

The Many Facets of Information in Networked Estimation and Control

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Annu. Rev. Control Robot. Auton. Syst. 2023.
6:233–59

First published as a Review in Advance on
November 2, 2022

The *Annual Review of Control, Robotics, and
Autonomous Systems* is online at
control.annualreviews.org

<https://doi.org/10.1146/annurev-control-042820-010811>

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Keywords

networked control systems, cyber-physical systems, control under communication constraints, decentralized inference, timing information, event triggering

Abstract

Networked control systems, where feedback loops are closed over communication networks, arise in several domains, including smart energy grids, autonomous driving, unmanned aerial vehicles, and many industrial and robotic systems active in service, production, agriculture, and smart homes and cities. In these settings, the two main layers of the system, control and communication, strongly affect each other's performance, and they also reveal the interaction between a cyber-system component, represented by information-based computing and communication technologies, and a physical-system component, represented by the environment that needs to be controlled. The information access and distribution constraints required to achieve reliable state estimation and stabilization in networked control systems have been intensively studied over the course of roughly two decades. This article reviews some of the cornerstone results in this area, draws a map for what we have learned over these years, and describes the new challenges that we will face in the future. Rather than simply listing different results, we present them in a coherent fashion using a uniform notation, and we also put them in context, highlighting both their theoretical

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insights and their practical significance. Particular attention is given to recent developments related to decentralized estimation in distributed sensing and communication systems and the information-theoretic value of event timing in the context of networked control.

1. INTRODUCTION

In this article we are concerned with networked control systems (NCSs) composed of a network of interacting elements, including sensors, actuators, and computing and communication devices connected in closed loop, with the objective of performing tasks that require interaction with the physical world. **Figure 1** presents a schematic representation of such a system. Examples include autonomous and remotely controlled robots, unmanned aerial vehicles, autonomous vehicles, and several industrial and consumer control systems. One of the key features of NCSs is the interaction between a cyber component (a networked computing and communication infrastructure composed of controllers, encoders, and decoders) and a physical component (a physical plant), which occurs through distributed sensors and actuators. For this reason, NCSs are also referred to as cyber-physical systems (1).

In this framework, two fundamental questions that we wish to address are (a) what is the minimum amount of information transfer among the different components of the system that is needed to keep the overall system stable, and (b) how can we design encoding, decoding, and control policies that best exploit the available information flow to reach stability? As we shall see, these questions are closely related to the ability to perform decentralized estimation through distributed sensing and communication, since achieving a reliable estimate of the state is key to determining the correct control action, and many of the results on control and stabilization also have counterparts in this setting. Deriving these results requires the development of a new

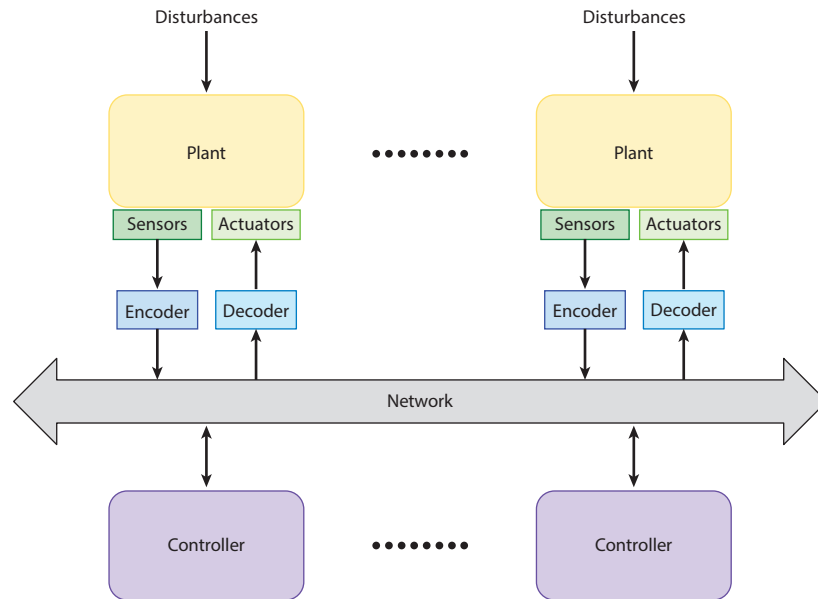


Figure 1

A distributed communication and control system.

information-theoretic paradigm in which the dynamical system aspects of the problem place strong constraints on the communication aspects. These constraints are typically ignored in classical information theory but must be taken into account in the context of control; in our setting, the utility of the received information at any given time, rather than merely the rate at which the information flows, determines the ability to perform control. This utility is associated with system-dependent parameters, as information quickly becomes outdated and thus unusable for control.

This new paradigm also leads to the realization that information can have many facets in NCSs. For example, the event corresponding to the availability of new data, along with the data themselves, can encode information that can be useful for control. In some cases, this allows one to perform control with a remarkably low data rate, which can be counterintuitive from a more classical information-theoretic perspective. The main theme of this article is to show how these considerations can be cast into a rigorous theory that lays the foundations for the development of next-generation NCSs. Related works investigating information decay over time include the age-of-information paradigm (2–5). However, as pointed out by Soleymani et al. (6, 7) and Uysal et al. (8), while these works are relevant for some specific applications (e.g., news feeds), they do not consider that when information is used in the context of control, a relevant age metric should be related to system parameters. We do not wish to review here all of the alternative measures of information utility that have been proposed in the literature; instead, we focus on the different aspects in which information can be encoded to be useful for control and identify the communication constraints needed for stability in NCSs.

In the last two decades, the research community has studied information constraints in NCSs by developing several mathematical abstractions of system components and interconnections. The results that have been obtained shed light on the behavior of real systems and provide guidelines to develop effective control policies. Surveys of this literature have appeared in numerous papers and books (9–17). We extend these reviews, focusing on data-rate requirements for stabilization and, in particular, on recent advancements and insights obtained through the study of event-triggered control policies and distributed inference systems. One key point is that the information flow in feedback systems is not only associated with data flowing through the links connecting the different devices but is also more generally encoded in events that occur over time. This new point of view leads to several extensions of classic results and to a broader perspective on the information constraints associated with control systems. Another point is that decentralized inference is an important building block for performing control in NCSs, and information constraints can be derived in this case for both single-plant and multinode networks. While in this article we focus primarily on system stabilization, we point out that related studies of optimal control under communication constraints have also been performed (6, 7, 18–20). Recently, the trade-offs between rate and linear-quadratic regulator cost for periodic control schemes have been evaluated (21).

The notation used in this article is as follows: Random variables are displayed in sans serif roman fonts, and their realizations are in serif italic fonts. Vectors and matrices are denoted by bold lowercase and uppercase letters, respectively. For example, a random variable and its realization are denoted by $\mathbf{x}$ and x , respectively; a random vector and its realization are denoted by $\mathbf{x}$ and $\mathbf{x}$, respectively; and a random matrix and its realization are denoted by $\mathbf{X}$ and $\mathbf{X}$, respectively. The expectation of $\mathbf{x}$ is denoted by $\mathbb{E}\{\mathbf{x}\}$. Given a discrete-time stochastic process $\{\mathbf{x}_t\}_{t \geq 0}$, the notation $\mathbf{x}_{s:t}$ represents the vertical concatenation of $\mathbf{x}_\tau$ for integers $s \leq \tau \leq t$. A logarithm of a positive number x with base 2 is denoted by $\log x$. The Euclidean norm of vector $\mathbf{x}$ is denoted by $\|\mathbf{x}\|$. $\mathcal{N}(a, b)$ denotes a Gaussian distribution with mean a and variance b .

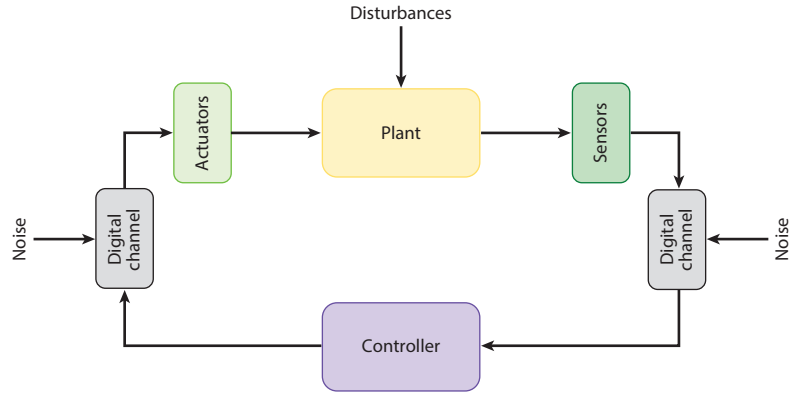


Figure 2

A high-level abstraction of a single-plant networked control system from a communication perspective.

2. THE DATA-RATE THEOREM

We start by considering a simple high-level abstraction of a single-plant network as a block diagram (as depicted in **Figure 2**) and later extend the treatment to the multinode case. A dynamical system evolves over time according to deterministic state equations, affected by stochastic disturbances. Sensors monitor the system's output, and their readings are encoded and sent through a digital communication channel to a controller, whose action is fed back to the actuators through another digital communication channel. We can further simplify this model by assuming that the controller is colocated with the actuators and that the only communication channel is between the sensors and the controller. This comes at no loss of generality so long as the information available to perform encoding and decoding is the same at the sensor, controller, and actuators. In this case, performing decoding and reencoding at the controller is redundant, and the bottleneck link determines the effective data rate. The information flow through the feedback loop can then be viewed as occurring over a single channel, which can also represent a multihop network connection, and in this case the effective data rate refers to the rate available at the endpoints of the connection. On the other hand, we point out that in practice the information available for encoding and decoding may be different at different points in the network, and solutions in this case are highly dependent on the assumed information pattern. Nevertheless, a global view of the network can be achieved by running a distributed consensus protocol (22) before attempting to perform control.

The first basic result regarding the information flow requirements for stabilization that we wish to describe is the so-called data-rate theorem (23, 24), which has also been the starting point for much of the research in the area of information constraints in NCSs. This theorem quantifies the effect that communication has on closed-loop stabilization of unstable systems by stating that the communication rate available in the feedback loop should be at least as large as the intrinsic entropy rate of the system. For continuous linear systems, the intrinsic entropy rate corresponds to the sum of the unstable modes of the system, and for discrete systems, it corresponds to the sum of the logarithms of the unstable modes. When this condition is satisfied, the controller can compensate for the growth of the state space occurring during the communication process and is able to keep the system stable.

To illustrate this result for linear systems, consider the set of equations

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{v}_k, & 1a. \\ y_k = \mathbf{C}\mathbf{x}_k + \mathbf{w}_k, & 1b. \end{cases}$$

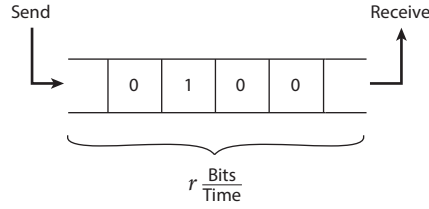


Figure 3

A bit-pipe channel.

where $k = 0, 1, \dots$ is time; $\mathbf{x}_k \in \mathbb{R}^d$ represents the state variable of the system; $\mathbf{u}_k \in \mathbb{R}^m$ is the control input; $\mathbf{v}_k \in \mathbb{R}^d$ is an additive disturbance; $\mathbf{y}_k \in \mathbb{R}^p$ is the sensor measurement; $\mathbf{w}_k \in \mathbb{R}^p$ is the measurement disturbance; and $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are constant real matrices of matching dimensions. Standard conditions on $(\mathbf{A}, \mathbf{B})$ to be reachable and on $(\mathbf{C}, \mathbf{A})$ to be observable are added to make the problems considered to be well posed. The equivalent continuous formulation is

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{v}(t), & 2a. \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{w}(t). & 2b. \end{cases}$$

In a first approximation, noise and bandwidth limitations in the communication channel can be captured by modeling the channel as a rate-limited bit pipe capable of transmitting only a fixed number r of bits in each time slot of the system's evolution (see **Figure 3**). In this way, the channel can represent a network connection with a limited available bit rate; when transmitting below this rate, communication errors are assumed to be negligible, and only the quantization of the transmitted messages is accounted for.

When the control objective is to keep the system's state bounded or asymptotically drive it to zero, the control law can be a linear function of the state estimate. Hence, for unstable linear systems under this rate-limited bit-pipe communication model, the central issue is to characterize the ability to perform a reliable estimate of the state at the receiving end of the communication channel. To keep the system stable, the data-rate theorem states that the information rate r supported by the channel must be high enough compared with the unstable modes of the system that it can compensate for the expansion of the state during the communication process—that is,

$$r > \sum_{|\lambda_i| \geq 1} \log_2 |\lambda_i| \text{ (bits/s)} \quad 3.$$

for discrete systems, where $\{\lambda_i\}$ represents the open-loop eigenvalues raised to their corresponding algebraic multiplicities, and

$$r > \sum_{\text{Re}\{\lambda_i\} > 0} \lambda_i \log_2 e \text{ (bits/s)} \quad 4.$$

for continuous systems. If the real parts of all the eigenvalues of $\mathbf{A}$ are positive (unstable), this can be written as

$$r > \text{tr}(\mathbf{A}) \log_2 e \text{ (bits/s)}. \quad 5.$$

The intuition behind the data-rate theorem is evident when considering a scalar system and noticing that while the volume of the state of the open-loop system increases by $|\lambda|$ in a unit time step in the discrete setting (or by $|e^\lambda|$ in the continuous setting), in closed loop this expansion is compensated by a factor 2^{-r} due to the partitioning induced by the coder providing r bits of information through the communication channel. By imposing the product to be less than one and

taking the logarithm base two, the results follow. Another interpretation arises if one identifies the right-hand sides of Equations 3 and 4 as measures of the rate at which information is generated by the unstable plant; then, the theorem essentially states that to achieve stability, the channel must be able to transport information as fast as it is produced.

Early incarnations of this fundamental result appeared in References 25–29 for undisturbed, scalar, unstable plants, when the objective is to keep the state bounded at all times. Improvements of the result from maintaining a bounded state to obtaining a state that asymptotically approaches zero were shown in References 30–32 and require an adaptive zoom-in, zoom-out strategy that adjusts the range of the quantizer so that it increases as the plant’s state approaches the target and decreases if the state diverges from the target. This follows the intuition that in order to drive the state to zero, the quantizer’s resolution should become higher close to the target.

In the presence of stochastic disturbances, asymptotic stability can only be guaranteed within the range of the disturbances. Tatikonda & Mitter (24) showed that for almost surely (a.s.) bounded disturbances and initial condition, the data-rate theorem holds, and we can have

$$\sup_{k \in \mathbb{N}} \|\mathbf{x}_k\|^2 < \infty, \quad \text{a.s.} \quad 6.$$

On the other hand, unbounded disturbances can drive the state arbitrarily far from zero, and one can only guarantee stability in a weaker, probabilistic sense. The typical approach is to consider mean-square (m.s.) stability, namely,

$$\sup_{k \in \mathbb{N}} \mathbb{E}\{\|\mathbf{x}_k\|^2\} < \infty. \quad 7.$$

Nair & Evans (23) proved the data-rate theorem using m.s. stability for systems with unbounded stochastic disturbances provided that higher moments are bounded—that is,

$$\exists \epsilon > 0 : \mathbb{E}\{\|\mathbf{x}_0\|^{2+\epsilon}\} < \infty, \sup_{k \in \mathbb{N}} \mathbb{E}\{\|\mathbf{v}_k\|^{2+\epsilon}\} < \infty, \sup_{k \in \mathbb{N}} \mathbb{E}\{\|\mathbf{w}_k\|^{2+\epsilon}\} < \infty. \quad 8.$$

A similar data-rate theorem formulation also holds for nonlinear systems. In this case, one may consider a partially observed, time-invariant, dynamical system,

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t)), \\ \mathbf{y}(t) = b(\mathbf{x}(t), \mathbf{w}(t)), \end{cases} \quad \begin{matrix} 9a. \\ 9b. \end{matrix}$$

where f denotes the state transition function; b denotes the observation measurement function; and $\mathbf{y}, \mathbf{x}, \mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ denote the observation, state, control input, state disturbance, and observation disturbance, respectively, as before. To express the data-rate theorem, one needs to quantify the rate at which the dynamical system generates information, which for the linear case corresponds to the right-hand sides of Equations 3 and 4. Obviously, this rate should be intrinsic to the nonlinear dynamical system and thus independent of encoding and decoding processes, controllers, and feedback communication constraints. One way to obtain such quantification is to refer to the topological entropy of the system (33), a construction spin-off from Kolmogorov’s entropy definition for completely deterministic nonlinear maps (34). The idea behind this definition is to first fix an open cover for the space, through which each iteration of the map is observed. As the number of iterations increases, the family of all possible intersections of initial-state open sets forms an increasingly fine open cover for the space. The topological entropy of the map is then obtained by taking the supremum of the asymptotic rate of increase of the cardinality of this open cover over all observation open covers. This measures the fastest rate at which uncertainty about the initial state can be reduced or, equivalently, the fastest rate at which initial-state information can be generated.

There is an analogy here with source coding in classical information theory (35), which Kolmogorov credited as the inspiration for his own work (36). Source coding is concerned with determining the lowest data rate at which a stochastic source can be encoded, transmitted, and reliably decoded over a noiseless digital channel. Shannon's source-coding theorem states that the lowest possible data rate is equal to the Shannon entropy of the source, independent of external constructs. Parallels between Kolmogorov's deterministic theory and Shannon's stochastic theory were further explored by Lim & Franceschetti (37) and Donoho (38). Nair et al. (33) considered fully observable, undisturbed systems closed over a bit-pipe communication channel and used the notion of topological entropy to determine necessary and sufficient bit rates for local uniform asymptotic stability. Hagihara & Nair (39) provided the extension for partially observable systems. Liberzon & Hespanha (40) and Sharon & Liberzon (41) followed a different approach, expressing sufficient conditions for stabilization in terms of the Lipschitz constant. De Persis (42) considered noiseless and fully observed nonlinear systems with a special upper triangular structure (i.e., feedforward systems), providing a tight condition for global stability that matches the topological entropy formulation of Nair et al. (33). Extensive compendiums of related results connecting different variants of topological entropy definitions to data-rate requirements for different stability notions have been published in works by Kawan (17) and, more recently, by Colonius & Hamzi (43 and references therein).

3. EXTENSIONS TO NOISY CHANNELS

Several generalizations of the simple bit-pipe communication model have been considered in the literature. An important body of work relates to extensions to stochastic channels, namely, channels whose behavior varies randomly due to noise. In this case, the rate available for transmission through the noisy channel must be defined in terms of information capacity. In this setting, a key result is that for undisturbed systems one can derive a data-rate theorem expressing the rate available for transmission in terms of the Shannon capacity of the channel (44, 45), which must be larger than the entropy rate of the system in order to guarantee stability. By contrast, when systems are subject to disturbances, the standard notion of Shannon capacity turns out to be insufficient to express the ability to stabilize the system in both the a.s. and m.s. sense. In this case, alternative notions of capacity that have stronger reliability constraints must be used to formulate data-rate theorems; that is, the anytime capacity must be used to express m.s. stability (46) (or, more generally, α -moment stability), and the zero-error capacity must be used to express a.s. stability (47).

The main difference between these notions of capacity is that while the Shannon capacity is defined as the supremum of the rates that can achieve an arbitrarily small probability of error, the anytime capacity has more stringent conditions on the probability of error, requiring anytime decoding of all code words every time a new symbol is received and imposing that the probability of having an error in any of the decoded code words tends to zero exponentially as more and more symbols are received. On the other hand, the zero-error capacity requires that the probability of error be exactly zero for every transmitted code word. In short, the Shannon capacity offers only weak reliability constraints, and it is generally insufficient to characterize the ability to stabilize the system in the presence of external disturbances, while the zero-error and anytime capacities offer stronger reliability constraints and can be used to characterize the ability to stabilize the system in the presence of external disturbances in an a.s. and moment setting, respectively. For a more extensive discussion of the relationship between the different capacity definitions, we direct readers to References 9 and 48.

To illustrate the main results for noisy channels, first consider stabilization of a scalar system over a simple stochastic erasure channel, where the rate varies randomly between the two values

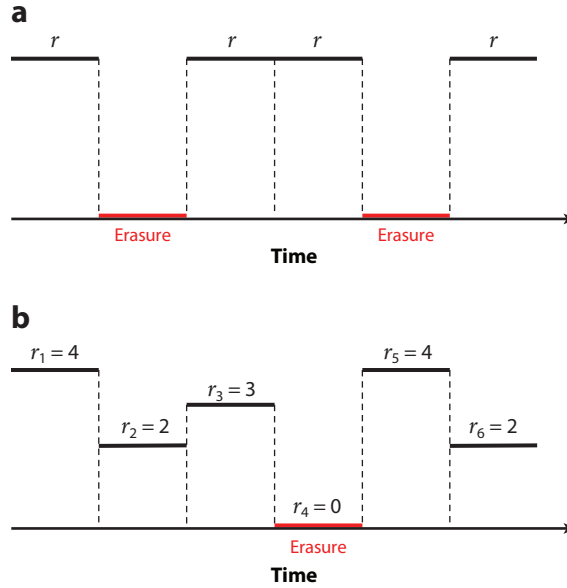


Figure 4

Examples of stochastic channels. (a) An r -bit packet erasure channel. (b) A stochastic-rate channel, including erasures.

$\{r, 0\}$ in an independent and identically distributed (i.i.d.) fashion. That is, for all k we have the stochastic-rate process

$$r_k = \begin{cases} 0 & \text{with probability } p, \\ r & \text{with probability } 1 - p, \end{cases} \quad 10.$$

and we assume that both encoder and decoder have causal knowledge of the channel realization. In information-theoretic terms, this is as an r -bit packet erasure channel with acknowledgment of packet reception and erasure probability p . This channel is illustrated in **Figure 4a**.

The condition to achieve m.s. stabilization (Equation 7) over this channel is expressed as

$$\mathbb{E} \left\{ \frac{|\lambda|^2}{2^{2r_k}} \right\} = p \frac{|\lambda|^2}{2^0} + (1 - p) \frac{|\lambda|^2}{2^{2r}} < 1. \quad 11.$$

Using the same interpretation of production and consumption of information mentioned above, this condition states that the average of the product of the open-loop state expansion and compensation through r -bit quantization should be kept less than one to ensure stability. By rearranging terms, we obtain an expression where the anytime capacity of the channel appears on the left-hand side of the inequality (46)

$$\check{C}(2) \equiv \frac{2^{2r}}{2^{2r}p + 1 - p} = \frac{2^{2r}}{p(2^{2r} - 1) + 1} > |\lambda|^2, \quad 12.$$

where the anytime capacity $\check{C}(\alpha)$ is parameterized by the m.s. stability exponent $\alpha = 2$. The expression in Equation 12 shows a clear trade-off between the reliability of the channel and the quantization rate—namely, that when the quantization rate $r \rightarrow \infty$, we obtain

$$\frac{1}{p} > |\lambda|^2, \quad 13.$$

indicating that the erasure probability p must be low enough to guarantee stability. By contrast, the Shannon capacity of the r -bit erasure channel is (49)

$$C = (1 - p)r, \quad 14.$$

which diverges as $r \rightarrow \infty$, independent of the value of p . It follows that in this case the Shannon capacity does not give any indication of the ability to achieve stabilization. In general, for any finite value of r both the quantization rate and the reliability of the channel play a role in determining the ability to stabilize the system. We also note that when $r \rightarrow 0$, we obtain $|\lambda| < 1$, meaning that the system cannot be stabilized regardless of how low the erasure probability p is.

The anytime capacity is the correct figure of merit to express data-rate theorems describing the ability to achieve α -moment stabilization (46) for more general noisy channels, besides the simple erasure one. The price to pay to have a complete characterization, however, is the computation of the anytime capacity, which becomes increasingly difficult. Only for a few channels have anytime capacity stabilization conditions similar to Equation 12 been obtained. These include time-varying rate channels (50, 51), where the rate process r_k varies randomly over time in an i.i.d. fashion, taking values in a subset of the nonnegative integers (see **Figure 4b**). The erasure channel considered above is a special case, when the rate process takes values in $\{0, r\}$. Results have also been obtained for stochastic channels, where the rate varies according to a Markov process (48, 52, 53). This allows arbitrary temporal correlations of the channel variations over time, and results rely on the theory of Markov jump linear systems (9).

In the special case of additive Gaussian channels, it turns out that the Shannon capacity is indeed sufficient to characterize m.s. stability (for a description of these results, see section 1.4.4 in Reference 9 and references therein). Another way to use the weaker notion of Shannon capacity to characterize the ability to stabilize the system is to relax the notion of stability. Matveev & Savkin (15, chapter 8) considered the weaker notion of stability in probability, requiring the state to be bounded with a probability of at least $(1 - \epsilon)$ by a constant K_ϵ that diverges as $\epsilon \rightarrow 0$, that is,

$$\mathbb{P}\left\{\sup_t |\mathbf{x}(t)| < K_\epsilon\right\} > 1 - \epsilon, \quad 15.$$

and showed that in this case it is possible to stabilize linear systems with bounded disturbances over noisy channels provided that the Shannon capacity of the channel is larger than the entropy rate of the system. At the opposite extreme, for general stochastic channels and systems with bounded disturbances, if instead of the m.s. stability shown in Equation 7 one wants to achieve the more stringent a.s. stabilization condition in Equation 6, a basic result from Matveev & Savkin (47) showed that the capacity notion to use is the zero-error one, and Nair (54) showed that the zero-error capacity can be written in terms of an information functional describing the flow of information through the feedback loop.

Some additional extensions relate to stabilization over channels with multiplicative noise that can be used to model fast-fading wireless communication channels or synchronization errors in system sampling. One example of this case is a paper by Ding et al. (55), who considered the following scalar system:

$$\begin{cases} \mathbf{x}_{k+1} = A\mathbf{x}_k - \mathbf{u}_k, \\ \mathbf{x}'_k = \mathbf{z}_k \mathbf{x}_k, \end{cases} \quad \begin{matrix} 16a. \\ 16b. \end{matrix}$$

where A is constant, $\mathbf{x}_0 \sim \mathcal{N}(0, 1)$, $\mathbf{u}_k$ can be any function of the current and previous observations, and $\mathbf{z}_k$ are random variables representing the multiplicative noise. In this setting, Ding et al. (55) showed a result that is reminiscent of a data-rate theorem: If $\mathbf{z}_k$ are i.i.d. with a known bounded density with unit mean and variance equal to σ^2 , letting $A^* = \sqrt{1 + (1/\sigma^2)}$, we have that a memoryless linear controller can stabilize the system in a second-moment sense if $A \leq A^*$. Moreover, if $A > A^*$, the system cannot be second-moment stabilized using a linear control strategy.

4. EVENT-TRIGGERED CONTROL

Event triggering (56–71) is a recent control paradigm that seeks to prescribe information exchange between the controller and the plant in an opportunistic manner. Rather than periodically communicating the control action, communication in event-triggered control occurs only when triggered by events that indicate the need to send fresh information to guarantee the correct execution of the task at hand (e.g., stabilization or tracking). The primary focus then is on minimizing the number of transmissions while guaranteeing the control objectives.

At a high level, one can view event triggering as sampling in time with the objective to identify the minimum sampling rate at which information may be transmitted through the feedback loop. Similarly, a bit-pipe communication model can be viewed as sampling in space (as quantization of the signal), and the data-rate theorem corresponds to the identification of the minimum quantization rate that can still guarantee stabilization. In the case of stochastic channels, we have both sampling in space, since we are transmitting through a digital channel messages of finite precision, and sampling in time, through errors and erasures. This view suggests that there should be a close connection between data-rate theorems and event-triggered control (69).

A first connection was revealed by Tallapragada & Cortés (72), who presented a data-rate theorem for event-triggering strategies for systems subject to bounded disturbances and controlled over a bit-pipe communication channel. Consider the system's equations,

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{v}(t), \\ \mathbf{y}(t) = \mathbf{x}(t), \end{cases} \quad \begin{matrix} 17a. \\ 17b. \end{matrix}$$

where the initial condition $\mathbf{x}(0)$ and the system disturbance $\mathbf{v}(t)$ are a.s. bounded. At each triggering event, the sensor transmits to the controller a packet of a fixed number of bits. If we let $b_s(t)$ be the total number of bits transmitted by the sensor to the controller up to time t , then the data-rate theorem is expressed in terms of the asymptotic average transmission bit rate of the sensor,

$$r \equiv \limsup_{t \rightarrow \infty} \frac{b_s(t)}{t} \quad (\text{bits/s}). \quad 18.$$

If we let $\|\mathbf{x}_\infty\|^2$ be a deterministic bound on the steady state and κ be a sufficiently large constant (depending on both the range of the disturbance and the initial condition), then it turns out that to obtain exponential stability at rate $\sigma \leq 0$, that is, requiring that for all $t \geq 0$ that

$$\|\mathbf{x}(t)\|^2 \leq (\kappa - \|\mathbf{x}_\infty\|^2)e^{-2\sigma t} + \|\mathbf{x}_\infty\|^2 \quad \text{a.s.}, \quad 19.$$

we need

$$r \geq (\text{tr}(\mathbf{A}) + \sigma d) \log_2 e \quad (\text{bits/s}), \quad 20.$$

where d indicates the dimension of the system.

The expressions in Equations 19 and 20 are consistent with Equations 5 and 6, where σd represents the extra bits required for exponential convergence to the steady state. It follows that the result from Tallapragada & Cortés (72) can be viewed as being analogous to the one from Tatikonda & Mitter (24) but is obtained here in the context of event triggering for continuous systems and with exponential convergence guarantees. However, while the result from Tatikonda & Mitter (24) is a data-rate theorem that is both necessary and sufficient for stabilization, the event-triggered controller design proposed by Tallapragada & Cortés (72) uses an asymptotic data rate that is within a constant factor of the necessary condition shown in Equation 20. This sufficient rate clearly depends on the triggering strategy. In particular, the event-triggering strategy utilized by Tallapragada & Cortés (72) is based on a Lyapunov function that ensures the desired convergence rate of the state. Nevertheless, the proposed design adjusts the communication rate

in accordance with state information in an opportunistic fashion, and it guarantees a uniform positive lower bound on the times between successive triggering events, thus avoiding degenerate cases where triggering occurs infinitely often in a finite interval.

5. TIMING INFORMATION IN EVENT TRIGGERING

The results from Tallapragada & Cortés (72) seem to indicate that the data-rate theorem is in complete harmony with event triggering, in the sense that regardless of whether the transmission rate is limited by channel conditions or is limited opportunistically through event triggering, the same fundamental limitation applies, which is dictated by the unstable modes of the system and by the desired convergence rate and is expressed by Equation 20. It turns out, however, that event triggering can also exploit an additional resource that is not accounted for in the current formulation and that allows stabilization to be achieved with a dramatically lower data rate.

Kofman & Braslavsky (73) revealed that if the channel does not introduce any delay and the controller is aware of the triggering strategy used by the sensor, then one can achieve stabilization by transmitting at a rate that is arbitrarily close to zero. To illustrate this point, consider the following undisturbed system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \quad \begin{matrix} 21a. \\ 21b. \end{matrix}$$

where $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in \mathbb{R}$ (i.e., $\mathbf{C} \in \mathbb{R}^{1 \times d}$), and the only uncertainty is due to the random initial condition. The channel connecting the sensor to the controller is assumed to be capable of transmitting one bit in an arbitrarily small time unit, so that communication of this binary symbol can be considered instantaneous when compared with the system dynamics. Let $\{t_s^k\}_{k \in \mathbb{N}}$ be the sequence of times at which the sensor transmits a bit to the controller. These times are set by event triggering according to a level-crossing strategy. If we let b be a given threshold, then a transmission occurs every time the absolute value of the difference between two successive output samples crosses that threshold—that is, the triggering condition is given by

$$|\mathbf{y}(t_s^k) - \mathbf{y}(t_s^{k+1})| = b. \quad 22.$$

The system initially evolves in open loop by letting $\mathbf{u} = 0$. Then, at each triggering time the sensor transmits a single bit to the controller, which encodes the sign of the b step change in the $\mathbf{y}$ value. By receiving at least $d + 1$ bits and solving the system of equations

$$\begin{bmatrix} \mathbf{C}(e^{\mathbf{A}t_s^1} - e^{\mathbf{A}t_s^0}) \\ \mathbf{C}(e^{\mathbf{A}t_s^2} - e^{\mathbf{A}t_s^1}) \\ \vdots \\ \mathbf{C}(e^{\mathbf{A}t_s^d} - e^{\mathbf{A}t_s^{d-1}}) \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} \mathbf{y}(t_s^1) - \mathbf{y}(t_s^0) \\ \mathbf{y}(t_s^2) - \mathbf{y}(t_s^1) \\ \vdots \\ \mathbf{y}(t_s^d) - \mathbf{y}(t_s^{d-1}) \end{bmatrix},$$

where the right-hand side is a column vector of $\pm b$ values, the controller can infer the initial condition provided that the matrix on the left-hand side is nonsingular. Once the initial condition is known, it can then stabilize the system in a closed-loop fashion. Since we have

$$\limsup_{t \rightarrow \infty} \frac{b_s(t)}{t} = \lim_{t \rightarrow \infty} \frac{(d+1)}{t} = 0, \quad 23.$$

it follows that we can stabilize the system with an arbitrarily low transmission rate.

The intuition behind the result follows by noting that, similarly to how pauses are used in spoken language to convey information, in the context of event-triggered control it is possible to transmit information in the feedback loop not only by the content of messages but also with

their timing. Specifically, in the absence of a delay in the communication channel, the mere act of sending one bit at a given time can reveal the state of the system with arbitrary precision, and transmitting a single data payload bit at every triggering event is enough to compute the appropriate control action. In fact, we may take this intuition one step further and also notice that under the same assumptions of Kofman & Braslavsky (73), we do not even need to transmit a bit at each triggering time. To reveal any component x of $x(0)$, we could transmit a single arbitrary symbol, $\clubsuit$, at a time equal to any bijective mapping of x into a point of the nonnegative reals. For example, we could transmit a $\clubsuit$ symbol at time $t = \tan^{-1}(x)$, where $t \in [0, \pi]$. Since there is no choice associated with the symbol selection, in principle receiving the symbol should not bring any information. However, its arrival time carries information and can reveal x with arbitrary precision. To communicate the whole vector $x(0)$, we could then send $d + 1$ identical $\clubsuit$ symbols at different times and encode all the components of $x(0)$ in their intertransmission times. In principle, one could even send a single $\clubsuit$ symbol to encode the whole $x(0)$ vector by using a d -dimensional space-filling curve and selecting a time of transmission for which a point on the curve is mapped onto $x(0)$.

The important message to be taken from Kofman & Braslavsky (73) is that, using event triggering, one can transmit information in the feedback loop not only by sending data but also by carefully selecting the times of transmission. Pearson et al. (74) made a similar observation. They considered the system to be fully observable; that is, C is the identity, and the sensor transmits, at a fixed sequence of transmission times, $\{t_k\}$ symbols from a finite alphabet over a delay-free and error-free communication channel. They further assumed that a special symbol in the alphabet can be transmitted without consuming any communication resources, effectively representing the absence of an explicit transmission, while the other symbols require one unit of communication resource per transmission. From an information-theoretic perspective, this setup is related to the silence-based communication paradigm of Dhulipala et al. (75). If we let $s(t_k)$ be the total number of nonfree symbols transmitted by the sensor to the controller up to time t_k , then the asymptotic average cost per unit time is given by

$$c \equiv \limsup_{k \rightarrow \infty} \frac{s(t_k)}{t_k} \quad (\text{symbols/s}), \quad 24.$$

where $t_k \rightarrow \infty$ as k tends to infinity. This can also be interpreted as an effective data rate, since it represents the rate accounting for only the nonfree transmissions, and it should be compared with Equation 18. In both cases, the rate depends on the transmission strategy.

Pearson et al. (74) showed that stabilization can be achieved with arbitrarily small values of Equation 24 by letting the transmission times $t_k = kT$, decreasing the sampling period T , and rarely transmitting nonfree symbols. In this regime, the transmission policy resembles an event-triggering strategy where the transmission of a nonfree symbol may occur at any given time, which depends on the encoding strategy, and can be chosen with arbitrary precision as $T \rightarrow 0$. At all other times, only free symbols are sent—which is analogous to sending nothing. As in the work by Kofman & Braslavsky (73), in this setup the act of transmitting a nonfree symbol now carries an amount of information that can be made arbitrarily large by decreasing the sampling time T . This allows one to decrease the number of transmitted nonfree symbols and drives the effective rate in Equation 24 arbitrarily close to zero.

6. TIMING INFORMATION IN THE PRESENCE OF DELAY

The works we have described suggest that a more general formulation of data-rate theorems should account for two distinct information flows: one through data payload (possibly corrupted by noise) and another through timing (possibly corrupted by delay). Traditionally, only the data

payload case has been considered, but the timing information can be very relevant, especially in the context of event triggering. Tallapragada & Cortés (72) considered only communication through data payload and did not attempt to exploit timing information. As a result, they recovered the traditional data-rate theorem formulation. By contrast, Kofman & Braslavsky (73) and Pearson et al. (74) showed that stabilization can be achieved with an arbitrarily low data payload rate by exploiting the timing information implicit in event-triggering schemes when the sender and receiver are perfectly synchronized.

At this point, one may suspect that the ability to stabilize the system with a zero payload rate is an artifact of the assumed perfect synchronization between the sensor and the controller achieved through a zero-delay channel. As we shall see next, this is not the case. In the presence of unknown delay, the value of the timing information in the triggering events decreases, because in this case the sensor may reveal only the state of the system with a finite precision, which depends on the range of the unknown delay. However, as long as the amount of information supplied by timing is above what is prescribed by the data-rate theorem for stabilization, it is still possible to stabilize the system with an arbitrarily low data payload rate. Next, we illustrate this point in more detail.

6.1. Information Access Rate Versus Information Transmission Rate

Khojasteh et al. (76, 77) made a key distinction between the information access rate and the information transmission rate. The former is the rate at which the controller needs to receive information, conveyed by both data payload and timing information; it is subject to the requirement expressed by the classic data-rate theorem. The latter is the rate at which the sensor needs to send data in the form of payload bits; it depends on the triggering scheme and can become arbitrarily low without affecting the ability to stabilize the system.

First, let us take the viewpoint of the sensor and examine the amount of information in the data payload transmissions to the controller. Let $b_s(t)$ be the number of bits in the data payload transmitted by the sensor up to time t and define the information transmission rate as

$$r_s \equiv \limsup_{t \rightarrow \infty} \frac{b_s(t)}{t}.$$

Let us now consider the viewpoint of the controller and examine the amount of information that it needs to receive in order to be able to select its stabilizing policy. This includes both payload and timing information. It is also the same as the number of bits needed to construct a reliable state estimate (78, theorem 1). We let $b_c(t)$ be the number of bits required at the controller to perform its selection at time t and define the information access rate as

$$r_c \equiv \limsup_{t \rightarrow \infty} \frac{b_c(t)}{t}.$$

In classic data-rate theorems, r_c coincides with r_s because the controller uses only data payload bits to select its control law. On the other hand, as discussed above, when exploiting timing information r_s and r_c can be substantially different; the classic data-rate limitation applies only to r_c , while we can achieve stabilization with r_s arbitrarily close to zero.

To view the limitation on r_c , we consider the same system's equations as in Equation 17. In this case, a necessary and sufficient condition to achieve exponential stabilization at rate σ is given by the usual data-rate theorem formula expressed in terms of r_c :

$$r_c \geq (\text{tr}(\mathbf{A}) + \sigma d) \log_2 e \quad (\text{bits/s}). \quad 25.$$

This result should be compared with Equation 20. It is important to stress that the limitation in Equation 25 describes what is required by the controller and does not depend on the feedback

structure—including aspects such as communication delays, information patterns at the sensor and controller, and whether the times at which transmissions occur are state dependent, as in event-triggered control, or periodic, as in time-triggered control. To obtain Equation 25, one considers for any control input trajectory $u(t)$ the subset of initial conditions for which the plant is stabilized by such input. Then, one constructs a cover of the set of all initial conditions by stabilizing control policies. This leads to a discrete set of choices for selecting the stabilization policy for any given realization of the initial condition. It follows that the logarithm of the covering number is the number of bits needed by the controller by time t to select a stabilizing control policy. A usual balance-of-information argument between the rate of expansion of the uncertainty in the state due to the random initial condition and the quantization due to the covering finally leads to the lower bound in Equation 25.

Having established the classic data-rate theorem result for the information access rate, we can now ask what the data-rate requirement is on the information transmission rate r_s , assuming that the sensor has access to causal feedback regarding what has been received by the controller and for different ranges of the possible delay. While we have already established that r_s can be arbitrarily close to zero in the absence of delay, the presence of unknown delay decreases the amount of information that can be communicated by timing, and this may require r_s to become positive.

To illustrate the results, we denote by $\{t_s^k\}_{k \in \mathbb{N}}$ the sequence of times when the sensor transmits a packet of a certain number $g(t_s^k)$ of bits to the controller. We assume the packet is delivered to the controller without error and entirely but with an unknown delay. Letting $\{t_c^k\}_{k \in \mathbb{N}}$ be the sequence of times when the controller receives the packets transmitted at times $\{t_s^k\}_{k \in \mathbb{N}}$, we assume that (a) for all $k \in \mathbb{N}$, the communication delay $\Delta_k = t_c^k - t_s^k$ satisfies

$$\Delta_k \leq \gamma, \quad 26.$$

where $\gamma \in \mathbb{R}_{\geq 0}$, and (b) both t_s^k and t_c^k tend to infinity as $k \rightarrow \infty$. We can then study how the rate r_s required for stabilization using event triggering varies as a function of γ . Khojasteh et al. (76) first considered the case of systems without disturbances, where the objective is to drive the state to zero at an exponential rate σ , and later (77) considered the case of systems with disturbances using a notion of input to state stability, which guarantees that the state is bounded at all times and that this bound, as usual, depends on the range of the disturbance. While results hold for both scalar and vector systems, for illustrative purposes, in the following we review the results from Khojasteh et al. (77) for scalar systems.

Figure 5 depicts a plot of the rate required to keep the state bounded when using any threshold-based event-triggering policy based on the value of the state estimation error. This plot shows that the required rate for stabilization undergoes a phase transition: For small values of the delay upper bound γ , the system can be stabilized with an arbitrarily low information transmission rate. However, when γ reaches the critical threshold

$$\gamma_c = \frac{\ln 2}{A}, \quad 27.$$

the required rate begins to increase, eventually surpassing the data-rate theorem requirement $A/\ln 2$, where A here is a positive scalar. This indicates that for $\gamma < \gamma_c$, the amount of information contained in the timing of the triggering events is large enough that the rate that must be supplied by the data payload to guarantee stability is zero. On the other hand, when $\gamma > \gamma_c$, the information contained in the timing of the triggering events is not enough to guarantee stability, and the rate must begin to increase. One way to interpret this result is that in the presence of delay the value of the timing information supplied by event triggering deteriorates and eventually becomes insufficient to be used alone for stabilization. On the other hand, increasing the delay also affects

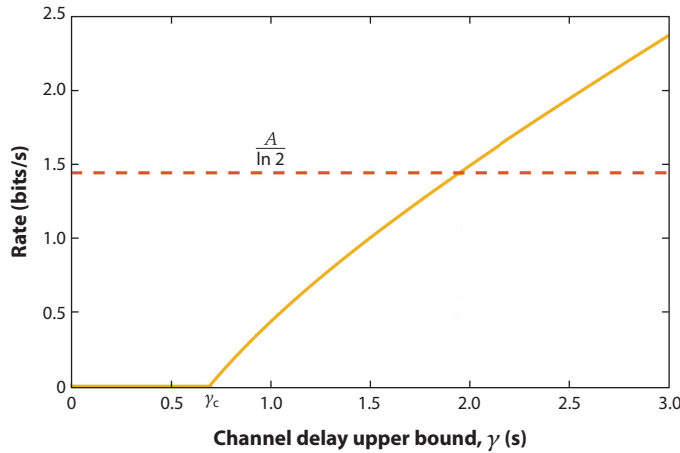


Figure 5

Phase transition of the necessary information transmission rate for stabilization. The graph is valid for any generic system. In this example, we have a scalar system with no disturbance, where $A = 1$ and $\gamma_c = \ln 2/A = 0.6931$; the rate dictated by the data-rate theorem is $r_c \geq A/\ln 2 = 1.4427$.

the rate at which the transmitted payload bits are received, which results in a higher transmission rate requirement that can surpass the data-rate theorem requirement.

We also note that the critical value γ_c at which the information transmission rate becomes positive equals the inverse of the entropy rate of the system—that is, Equation 27 is the inverse of the critical rate in the data-rate theorem formula in Equation 5. Recalling the production-and consumption-of-information analogy discussed in Section 2, we have that for $\gamma = \gamma_c$ the entropy of the system can expand by one new bit at every delay occurrence, and this amount of information cannot be counterbalanced by the information carried by the event-triggering times. In other words, the information supplied by the triggering events can always be one bit short due to the uncertainty introduced by the delay, and this bit must be supplied by the data payload to ensure stabilization. For this reason, the rate r_s begins to increase once γ reaches the critical value γ_c . **Figure 6** also shows a sufficient condition for stabilization obtained using a given triggering strategy described by Khojasteh et al. (77) that employs a fixed-threshold policy and compares it with the necessary condition and with the data-rate theorem requirement. The results have been validated in a real system configuration (79).

6.2. The Information Value of Event Timing

We have shown that information useful for control can be carried through the feedback loop by both data packets and event timing. Event timing may allow one to achieve stabilization by sending packets at a bit rate that is lower than what the classic data-rate theorem prescribes. We now ask whether it is possible to provide an information-theoretic characterization of event timing, and whether it is possible to recover the classic data-rate theorem formulation that relates the amount of information carried by event timing alone to the intrinsic entropy rate of the system.

To quantify the amount of timing information, Khojasteh et al. (80) considered a channel carrying $\clubsuit$ symbols from a unitary alphabet, where each transmission is received after a random delay. Since the alphabet is composed of a single symbol, the data packets convey no information, and communication can only occur by selecting the times of transmission of the unitary symbols. Every time a symbol is received, the sender is notified of the reception by an instantaneous

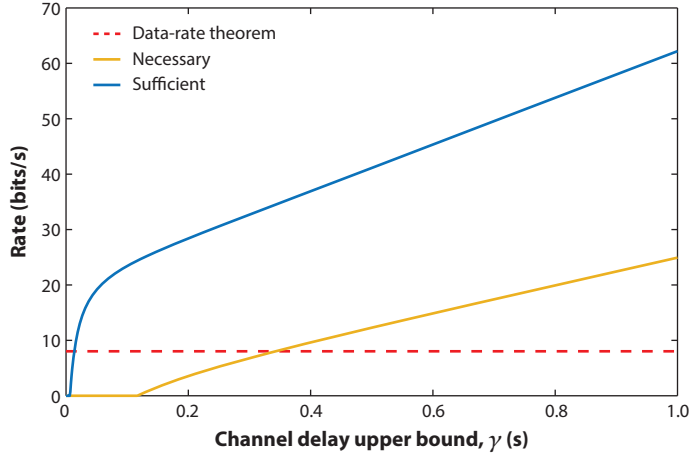


Figure 6

Sufficient and necessary transmission rates for stabilization. Here, we have a scalar system with disturbances bounded by 0.4, with $A = 5.5651$; the rate dictated by the data-rate theorem is $r_c \geq A/\ln 2 = 8.02874$.

acknowledgment. The channel is initialized with a $\spadesuit$ received at time $t = 0$. After receiving the acknowledgment for the i th $\spadesuit$, the sender waits for $\tilde{w}_{i+1}$ seconds and then transmits the next $\spadesuit$. Transmitted symbols are subject to i.i.d. random delays $\{s_i\}$. Letting d_i be the interreception time between two consecutive symbols, we have

$$d_i = \tilde{w}_i + s_i. \quad 28.$$

The operation of this channel is analogous to that of a telephone system where a transmitter signals a phone call to the receiver through a ring and, after a random time required to establish the connection, is aware of the ring being received. Communication between transmitter and receiver can then occur without any vocal exchange, but by encoding messages in the waiting times between consecutive calls. **Figure 7** provides an example of the timing channel in action. Anantharam & Verdú (81) defined the timing capacity for this telephoning signaling system in terms of mutual information between transmitter and receiver. This notion is analogous to the Shannon capacity for the timing channel. Khojasteh et al. (80) considered the stabilization of the following scalar continuous-time system with no disturbance over the timing channel described above:

$$\dot{x} = Ax(t) + Bu(t). \quad 29.$$

The constants $A, B \in \mathbb{R}$ are such that $A > 0$ and $B \neq 0$. Since the system in Equation 29 is not subject to disturbances, we expect that a notion of capacity analogous to the Shannon one is sufficient to characterize the ability to stabilize the system, as discussed in Section 2. Indeed, by

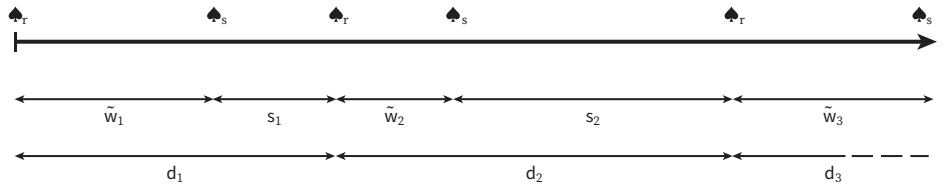


Figure 7

The timing channel. Subscripts s and r denote sent and received symbols, respectively.

using the notion of timing capacity described by Anantharam & Verdú (81), Khojasteh et al. (80) showed that for the state to converge to zero in probability, the timing capacity of the channel should be at least as large as the entropy rate of the system. Conversely, in the case where the random delays are exponentially distributed, when the timing capacity is strictly greater than the entropy rate of the system, we can drive the state to zero in probability by using a decoder that refines its estimate of the transmitted message every time a new symbol is received. Finally, since the timing capacity depends on the distribution of the delay, it was also shown that in the case of exponentially distributed delay it is possible to achieve stabilization at a zero data rate only for a sufficiently small average delay, namely, when

$$\mathbb{E}\{\mathbf{s}\} < (eA)^{-1}, \quad 30.$$

which confirms the intuition from the event-triggering results that to achieve stabilization at a zero data rate, the delay should be sufficiently small.

7. ESTIMATION UNDER COMMUNICATION CONSTRAINTS

In closed-loop systems, the ability to select the correct control action to keep the system stable boils down to that of constructing a reliable state estimate that can be used for stabilization. As discussed in Section 2, to keep the system stable, the amount of information that must flow through the feedback loop must compensate for the expansion in the uncertainty of the state, and this dictates the communication constraints expressed by the various data-rate theorem formulations.

In the absence of the controller, the problem of estimating the state of an open-loop dynamical system observed over a communication channel is also of interest and is further motivated by additional applications, such as situation awareness (82–84), asset tracking (85–87), smart cities (88–90), the internet of things (91–93), and network localization and navigation (94–97), where nodes in a network aim to infer their positions and possibly other position-related quantities using observations obtained via different types of sensors. In this case, results analogous to the data-rate theorem for stabilization have been obtained, and in what follows we wish to compare them and put them in the context of those that we have already described.

References 98–100 exploited notions of dynamical system entropy for estimation that were inspired by the topological entropy approach in References 17, 33, and 43, which was used to determine stabilization conditions for nonlinear systems over bit-pipe communication channels. In particular, Liberzon & Mitra (100) introduced the notion of estimation entropy in terms of the number of system trajectories that approximate all other trajectories up to an exponentially decaying error. In the case of linear systems of the form $\dot{\mathbf{x}} = A\mathbf{x}(t)$ and exponential error decay rate $\sigma \geq 0$, the estimation entropy reduces to $(\text{tr}(A) + \sigma d) \log_2 e$ (bits/s), which is analogous to the stabilization result in Equation 25. Furthermore, for more general nonlinear systems of the form $\dot{\mathbf{x}} = f(\mathbf{x})$, where $\mathbf{x}(t) \in \mathbb{R}^d$, Liberzon & Mitra (100) showed upper and lower bounds on the estimation entropy.

In the context of exploiting timing information for estimation in event-based transmission, Yu et al. (101) considered estimation over a finite-size packet communication channel with delay analogous to the one in the works by Khojasteh et al. (76, 77). Here, the aim is to remotely estimate a discrete-event process,

$$\mathbf{x}(t) = \sum_{k=0}^{\infty} \boldsymbol{\eta}_k \mathbb{1}(\boldsymbol{\tau}_k \leq t < \boldsymbol{\tau}_{k+1}), \quad 31.$$

where the system states $\boldsymbol{\eta}_k$ are discrete i.i.d. random variables that belong to a finite set, $\boldsymbol{\tau}_k \in \mathbb{N}$ denotes the random time when a state transition to $\boldsymbol{\eta}_k$ occurs, $0 < \boldsymbol{\tau}_0 < \boldsymbol{\tau}_1 < \dots$, and $\mathbb{1}(\cdot)$ is the

indicator function. It follows that $\mathbf{x}(t)$ describes the state evolution in continuous time and that the random duration time of each state is $t_k = \tau_{k+1} - \tau_k$. Since the process $\mathbf{x}(t)$ remains constant during interevent times, it is sufficient to describe $\mathbf{x}(t)$ with $\{\mathbf{x}(k) \mid k \in \mathbb{N}\}$.

To perform remote estimation, when a state transition to $\boldsymbol{\eta}_k$ occurs, a packet with a finite number of bits is transmitted over a channel. Like in the communication setting of Khojasteh et al. (76, 77), the packet is delivered to the receiver without error but with an unknown delay, denoted by $\tilde{\Delta}_k$. The transmission delays $\{\tilde{\Delta}_k\}$ are assumed to be random, i.i.d., and independent of the states $\{\boldsymbol{\eta}_k\}$.

The amount of information that the system produces can be expressed in terms of the Shannon entropy of two stochastic sources, namely, the unknown state value $\boldsymbol{\eta}$ and the unknown interevent time t . Letting $H\{\cdot\}$ be the joint Shannon entropy of an ensemble of random variables (49), we have that the entropy rate of the information produced is

$$r_0 = \lim_{k \rightarrow \infty} \frac{H\{\mathbf{x}(0), \mathbf{x}(1), \dots, \mathbf{x}(k)\}}{k} = \frac{H\{\boldsymbol{\eta}\} + H\{t\}}{\mathbb{E}\{t\}} \quad (\text{bits/s}). \quad 32.$$

This represents the average rate at which the system generates information, and the receiver needs to have access to at least this amount of information to construct a correct estimate of the state. It can also be interpreted as the stochastic analogue of the information access rate defined in Section 6.1 for closed-loop stabilization. The information access rate was defined in a deterministic setting under worst-case delay conditions, while here we have a stochastic setting and an average transmission rate expressed in terms of entropy.

On the other hand, by exploiting knowledge of the reception times $t_k + \tilde{\Delta}_k$, the receiver may be able to perform estimation with a rate lower than Equation 32. It turns out that the information rate required by the receiver for real-time estimation of the process in Equation 31 is given by

$$r'_0 = \frac{H\{\boldsymbol{\eta}\} + H\{t|t + \tilde{\Delta}\}}{\mathbb{E}\{t\}} \quad (\text{bits/s}). \quad 33.$$

Since conditioning reduces the entropy, we immediately deduce that $r'_0 \leq r_0$. There is an analogy in this case with the information transmission rate defined in Section 6.1, which is the deterministic counterpart of this reduced entropy rate for the case of stabilization. In the case of a small delay, we have $H\{t|t + \tilde{\Delta}\} \approx H\{t\} = 0$, and the rate required for estimation reduces to

$$r'_0 = \frac{H\{\boldsymbol{\eta}\}}{\mathbb{E}\{t\}} \quad (\text{bits/s}), \quad 34.$$

meaning that we need to transmit only the number of bits required on average to describe the value of the process $\{\boldsymbol{\eta}_k\}$ and do not need to encode any information regarding the intertransmission times $\{t_k\}$. We also point out that in the event-triggering results discussed in Section 6.1, the transmission time was a function of the system state. Here, the transmission time and the symbol are assumed to be independent. Consequently, the timing information cannot further reduce the uncertainty in the state, and Equation 34 remains a nonzero lower bound on the required rate. By contrast, in the event-triggering results described in Section 6.1, since timing information can also encode information about the state, the minimum required transmission rate can become arbitrarily low for small delay values.

8. ESTIMATION OVER NOISY CHANNELS

In a decentralized estimation problem, a node can exchange messages that contain information on the states of interest with other nodes via noisy communication channels. In Section 3, we have seen that to guarantee moment stability over noisy channels, a suitable metric for characterizing the quality of a communication channel is anytime capacity, a notion introduced by Sahai

& Mitter (46) that is parameterized by a positive scalar α , which specifies the requirements on the communication reliability. In particular, this notion is used to establish tight necessary and sufficient conditions for stabilizing a system over a noisy channel in the presence of bounded disturbances (46).

In the following, we present the results of Liu and colleagues (102–105), who showed that the anytime capacity is also a relevant measure of information transmission for estimation of open-loop systems over noisy channels in both single-node and multiple-node scenarios. Specifically, while a necessary condition for bounded moment error is expressed in terms of the Shannon capacity, a sufficient condition is obtained in terms of the anytime capacity. Since the anytime capacity expresses communication with stronger reliability constraints, for any $\alpha > 0$, the α -anytime capacity of a channel is greater than or equal to its Shannon capacity. It follows that these conditions are not tight in general. Nevertheless, their form resembles analogous results for closed-loop stabilization and can also be extended to multiple-node networks settings.

8.1. Two-Node System

Consider a system consisting of node i and node j in discrete-time scenarios. In particular, node j is associated with a time-varying state that this node aims to infer. The state of node j at time step t is denoted by a d -dimensional random vector $\mathbf{x}_t^{(j)}$ (see **Figure 8**), which satisfies

$$\mathbf{x}_t^{(j)} = \mathbf{A}^{(j)}\mathbf{x}_{t-1}^{(j)} + \mathbf{v}_t^{(j)}, \quad t = 1, 2, \dots, \quad 35.$$

where $\mathbf{A}^{(j)} \in \mathbb{R}^{d \times d}$ is a deterministic matrix known to both nodes, and $\mathbf{v}_t^{(j)} \in \mathbb{R}^d$ is a zero-mean random vector representing the disturbance to the state.

The other node in the system, node i , obtains an observation $\mathbf{y}_t$ of $\mathbf{x}_t^{(j)}$ at each time step t given by

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t^{(j)} + \mathbf{w}_t, \quad t = 0, 1, \dots,$$

where $\mathbf{C}$ is the sensor gain matrix known to both nodes, and $\mathbf{w}_t$ is a zero-mean random vector representing the observation noise at time step t . Moreover, node i generates an encoded message represented by a random vector $\mathbf{m}_t$ at time step t based on its observations $\mathbf{y}_{0:t}$ —that is, $\mathbf{m}_t$ is a function of $\mathbf{y}_{0:t}$. The concatenation of such functions from time step 0 to the last time step of interest is referred to as an encoding strategy, which is designed by node i . Message $\mathbf{m}_t$ is transmitted via a memoryless channel to node j , and the received message is represented by a random vector $\mathbf{r}_t$ that may be different from $\mathbf{m}_t$ due to noise, fading, and interference in the channel.

The following assumptions are made on the state disturbance and the observations:

1. $\mathbf{v}_t^{(j)}$ are independent over time steps t for all $j \in \mathcal{V}$ —that is, random vectors $\mathbf{v}_{t_0}^{(j)}, \mathbf{v}_{t_1}^{(j)}, \dots, \mathbf{v}_{t_n}^{(j)}$ are independent for any positive integer n and $0 \leq t_0 < t_1 < \dots < t_n$. Similarly, $\mathbf{w}_t$ are independent over t . In addition, the processes $\{\mathbf{v}_t^{(j)}\}_{t \geq 0}$ and $\{\mathbf{w}_t\}_{t \geq 0}$ are independent.



Figure 8

Block diagram for decentralized inference in a two-node system. The state $\mathbf{x}_t^{(j)}$ is measured by a sensor to generate observation $\mathbf{y}_t$ at time step t . Observations $\mathbf{y}_{0:t}$ are used by the encoder to generate a message $\mathbf{m}_t$, which is then transmitted via the channel. Using the received messages $\mathbf{m}_{0:t}$, a decentralized estimator $\hat{\mathbf{x}}_t^{(j)}$ of $\mathbf{x}_t^{(j)}$ is evaluated by node j .

2. There exists a scalar $a > 2$ such that sequences $\{\mathbb{E}\{\|\mathbf{v}_t^{(j)}\|^a\}\}_{t \geq 0}$ and $\{\mathbb{E}\{\|\mathbf{w}_t\|^a\}\}_{t \geq 0}$ are bounded over time—that is,

$$\sup_{t \geq 0} \mathbb{E}\{\|\mathbf{v}_t^{(j)}\|^a\} < \infty, \quad \sup_{t \geq 0} \mathbb{E}\{\|\mathbf{w}_t\|^a\} < \infty.$$

3. There exists a constant $\underline{b} > -\infty$ such that the differential entropy $h(\mathbf{v}_t^{(j)})$ of $\mathbf{v}_t^{(j)}$ satisfies $h(\mathbf{v}_t^{(j)}) > \underline{b}$ for all $t \geq 0$.

The above assumptions are mild, and the adopted models for the states, observations, and message transmission are general. In particular, assumption 1 is widely adopted in the literature on inference and filtering. Assumption 2 holds if the tail of the distribution for each entry of $\mathbf{v}_t^{(j)}$ and $\mathbf{w}_t$ is not heavy. As an example, if both $\mathbf{v}_t^{(j)}$ and $\mathbf{w}_t$ have an identical distribution for different time steps t , and entries of $\mathbf{v}_t^{(j)}$ and $\mathbf{w}_t$ are subexponential random variables,¹ then assumption 2 holds for all a . Assumption 2 indicates that the uncertainty in the state disturbance does not vanish as time approaches infinity.

Node j evaluates a decentralized estimator $\hat{\mathbf{x}}_t^{(j)}$ of $\mathbf{x}_t^{(j)}$ at time step t using its received messages $\mathbf{r}_{0:t}$ —that is, $\hat{\mathbf{x}}_t^{(j)}$ is a function of $\mathbf{r}_{0:t}$. The optimal design of the estimator depends on the metric for inference error. The metric we consider for the inference error at time step t is the b th moment of $\|\hat{\mathbf{x}}_t^{(j)} - \mathbf{x}_t^{(j)}\|$, namely,

$$\mathbb{E}\{\|\hat{\mathbf{x}}_t^{(j)} - \mathbf{x}_t^{(j)}\|^b\} =: e_t^{(j)}, \quad 36.$$

where $b \geq 2$. In the special case where $b = 2$, this metric becomes the mean squared error. We establish conditions under which there exist an encoder at node i and a decentralized estimator at node j such that $e_t^{(j)}$ is bounded over time—that is,

$$\sup_{t \geq 0} e_t^{(j)} < \infty. \quad 37.$$

Intuitively, whether Equation 37 holds depends on the quality of the channel from node i to node j : A better channel allows a higher rate of messages to be transmitted reliably, thus increasing the amount of information of the unknown state obtained by node j .

Liu et al. (102) showed the following:

1. If there exist an encoder at node i and a decentralized estimator at node j such that Equation 37 holds, then the Shannon capacity C of the channel satisfies

$$C > \sum_{|\lambda_i| \geq 1} \log_2 |\lambda_i^{(j)}|. \quad 38.$$

2. Conversely, if there exists a parameter $\alpha > \frac{ab}{a-b} \log |\rho(\mathbf{A}^{(j)})|$ such that the α -anytime capacity $\check{C}(\alpha)$ of the channel satisfies

$$\check{C}(\alpha) > \sum_{|\lambda_i| \geq 1} \log_2 |\lambda_i^{(j)}|, \quad 39.$$

then there exist an encoder at node i and a decentralized estimator at node j such that Equation 37 holds. Here, $\rho(\mathbf{A}^{(j)})$ represents the spectral radius, that is, the largest of the magnitudes of all the eigenvalues of $\mathbf{A}^{(j)}$.

This result specifies a necessary condition and a sufficient condition in Equations 38 and 39, respectively, for the inference error to be bounded over time and is parallel to the data-rate theorem

¹A random variable $\mathbf{x}$ is subexponential if there exists a constant $c > 0$ such that $\mathbb{P}\{|\mathbf{x}| > x\} \leq 2 \exp\{-cx\}$ for any $x \geq 0$ (106, chapter 2). For example, a random variable is subexponential if it has a Gaussian distribution or if it has a bounded support.

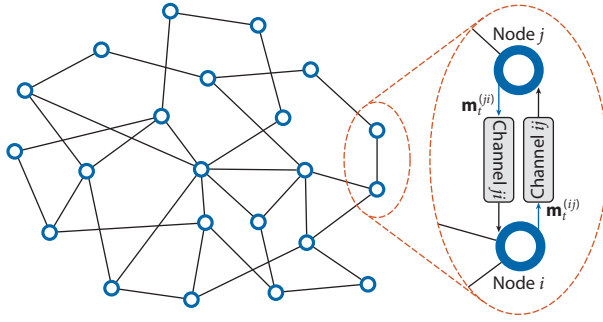


Figure 9

Decentralized inference via sensing and communication in a multinode network. A pair of nodes in the network are connected by an edge if they are neighbors to each other. Figure adapted with permission from Reference 105.

for stabilization over noisy channels, as both Equations 38 and 39 compare the capacity of the channel with a threshold determined by eigenvalues of $\mathcal{A}^{(j)}$.

Since the right-hand sides of Equations 38 and 39 are equal, the necessary and sufficient conditions become tight for channels whose Shannon capacity and anytime capacity coincide. Such channels include the noiseless bit-pipe channels and the Gaussian channel with feedback (46). As a final remark, we also point out that assumption 2 is required only for establishing the sufficient condition in Equation 39 and is not required for the necessary condition in Equation 38, whereas assumption 3 is required only for Equation 38 and not for Equation 39.

8.2. Multinode Network

Conditions for the boundedness of the inference error for decentralized inference can also be established in a general network with multiple nodes. Specifically, consider a network comprising a set $\mathcal{V}$ of nodes where each node is associated with a time-varying unknown state (see **Figure 9**). In particular, the state of node j at time step t is represented by $\mathbf{x}_t^{(j)}$ and satisfies Equation 35. Each node in the network can perform observations and exchange messages with other nodes within its sensing and communication range, which are referred to as neighbors of the node. Specifically, node i obtains an intranode observation $\mathbf{y}_t^{(ii)}$ as well as an internode observation $\mathbf{y}_t^{(ij)}$ for each neighbor $j \in \mathcal{N}^{(i)}$ at time step t , where $\mathcal{N}^{(i)}$ represents the set of neighbors of node i . Observations $\mathbf{y}_t^{(ii)}$ and $\mathbf{y}_t^{(ij)}$ are given by

$$\mathbf{y}_t^{(ii)} = \mathbf{C}^{(ii)}\mathbf{x}_t^{(i)} + \mathbf{w}_t^{(ii)}, \quad \mathbf{y}_t^{(ij)} = \mathbf{C}_1^{(ij)}\mathbf{x}_t^{(i)} + \mathbf{C}_2^{(ij)}\mathbf{x}_t^{(j)} + \mathbf{w}_t^{(ij)}, \quad t = 0, 1, \dots,$$

where $\mathbf{C}^{(ii)}$, $\mathbf{C}_1^{(ij)}$, and $\mathbf{C}_2^{(ij)}$ are sensor gain matrices, whereas random vectors $\mathbf{w}_t^{(ij)}$ and $\mathbf{w}_t^{(ii)}$ represent observation noise. Moreover, node i transmits an encoded message $\mathbf{m}_t^{(ij)}$ to each neighbor $j \in \mathcal{N}^{(i)}$ at time step t . In particular, $\mathbf{m}_t^{(ij)}$ is generated based on the observations obtained by node i up to time step t and messages received by node i up to time step $t - 1$. In other words, $\mathbf{m}_t^{(ij)}$ is a function of $\mathbf{y}_{0:t}^{(ii)}$, $\{\mathbf{y}_{0:t}^{(ij)} : j \in \mathcal{N}^{(i)}\}$, and $\{\mathbf{r}_{0:t-1}^{(ki)} : k \in \mathcal{N}^{(i)}\}$. Here, $\mathbf{r}_\tau^{(ki)}$ is the message received by node i from node k at any time step $\tau \geq 0$.

A subset $\mathcal{V}_a \subseteq \mathcal{V}$ of nodes in the network, referred to as agents, aim to infer their states in real time.² In particular, each agent j evaluates an estimator $\hat{\mathbf{x}}_t^{(j)}$ of $\mathbf{x}_t^{(j)}$ at time step t using the

²The subset $\mathcal{V}_a$ can be chosen arbitrarily, from a singleton $\{j\}$ to the entire set $\mathcal{V}$.

observations and received messages obtained up to time t . In other words, $\hat{\mathbf{x}}_t^{(j)}$ is a function of $\mathbf{y}_{0:t}^{(ji)}$, $\{\mathbf{y}_{0:t}^{(ji)} : i \in \mathcal{N}^{(j)}\}$, and $\{\mathbf{r}_{0:t}^{(ij)} : i \in \mathcal{N}^{(j)}\}$. The metric $e_t^{(j)}$ defined in Equation 36 is adopted for the inference error of agent j at time step t .

Liu (105) presented a necessary condition and a sufficient condition for the inference error of all the agents to be bounded over time (i.e., Equation 37 holds for all $j \in \mathcal{V}_a$). Both the necessary condition and the sufficient condition consist of a sensing subcondition and a communication subcondition. The sensing subcondition in the necessary condition is the same as that in the sufficient condition. In particular, this subcondition is stated in terms of the sensor gain matrices of nodes in the network. On the other hand, the communication subcondition is stated in terms of the Shannon capacities and anytime capacities of channels in the network in the necessary condition and the sufficient condition, respectively. Liu (105) discussed the gap between the established necessary condition and sufficient condition and showed that such a gap is small in certain scenarios.

8.3. Decentralized Estimation in Continuous Time

Results for decentralized inference are also established for continuous-time scenarios. Specifically, consider a system consisting of nodes i and j . The unknown state of node j at time t is represented by a random variable $\mathbf{x}_t^{(j)}$, which satisfies the following stochastic differential equation:

$$d\mathbf{x}_t^{(j)} = \mathbf{A}^{(j)} \mathbf{x}_t^{(j)} dt + \mathbf{B}^{(j)} d\mathbf{v}_t^{(j)}, \quad t \in [0, \infty),$$

where $\mathbf{A}^{(j)} \in \mathbb{R}$ satisfies $|\mathbf{A}^{(j)}| \geq 1$ and is known to both nodes. Quantity $\mathbf{B}^{(j)}$ is a row vector and is also known to the two nodes. Process $\{\mathbf{v}_t^{(j)}\}_{t \geq 0}$ is a Brownian motion corresponding to the disturbance to the state of node j .

Node i obtains an observation of node j 's state at each time. The observation obtained by node i at time t is represented by a random vector $\mathbf{y}_t$, which satisfies

$$d\mathbf{y}_t = \mathbf{C} \mathbf{x}_t^{(j)} dt + \mathbf{\Xi} d\mathbf{w}_t, \quad t \in [0, \infty),$$

where the sensor gain vector $\mathbf{C}$ and matrix $\mathbf{\Xi}$ are deterministic and are known to both nodes. Process $\{\mathbf{w}_t\}_{t \geq 0}$ is a Brownian motion corresponding to the observation noise. Moreover, node i generates an encoded message $\mathbf{r}_t$ at each time t and transmits the message via a scalar Gaussian channel with noiseless feedback. The message $\mathbf{r}_t$ is a function of $\{\mathbf{y}_\tau\}_{\tau \in [0, t]}$ and $\{\mathbf{r}_\tau\}_{\tau \in [0, t]}$, where $\mathbf{r}_\tau$ represents the message received by node j at time τ . Specifically, $\mathbf{r}_t$ satisfies

$$d\mathbf{r}_t = \mathbf{m}_t dt + \kappa d\mathbf{w}_t, \quad t \in [0, \infty),$$

where κ is a known scalar, and $\{\mathbf{w}_t\}_{t \geq 0}$ is a one-dimensional Brownian motion corresponding to the additive Gaussian noise in the channel.

Node j evaluates a decentralized estimator $\hat{\mathbf{x}}_t^{(j)}$ of its state $\mathbf{x}_t^{(j)}$ at each time t . Consider the mean squared error $e_t^{(j)} := \mathbb{E}\{\|\hat{\mathbf{x}}_t^{(j)} - \mathbf{x}_t^{(j)}\|^2\}$ as the metric for the inference error at time t . Liu et al. (104) established a necessary and sufficient condition under which $e_t^{(j)}$ is bounded over time, namely, that there exist an encoder at node i and a decentralized estimator at node j that achieve Equation 37 if and only if the Shannon capacity C of the channel satisfies $C > \log(|\mathcal{A}|)$.

Liu and colleagues (104, 105) have also studied decentralized inference from an information-theoretic perspective. Specifically, building on the pioneering work of Mitter & Newton (107) and Newton (108), they established a relationship between mutual information and Fisher information for decentralized inference.

9. DISCUSSION AND OUTLOOK ON THE FIELD

Research at the intersection of information theory and control theory has now entered its third decade, and we can reflect on the body of knowledge that has been developed so far, as well as on the new challenges that we see appearing at the horizon. Examining the large body of research conducted, we draw the following basic conclusions. First, there is the realization that in order to describe the ability to stabilize an NCS, a notion of information capacity must be related to system parameters expressing the value or utility of the information available for control. Second, we have that the information produced by the dynamical system can be quantified in terms of the intrinsic entropy of the system. Since the intrinsic entropy can grow exponentially over time whenever there is no information available at the decoder, or in the presence of decoding errors, we compare the intrinsic entropy of the system to a notion of information capacity available through the system loop that takes these dynamics into account. It follows that, depending on the notion of stability employed, we can use different capacity notions that range from the Shannon one to the anytime and the zero-error ones. Third, there is the realization that in NCSs, information useful for control can be transmitted not only through data packets but also through events that occur over time, as in the case of event-triggering strategies, and in this case capacity notions should include timing information. Finally, there is a certain duality between results obtained in the context of stabilization and those obtained in the context of estimation. In the latter case, recent advancements have also shown results for multinode networks.

Moving forward, we expect that a complete theory of communication over feedback loops can be constructed by considering encoding and decoding strategies that account for both communication by timing and communication by data payload, as well as accounting for the distributed nature of many system implementations. This theory will make an impact on practical developments that will take into account the information constraints that need to be satisfied to achieve different objectives. Some experimental platforms have already demonstrated the applicability of the theoretical results obtained so far, and we expect more impacts to emerge as the theory is used to develop industrial systems.

A grand challenge will be to extend the treatment to distributed networks where partial state information is available at the different nodes. In this case, since neither network information theory nor distributed control theory is fully developed, studying the intersection between the two will present additional challenges. Nevertheless, while recognizing that much needs to be done, given the amount of progress we have witnessed in the last two decades, we can look to the future with a positive outlook and dare to say: Much *will* be done.

DISCLOSURE STATEMENT

The authors are not aware of any affiliations, memberships, funding, or financial holdings that might be perceived as affecting the objectivity of this review.

ACKNOWLEDGMENTS

This work was supported in part by National Science Foundation awards ECCS-2127946, CNS-1446891, and ECCS-1917177 and in part by the Army Research Office through the MIT Institute for Soldier Nanotechnologies under contract W911NF-13-D-0001.

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