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Research Article

Combination Synchronization of Three Different Fractional-Order Delayed Chaotic Systems

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Time delay is a frequently encountered phenomenon in some practical engineering systems and introducing time delay into a system can enrich its dynamic characteristics. There has been a plenty of interesting results on fractional-order chaotic systems or integer-order delayed chaotic systems, but the problem of synchronization of fractional-order chaotic systems with time delays is in the primary stage. Combination synchronization of three different fractional-order delayed chaotic systems is investigated in this paper. It is an extension of combination synchronization of delayed chaotic systems or combination synchronization of fractional-order chaotic systems. With the help of stability theory of linear fractional-order systems with multiple time delays, we design controllers to achieve combination synchronization of three different fractional-order delayed chaotic systems. In addition, numerical simulations have been performed to demonstrate and verify the theoretical analysis.

1. Introduction

Since it was reported that nature and engineering fields existed in many fractional dimensions in 1983 [1], fractional calculus has attracted researchers from academia and industry. More and more researches have shown that fractional-order differential equations are useful tools to investigate complex dynamical behaviors and describe various physical and engineering systems. Time delays are found to exist widely in real world systems, such as electronic circuits, chemical, and economical systems [2–7]. It is very necessary to include time delays into a system to model a real-world application. Therefore, scientists from various fields begin to focus on the study of fractional-order delayed differential equations (FDDEs), due to their wide potential applications. Many chaotic systems of FDDEs were proposed and their synchronizations were studied. The fractional-order delayed Liu system was presented and the existence of chaos was investigated in [8], and the impulsive synchronization and robust predictive synchronization were investigated in [9, 10], respectively. The nonlinear dynamics and chaos were studied for the fractional-order delayed financial system in [11], and the sliding-mode synchronization was investigated in [12]. In [13], hybrid projective synchronization between the two aforementioned systems was done. The fractional-order delayed Chen system was considered in [14], and its adaptive synchronization was investigated in [15].

All the synchronization schemes mentioned above are based on the usual drive-response method, which only has one drive system and one response system. In [16], Luo et al. generalized the usual drive-response synchronization scheme to combination synchronization, which has two drive systems and one response system. Combination synchronization has stronger antidecode and antiattack ability than that of the drive-response synchronization in secure communication, because the origin message can be divided into two segments and each segment can be separated into two distinct drive systems. The authors [17] applied robust adaptive sliding-mode control method to investigate combination synchronization of Lorenz system with time delay. In [18], phase and antiphase combination synchronization of three delayed systems were studied using active control. The adaptive function projective combination synchronization of three fractional-order chaotic systems was investigated in [19]. Jiang et al. [20] analyzed complex combination synchronization of three fractional-order chaotic complex-variable systems. Delavari and Mohadeszadeh [21] proposed adaptive sliding-mode control method for synchronization of nonidentical fractional-order chaotic and hyperchaotic systems. In [22], combination synchronization of a new fractional-order Lorenz-like system with two stable node-foci was analyzed with the help of nonlinear feedback control method. Although fractional-order delayed chaotic systems were considered in the literature [23], the method used for synchronization was not combination synchronization. The generalization of combination-combination synchronization of chaotic n-dimensional fractional-order dynamical systems is studied in [24]. There exist many works focusing on the combination synchronization of integer-order delayed chaotic systems; however, the conclusions on those works cannot be used on fractional-order delayed chaotic system directly. The problem of combination synchronization of fractional-order delay chaotic system is still an open challenging problem.

Motivated by the above analysis, we consider combination synchronization of three fractional-order delayed chaotic systems, which is an extension of combination synchronization of delayed chaotic systems or combination synchronization of fractional-order chaotic systems. The Adams-Bashforth-Mounton method is used for numerical solutions of fractional-order delay chaotic system.

2. Preliminaries

Fractional calculus is an old mathematical topic and is an extension of integration and differentiation to noninteger-order fundamental operator ${}_aD_t^r$, which is described by

$$ {}_aD_t^r = \begin{cases} \dfrac{\mathrm{d}^r}{\mathrm{d}t^r}, & r > 0, \\ 1, & r = 0, \\ \displaystyle\int_a^t (\mathrm{d}\tau)^{-r}, & r < 0. \end{cases} \quad (1) $$

One of the commonly used definitions for the fractional-order differential operator is the Caputo definition [25, 26], which is defined as

$$ {}_aD_t^r f(t) = \frac{1}{\Gamma(n-r)} \int_a^t \left(\frac{f(\tau)}{t-\tau}\right)^{r-n+1} \mathrm{d}\tau, \quad (2) $$

where $1 < r < n$.

The following is the n-dimensional linear fractional-order differential system with multiple time delays:

$$ \begin{cases} D^{\alpha_1} x_1(t) = a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12}) + \cdots + a_{1n}x_n(t-\tau_{1n}), \\ D^{\alpha_2} x_2(t) = a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22}) + \cdots + a_{2n}x_n(t-\tau_{2n}), \\ \vdots \\ D^{\alpha_n} x_n(t) = a_{n1}x_1(t-\tau_{n1}) + a_{n2}x_2(t-\tau_{n2}) + \cdots + a_{nn}x_n(t-\tau_{nn}), \end{cases} \quad (3) $$

where α_i is the order of the fractional derivative, which is real and lies in $(0,1)$, $x_i(t)$ is the state variable, $\tau_{ij} > 0$ is the time delay, the initial value $x_i(t) = \phi_i(t)$ is given by $-\max\tau_{ij} = -\tau_{\max} \le t \le 0$, $A = [a_{ij}] \in R_{n\times n}$ is the coefficient matrix.

In order to study the stability of system (3), we first take Laplace transform on system (3) and have

$$ \Delta(s) \cdot X(s) = b(s), \quad (4) $$

where $X(s) = (X_1(s), X_2(s), \ldots, X_n(s))^T$ is the Laplace transform of $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $b(s) = (b_1(s), b_2(s), \ldots, b_n(s))^T$ is the remaining nonlinear part, and the characteristic matrix of system (3) is

$$ \Delta(s) = \begin{pmatrix} s^{\alpha_1} - a_{11}e^{-s\tau_{11}} & -a_{12}e^{-s\tau_{12}} & \cdots & -a_{1n}e^{-s\tau_{1n}} \\ -a_{21}e^{-s\tau_{21}} & s^{\alpha_2} - a_{22}e^{-s\tau_{22}} & \cdots & -a_{2n}e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}e^{-s\tau_{n1}} & -a_{n2}e^{-s\tau_{n2}} & \cdots & s^{\alpha_n} - a_{nn}e^{-s\tau_{nn}} \end{pmatrix}. \quad (5) $$

Here are some results for system (3).

Theorem 1 (see [27]). *If all the roots of the characteristic equation det$(\Delta(s)) = 0$ have negative real parts, then the zero solution of system (3) is Lyapunov globally asymptotically stable.*

Corollary 1 (see [27]). *If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \beta \in (0,1)$, all the eigenvalues λ of the coefficient matrix A satisfy $|\arg(\lambda)| > \beta\pi/2$, and the characteristic equation det$(\Delta(s)) = 0$ has no purely imaginary roots for any $\tau_{ij} > 0, i, j = 1, 2, \ldots, n$, then the zero solution of system (3) is Lyapunov globally asymptotically stable.*

3. Combination Synchronization of Three Fractional-Order Delayed Systems

In this section, we investigate combination synchronization of three different fractional-order delayed systems.

The following system is considered as the first drive system:

$$ \begin{aligned} D^{\alpha}x(t) &= x(t) + x(t-\tau) + A(x(t), x(t-\tau)), \\ x(t) &= x(0), \quad t \in [-\tau, 0]. \end{aligned} \quad (6) $$

The following system is taken as the second drive system:

$$D^{\alpha} y(t) = y(t) + y(t-\tau) + B(y(t), y(t-\tau)),$$
$$y(t) = y(0), \quad t \in [-\tau, 0]. \tag{7}$$

And the following system is used as the response system:

$$D^{\alpha} z(t) = z(t) + z(t-\tau) + C(z(t), z(t-\tau)) + U,$$
$$z(t) = z(0), \quad t \in [-\tau, 0], \tag{8}$$

in which $\alpha \in (0,1)$ is the order of the fractional differential equations, $\tau > 0$ is the time delay, $U = (U_1, U_2, U_3)$ is the controller vector to be designed later, $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, $y = (y_1, y_2, \ldots, y_n)^T \in R^n$, and $z = (z_1, z_2, \ldots, z_n)^T \in R^n$ are state vectors, and $A: R^{2n} \longrightarrow R^n$, $B: R^{2n} \longrightarrow R^n$, and $C: R^{2n} \longrightarrow R^n$ are continuous vector functions.

The error state vector is defined as

$$e(t) = Fz(t) - Gx(t) - Hy(t), \tag{9}$$

where $e(t) = (e_1, e_2, \ldots, e_n)^T \in R^n$, $F = \text{diag}\{f_1, f_2, \ldots, f_n\} \in R^{n\times n}$, $G = \text{diag}\{g_1, g_2, \ldots, g_n\} \in R^{n\times n}$, and $H = \text{diag}\{h_1, h_2, \ldots, h_n\} \in R^{n\times n}$ are real scaling matrix.

Definition 1 (see [21]). The drive systems (6) and (7) and the response system (11) are defined to be combination synchronization if there are three constant matrixes, F, G, $H \in R^n$ and $F \neq 0$ such that

$$\lim_{t\longrightarrow+\infty} \|Fz(t) - Gx(t) - Hy(t)\| = 0, \tag{10}$$

where $\|\cdot\|$ stands for the matrix norm.

Remark 1. If the scaling matrix $G = 0$ or $H = 0$, the combination synchronization mentioned above is correspondingly simplified to hybrid synchronization.

Remark 2. When $\tau = 0$, the combination synchronization scheme of fractional-order delayed systems is simplified to the combination synchronization scheme of fractional-order systems.

To achieve combination synchronization of the above systems, a nonlinear controller is constructed:

$$U = \widetilde{K} e(t) + GA(x(t), x(t-\tau)) + HB(y(t), y(t-\tau))$$
$$- FC(z(t), z(t-\tau)), \tag{11}$$

where $\widetilde{K} = K - I$, I is an n-dimensional identity matrix, and $K = \text{diag}\{k_1, k_2, \ldots, k_n\}$ is a feedback gain matrix.

From equations (6)–(8) and (11), we can get the following error system:

$$D^{\alpha} e(t) = (\widetilde{k} + I)e(t) + e(t-\tau) = Ke(t) + e(t-\tau). \tag{12}$$

When we use the controller U to control fractional-order delay-delayed response system, the combination synchronization problem of the two fractional-order delayed drive systems (6) and (7) and fractional-order delayed response system (8) is changed into the analysis of the asymptotical stability of system (15).

According to Corollary 1, we can have the following sufficient condition to achieve combination synchronization between systems (6) and (7) and system (8).

Theorem 2. *Combination synchronization between the drive systems (6) and (7) and the response system (8) can be achieved if there exists a matrix* $K = diag\{k_1, k_2, \ldots, k_n\}$ *in equation (15) such that* $k_i < (-1/\sin(\alpha\pi/2))$ $(i = 1, 2, \ldots, n)$.

Proof. $A = K + I$ is the coefficient matrix for the fractional-order delayed error system (12). Because $k_i < (-1/\sin(\alpha\pi/2)), \alpha \in (0,1)$, the eigenvalues of A are $\lambda_i = k_i + 1 < 0$ $(i = 1, 2, \ldots, n)$. Therefore, $|\arg(\lambda)| > \pi/2 > \alpha\pi/2$ holds.

Taking Laplace transform on equation (15) gives

$$\Delta(s) \cdot E(s) = s^{\alpha-1} e(0) + e(0)e^{-s\tau} \int_{-\tau}^{0} e^{-s\tau} dx, \tag{13}$$

where $E(s)$ is the Laplace transform of $e(t)$, $e(0) = Fz(0) - Gx(0) - Hy(0)$, and $\Delta(s) = s^a I - K - e^{-s\tau} I$ is the characteristic matrix. Consequently

$$\det(\Delta(s)) = \left|s^{\alpha} I - K - e^{-s\tau} I\right| = \left(s^a - k_1 - e^{-s\tau}\right)$$
$$\cdot \left(s^a - k_2 - e^{-s\tau}\right) \ldots \left(s^a - k_n - e^{-s\tau}\right) = 0. \tag{14}$$

Suppose

$$\left(s^a - k_i - e^{-s\tau}\right) = 0, \quad i = 1, 2, \ldots, n, \tag{15}$$

has a root $s = wi = |w|(\cos(\pi/2) + i\sin(\pm\pi/2))$. Then

$$|w|^{\alpha}\left(\cos\left(\frac{\alpha\pi}{2}\right) + i\sin\left(\frac{\pm\alpha\pi}{2}\right)\right) - k_i - \cos(\omega\tau) + i\sin(\omega\tau) = 0. \tag{16}$$

Separating the real and imaginary parts in system (16) yields

$$|w|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right) - k_i = \cos(\omega\tau),$$
$$|w|^{\alpha}\sin\left(\frac{\pm\alpha\pi}{2}\right) = -\sin(\omega\tau). \tag{17}$$

From system (17), we have

$$|w|^{2\alpha} - 2k_i\cos\left(\frac{\alpha\pi}{2}\right)|w|^{\alpha} + k_i^2 - 1 = 0. \tag{18}$$

Because $k_i < (-1/\sin(\alpha\pi/2)), \alpha \in (0,1)$, then the discriminant of the roots satisfies

$$\begin{aligned} \Delta &= \left(-2k_i\cos\left(\frac{\alpha\pi}{2}\right)\right)^2 - 4\left(k_i^2 - 1\right) \\ &= 4\left(1 - k_i^2\sin^2\left(\frac{\alpha\pi}{2}\right)\right) \\ &< 0, \end{aligned} \tag{19}$$

which means that equation (18) has no real solutions. Consequently, equation (14) has no purely imaginary roots.

According to Corollary 1, the zero solution of the fractional-order delayed error system (12) is globally asymptotically stable; i.e., combination synchronization is obtained between the drive systems (6) and (7) and the response system (8).

The completes the proof. □

4. Numerical Simulations

In what follows, numerical simulations are performed to illustrate the above-proposed combination synchronization of three different fractional-order delayed systems.

The fractional-order delayed financial system [14] is considered as the first drive system:

$$\begin{cases} D^{\alpha}x_1 = x_3 - a_1x_1 + x_1x_2(t-\tau), \\ D^{\alpha}x_2 = 1 - b_1x_2 - x_1^2(t-\tau), \\ D^{\alpha}x_3 = -x_1(t-\tau) - c_1x_3. \end{cases} \tag{20}$$

System (20) exhibits a chaotic attractor, as shown in Figure 1.

System (20) can be rewritten as

$$\begin{aligned} &D^{\alpha}x(t) = x(t) + x(t-\tau) + A(x(t), x(t-\tau)), \\ &\quad x(t) = x(0), \quad t \in [-\tau, 0], \end{aligned} \tag{21}$$

where

$$A(x(t),x(t-\tau)) = \begin{pmatrix} x_3 - (a_1+1)x_1 + x_1x_2(t-\tau) - x_1(t-\tau) \\ 1 - (b_1+1)x_2 - x_1^2(t-\tau) - x_2(t-\tau) \\ -x_1(t-\tau) - (c_1+1)x_3 - x_3(t-\tau) \end{pmatrix}. \tag{22}$$

The fractional-order delayed Liu system [11] is considered as the second drive system:

$$\begin{cases} D^{\alpha}y_1 = a_2(y_2 - y_1), \\ D^{\alpha}y_2 = b_2y_1(t-\tau) - y_1y_3, \\ D^{\alpha}y_3 = -c_2y_3(t-\tau) + 4y_1^2. \end{cases} \tag{23}$$

System (23) displays a chaotic attractor, as shown in Figure 2.

System (23) can be rewritten as

$$\begin{aligned} &D^{\alpha}y(t) = y(t) + y(t-\tau) + B(y(t), y(t-\tau)), \\ &\quad y(t) = y(0), \quad t \in [-\tau, 0], \end{aligned} \tag{24}$$

where

$$B(y(t), y(t-\tau)) = \begin{pmatrix} a_2(y_2 - y_1) - y_1 - y_1(t-\tau) \\ b_2y_1(t-\tau) - y_1y_3 - y_2 - y_2(t-\tau) \\ -(c_2+1)y_3(t-\tau) + 4y_1^2 - y_3 \end{pmatrix}. \tag{25}$$

The fractional-order delayed Lorenz system [15] is the response system given by

$$\begin{cases} D^{\alpha}z_1 = a_3(z_2 - z_1) + U_1, \\ D^{\alpha}z_2 = c_3z_1 - z_2 - z_1z_3 + U_2, \\ D^{\alpha}z_3 = z_1z_2 - b_3z_3(t-\tau) + U_3, \end{cases} \tag{26}$$

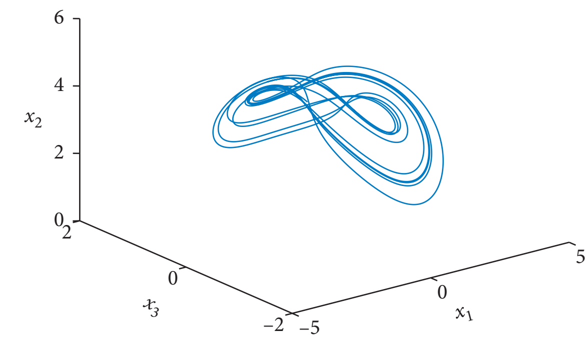

Figure 1: Chaotic attractor of financial system: $\alpha = 0.92$, $\tau = 0.01$.

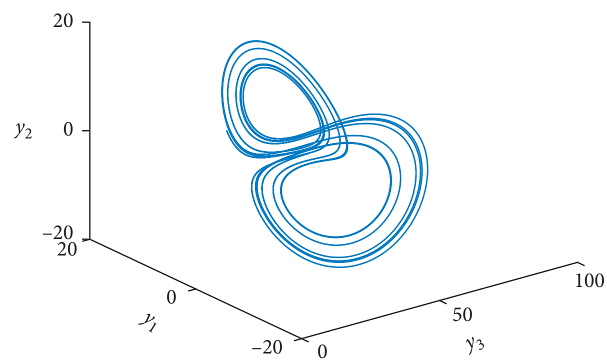

Figure 2: Chaotic attractor of Liu system: $\alpha = 0.92$, $\tau = 0.01$.

where U_1, U_2, and U_3 are controllers to be determined later. Without the controllers, system (26) displays a chaotic attractor, as shown in Figure 3.

System (26) can be rewritten as

$$\begin{aligned} &D^{\alpha}z(t) = z(t) + z(t-\tau) + C(z(t), z(t-\tau)) + U, \\ &\quad z(t) = z(0), \quad t \in [-\tau, 0], \end{aligned} \tag{27}$$

where

$$C(z(t), z(t-\tau)) = \begin{pmatrix} a_3(z_2 - z_1) - z_1 - z_1(t-\tau) \\ c_3z_1 - 2z_2 - z_1z_3 - z_2(t-\tau) \\ z_1z_2 - (b_3+1)z_3(t-\tau) - z_3 \end{pmatrix}. \tag{28}$$

In the following analysis, we suppose that $F = \mathrm{diag}(f_1, f_2, f_3)$, $G = \mathrm{diag}(g_1, g_2, g_3)$, and $H = \mathrm{diag}(h_1, h_2, h_3)$.

The error states are defined by

$$\begin{cases} e_1 = f_1z_1 - g_1x_1 - h_1y_1, \\ e_2 = f_2z_2 - g_2x_2 - h_2y_2, \\ e_3 = f_3z_3 - g_3x_3 - h_3y_3, \end{cases} \tag{29}$$

such that

$$\begin{cases} \lim_{t\longrightarrow\infty} \left\| f_1z_1 - g_1x_1 - h_1y_1 \right\| = 0, \\ \lim_{t\longrightarrow\infty} \left\| f_2z_2 - g_2x_2 - h_2y_2 \right\| = 0, \\ \lim_{t\longrightarrow\infty} \left\| f_3z_3 - g_3x_3 - h_3y_3 \right\| = 0. \end{cases} \tag{30}$$

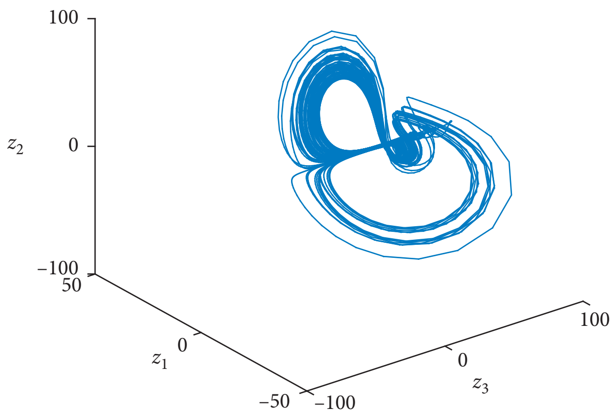

FIGURE 3: Chaotic attractor of Lorenz system: $\alpha = 0.92$, $\tau = 0.1$.

Subtracting (20) and (23) from (26), the error dynamical systems are obtained as follows:

$$\begin{cases} D^\alpha e_1 = f_1 D^\alpha z_1 - g_1 D^\alpha x_1 - h_1 D^\alpha y_1, \\ D^\alpha e_2 = f_2 D^\alpha z_2 - g_2 D^\alpha x_2 - h_2 D^\alpha y_2, \\ D^\alpha e_3 = f_3 D^\alpha z_3 - g_3 D^\alpha x_3 - h_3 D^\alpha y_3. \end{cases} \tag{31}$$

Substituting equations (20), (23), and (26) into equation (31) gives

$$\begin{cases} D^\alpha e_1 = f_1 [a_3 (z_2 - z_1)] - g_1 [x_3 - a_1 x_1 + x_1 x_2 (t-\tau)] \\ \qquad - h_1 [a_2 (y_2 - y_1)] + f_1 U_1, \\ D^\alpha e_2 = f_2 (c_3 z_1 - z_2 - z_1 z_3) - g_2 (1 - b_1 x_2 - x_1^2 (t-\tau)) \\ \qquad - h_2 (b_2 y_1 (t-\tau) - y_1 y_3) + f_2 U_2, \\ D^\alpha e_3 = f_3 (z_1 z_2 - b_3 z_3 (t-\tau)) - g_3 (-x_1 (t-\tau) - c_1 x_3) \\ \qquad - h_3 (-c_2 y_3 (t-\tau) + 4 y_1^2) + f_3 U_3. \end{cases} \tag{32}$$

Here are our results.

Theorem 3. *Combination synchronization between the driven systems (20) and (23) and the response system (26) can be obtained by presenting controllers as follows:*

$$\begin{cases} U_1 = \dfrac{1}{f_1} \{(k_1 - 1)(f_1 z_1 - g_1 x_1 - h_1 y_1) + g_1 (x_3 - (a_1 + 1) x_1 + x_1 x_2 (t-\tau) - x_1 (t-\tau)) \\ \qquad + h_1 (a_2 (y_2 - y_1) - y_1 - y_1 (t-\tau)) - f_1 (a_3 z_2 - (a_3 + 1) z_1 - z_1 (t-\tau))\}, \\ U_2 = \dfrac{1}{f_2} \{(k_2 - 1)(f_2 z_2 - g_2 x_2 - h_2 y_2) + g_2 \left(1 - (b_1 + 1) x_2 - x_1^2 (t-\tau) - x_2 (t-\tau)\right) \\ \qquad + h_2 (b_2 y_1 (t-\tau) - y_1 y_3 - y_2 - y_2 (t-\tau)) - f_2 (c_3 z_1 - 2 z_2 - z_1 z_3 - z_2 (t-\tau))\}, \\ U_3 = \dfrac{1}{f_3} \{(k_3 - 1)(f_3 z_3 - g_3 x_3 - h_3 y_3) + g_3 (-x_1 (t-\tau) - (c_1 + 1) x_3 - x_3 (t-\tau)) \\ \qquad + h_3 (-(c_2 + 1) y_3 (t-\tau) + 4 y_1^2 - y_3) - f_3 (z_1 z_2 - (b_3 + 1) z_3 (t-\tau) - z_3)\}. \end{cases} \tag{33}$$

Corollary 2

(i) Suppose that $g_1 = g_2 = g_3 = 0$, and f_1, f_2, and f_3 are nonzero, projective synchronization between the drive system (23) and the response system (26) can be obtained by presenting controllers as follows:

$$\begin{cases} U_1 = \dfrac{1}{f_1} \{(k_1 - 1)(f_1 z_1 - h_1 y_1) + h_1 (a_2 (y_2 - y_1) - y_1 - y_1 (t-\tau)) - f_1 (a_3 z_2 - (a_3 + 1) z_1 - z_1 (t-\tau))\}, \\ U_2 = \dfrac{1}{f_2} \{(k_2 - 1)(f_2 z_2 - h_2 y_2) + h_2 (b_2 y_1 (t-\tau) - y_1 y_3 - y_2 - y_2 (t-\tau)) - f_2 (c_3 z_1 - 2 z_2 - z_1 z_3 - z_2 (t-\tau))\}, \\ U_3 = \dfrac{1}{f_3} \left\{(k_3 - 1)(f_3 z_3 - h_3 y_3) + h_3 \left(-(c_2 + 1) y_3 (t-\tau) + 4 y_1^2 - y_3\right) - f_3 (z_1 z_2 - (b_3 + 1) z_3 (t-\tau) - z_3)\right\}. \end{cases} \tag{34}$$

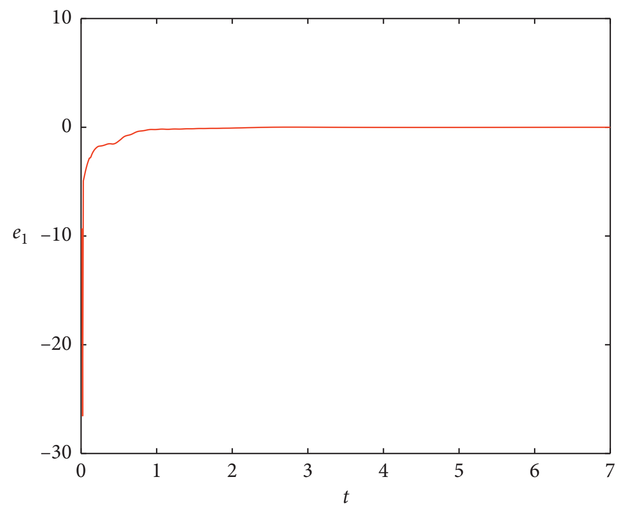

FIGURE 4: Combination synchronization errors e_1 between drive systems (20) and (23) and response system (26).

(ii) Accordingly, suppose that $h_1 = h_2 = h_3 = 0$, and f_1, f_2, and f_3 are nonzero, projective synchronization between the drive system (20) and the response system (26) can be obtained by presenting controllers as follows:

$$\begin{cases} U_1 = \dfrac{1}{f_1}\{(k_1 - 1)(f_1 z_1 - g_1 x_1) + g_1(x_3 - (a_1 + 1)x_1 + x_1 x_2(t-\tau) - x_1(t-\tau)) - f_1(a_3 z_2 - (a_3 + 1)z_1 - z_1(t-\tau))\}, \\ U_2 = \dfrac{1}{f_2}\left\{(k_2 - 1)(f_2 z_2 - g_2 x_2) + g_2\left(1 - (b_1 + 1)x_2 - x_1^2(t-\tau) - x_2(t-\tau)\right) - f_2(c_3 z_1 - 2z_2 - z_1 z_3 - z_2(t-\tau))\right\}, \\ U_3 = \dfrac{1}{f_3}\{(k_3 - 1)(f_3 z_3 - g_3 x_3) + g_3(-x_1(t-\tau) - (c_1 + 1)x_3 - x_3(t-\tau)) - f_3(z_1 z_2 - (b_3 + 1)z_3(t-\tau) - z_3)\}. \end{cases} \tag{35}$$

Corollary 3. *Suppose that $g_1 = g_2 = g_3 = 0, h_1 = h_2 = h_3 = 0$, and f_1, f_2, and f_3 are nonzero, system (26) can be stabilized to its equilibrium $O(0, 0, 0)$ with the following controllers:*

$$\begin{cases} U_1 = \dfrac{1}{f_1}\{(k_1 - 1)(f_1 z_1) - f_1(a_3 z_2 - (a_3 + 1)z_1 - z_1(t-\tau))\}, \\ U_2 = \dfrac{1}{f_2}\{(k_2 - 1)(f_2 z_2) - f_2(c_3 z_1 - 2z_2 - z_1 z_3 - z_2(t-\tau))\}, \\ U_3 = \dfrac{1}{f_3}\{(k_3 - 1)(f_3 z_3) - f_3(z_1 z_2 - (b_3 + 1)z_3(t-\tau) - z_3)\}. \end{cases} \tag{36}$$

In the numerical simulations, the system parameters are set as $a_1 = 3$, $b_1 = 0.1, c_1 = 1$, $a_2 = 10$, $b_2 = 40$, $c_2 = 2.5$, $a_3 = 10$, $b_3 = 8/3$, and $c_3 = 28$, respectively. For simplicity, suppose $f_1 = f_2 = f_3 = 1$, $g_1 = g_2 = g_3 = 1$, and $h_1 = h_2 = h_3 = 1$. The initial values for all of the systems are set as $(x_1(0), x_2(0), x_3(0)) = (0.1, 4, 0.5)$, $(y_1(0), y_2(0), y_3(0)) = (1.2, 2.4, 11)$, and $(z_1(0), z_2(0), z_3(0)) = (-8, 2, 3)$, respectively. Figures 4–6 display time responses of the combination synchronization errors e_1, e_2, and e_3. From Figures 4–6, we can observe that error states converge to zero;

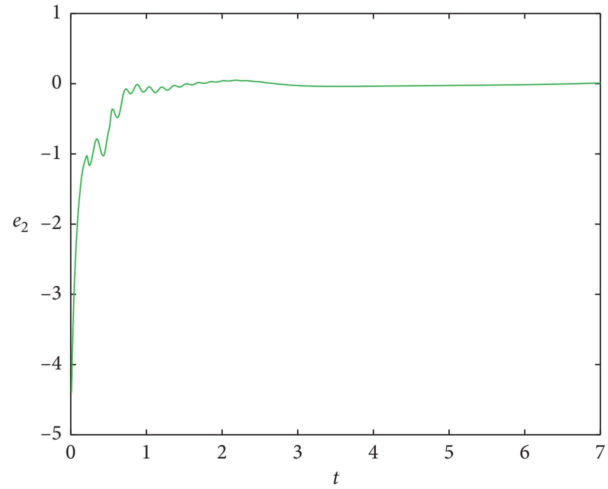

Figure 5: Combination synchronization errors e_2 between drive systems (20) and (23) and response system (26).

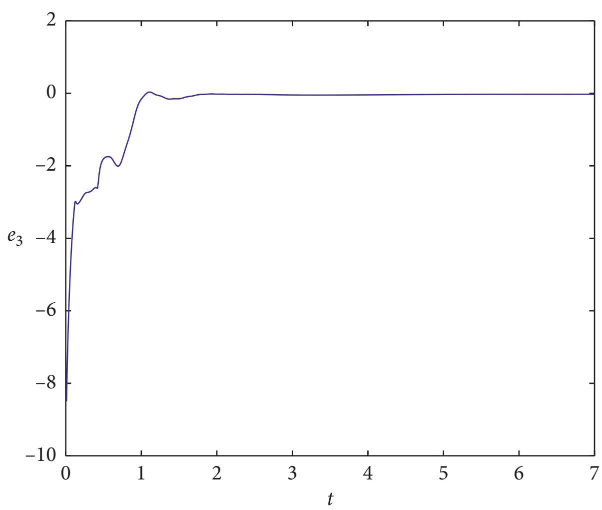

Figure 6: Combination synchronization errors e_3 between drive systems (20) and (23) and response system (26).

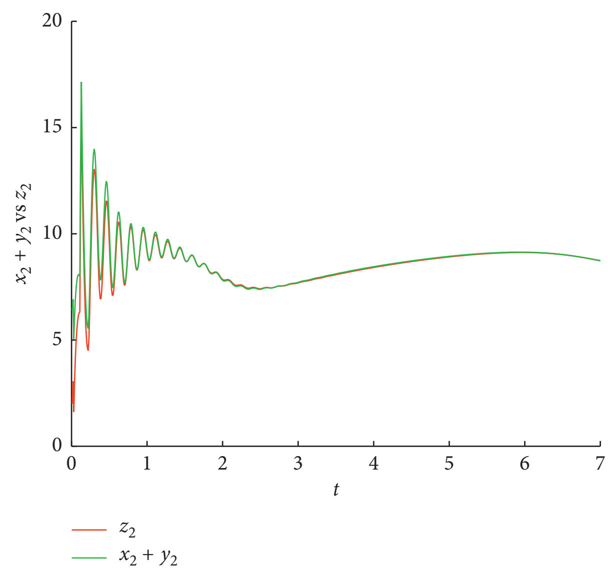

Figure 8: Time responses for state $x_2 + y_2$ versus z_2.

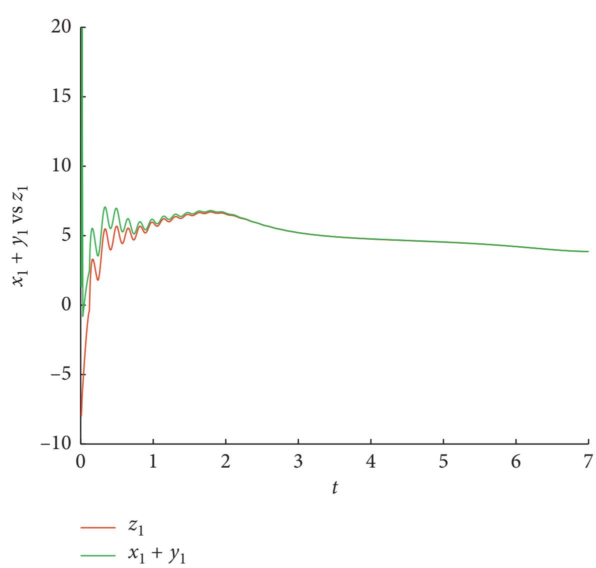

Figure 7: Time responses for state $x_1 + y_1$ versus z_1.

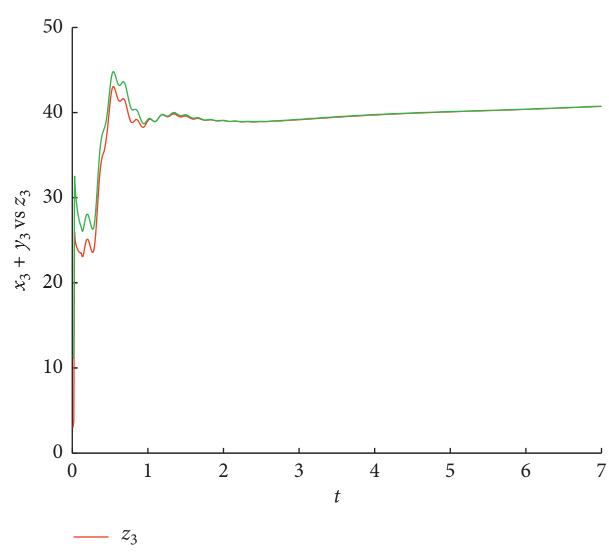

Figure 9: Time responses for state $x_3 + y_3$ versus z_3.

i.e., combination synchronization is achieved. Figures 7–9 illustrate the time responses of the states $x_1 + y_1$ versus z_1, $x_2 + y_2$ versus z_2, $x_3 + y_3$ versus z_3, respectively.

5. Conclusions

In this paper, we investigate combination synchronization of three different fractional-order delayed chaotic systems by generalizing combination synchronization of delayed chaotic systems or combination synchronization of fractional-order chaotic systems. With the help of the stability theory for linear fractional-order systems with multiple time delays, controllers are proposed to achieve combination synchronization of three different fractional-order delayed chaotic systems. In addition, projective synchronization [28] of three different fractional-order delayed chaotic systems is a special case of our work. Numerical simulations are presented to demonstrate and verify the applicability and feasibility of our theoretical analysis.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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