_t(G)$, and such that for every $t \geq i > 1$, $cr_i(G) < cr_{i-1}(G)$. ∎

3.2. The upper bound.

We will next outline a proof of the upper bound $t(k) \leq O(k^{1/2})$ (a complete proof is given in the full paper). Let D be a drawing of G with $cr(G) = k$ crossings. We partition the edges of G into (at most) two classes, H and L, where

H = set of edges with more than $2k^{1/2}$ crossings in D, and

L = set of edges with at most $2k^{1/2}$ crossings in D.

(For convenience, we will also use H and L to refer to the corresponding subgraphs of G). Now if H is empty, then we are essentially done: we obtain, from D, a planar graph by placing a new vertex at each crossing point, in addition to the vertices of G. Since any planar graph has a straight-line drawing (see [Th2] for a short proof), we will obtain a $(2k^{1/2})$-polygonal drawing of G with cr(G) crossings, as desired. In general, H is of course nonempty, but still this basic construct is the appropriate idea to use.

Our procedure is to first draw H, and then L, always obtaining drawings homeomorphic to those in D. The key fact here is that $|H| < k^{1/2}$. So if we draw H, ignoring L, and using the planar graph construction as in the previous paragraph, the members of H will be $(k^{1/2} + 1)$-polygonal lines (we stress that the crossings of edges in H with edges in L are ignored here). Similarly, if we independently draw L, the edges will be $O(k^{1/2})$-polygonal lines (by definition of L). As sketched below, by building more structure especially into the drawing of H, we will be able to piece together the two drawings as desired.

Consider the drawing of H provided by D. In general, this drawing will have several arc-connected components, which we call *pieces*. Each piece consists of possibly more than one graph-theoretic component of H, and each edge in H appears in one piece. We will draw each piece separately, with some added structure.

Let Z be a piece. Then the complement in $\mathbb{R}^2$ of Z consists of several connected regions, or *faces* (so if Z is planar these are faces in the standard sense). [Notice that the boundary of a face may not be simple]. One of the faces is unbounded, in the sense that it is homeomorphic to the complement of a closed disk. Further, the complement in $\mathbb{R}^2$ of H is partitioned into several connected regions, which we call *plots*. Each plot is in general the intersection of several faces corresponding to different pieces, as discussed above. Thus, in general, each plot P will be incident to several pieces, all of whom, with at most one exception, are enclosed by P (and the one exception encloses P). In other words, P is homeomorphic to an open disk (or to $\mathbb{R}^2$) with several holes cut out (where each boundary component of P corresponds to a different piece). How do H and L fit together? As follows: each edge in L is partitioned into sections, each contained in a separate plot.

The basic idea is that we can draw any piece of H, using the planar graph method described above, so that in addition each face has "bounded link distance" (that is, any two points in the face can be joined by a 2-polygonal line). This is achieved (roughly) by adding, to each face, an additional vertex to act as a "center", with edges joining this point to all vertices on the boundary of the face, and then drawing the new graph.

We next draw each "chunk" of L contained in a plot, as follows. Consider any given plot P. Regard it as drawn on the sphere. Then contract to a point every piece of H incident to P. We obtain a drawing which is (essentially) homeomorphic to the subset of L contained in P. As above, we now construct a planar graph by placing a vertex at each crossing, and also at each "shrunken piece".

It is not difficult to see that, having drawn all of L as in the previous paragraph, we can now integrate this drawing with small copies of the drawings of the pieces of H produced before, to obtain a drawing of G homeomorphic to that in D. Further, because of the bounded link distance property of the faces (of pieces of H), the drawing of any edge of L requires O(1) additional breakpoints as it crosses from one plot to another. Thus, in total, the number of breakpoints in any edge e in L is in the worst case proportional to the number of crossings e has in D. Hence every edge is drawn as an $O(k^{1/2})$-polygonal line, as desired.

This concludes the sketch of the proof of the upper bound $t(k) \leq O(k^{1/2})$.

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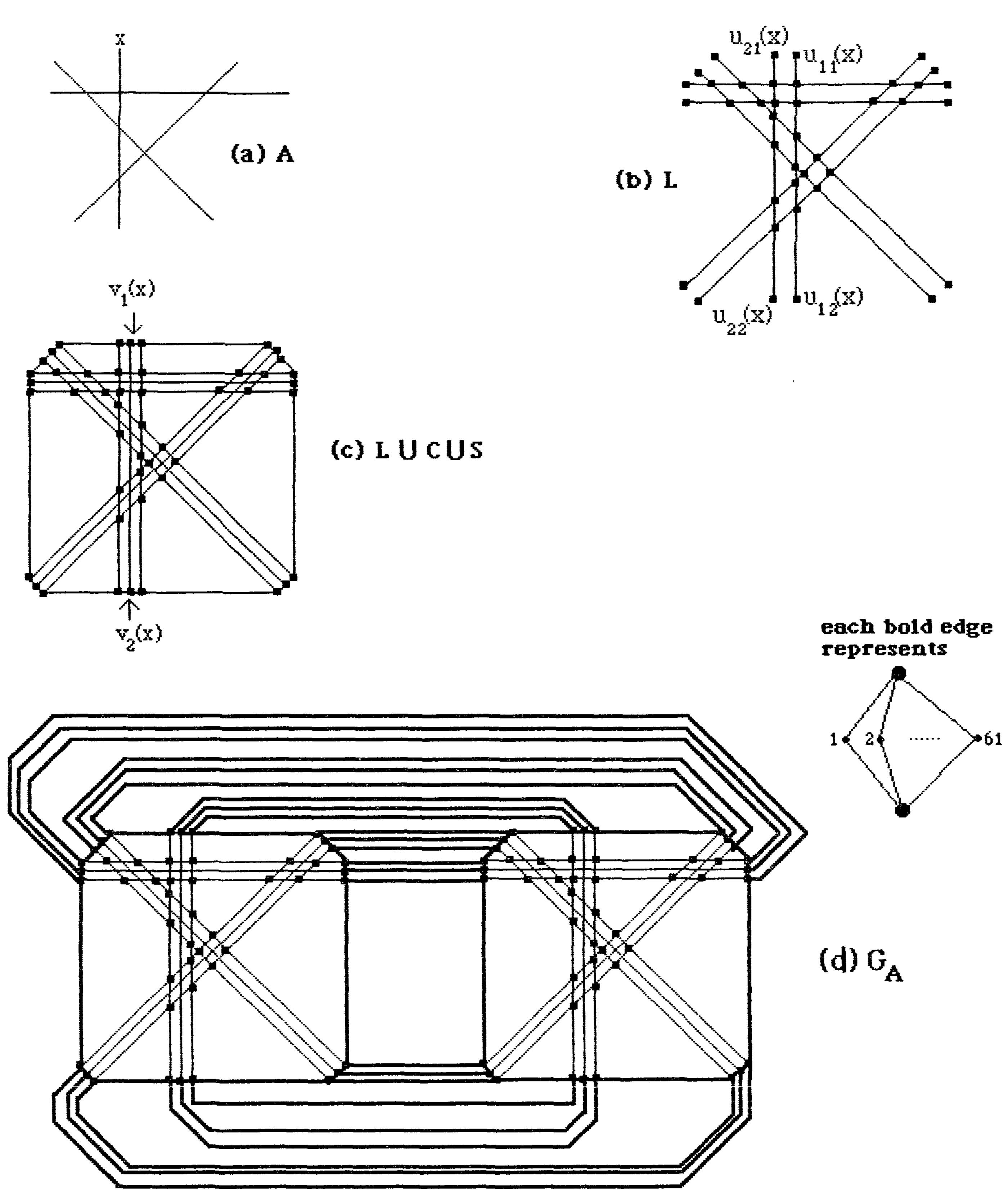
x
(a) A
$u_{21}(x)$
$u_{11}(x)$
(b) L
$u_{22}(x)$
$u_{12}(x)$
$v_1(x)$
(c) L U C U S
$v_2(x)$
each bold edge represents
1
2
......
61
(d) G_A

FIGURE 1