

Risk-sensitive Reinforcement Learning*

Yun Shen¹, Michael J. Tobia³, Tobias Sommer³, Klaus Obermayer^{1, 2}

¹Technical University Berlin, Germany

²Bernstein Computational Neuroscience Center, Berlin, Germany

³University Medical Center Hamburg-Eppendorf, Germany

January 23, 2014

Abstract

We derive a family of risk-sensitive reinforcement learning methods for agents, who face sequential decision-making tasks in uncertain environments. By applying a utility function to the temporal difference (TD) error, nonlinear transformations are effectively applied not only to the received rewards but also to the true transition probabilities of the underlying Markov decision process. When appropriate utility functions are chosen, the agents' behaviors express key features of human behavior as predicted by prospect theory (Kahneman and Tversky, 1979), for example different risk-preferences for gains and losses as well as the shape of subjective probability curves. We derive a risk-sensitive Q-learning algorithm, which is necessary for modeling human behavior when transition probabilities are unknown, and prove its convergence. As a proof of principle for the applicability of the new framework we apply it to quantify human behavior in a sequential investment task. We find, that the risk-sensitive variant provides a significantly better fit to the behavioral data and that it leads to an interpretation of the subject's responses which is indeed consistent with prospect theory. The analysis of simultaneously measured fMRI signals show a significant correlation of the risk-sensitive TD error with BOLD signal change in the ventral striatum. In addition we find a significant correlation of the risk-sensitive Q-values with neural activity in the striatum, cingulate cortex and insula, which is not present if standard Q-values are used.

1 Introduction

Risk arises from the uncertainties associated with future events, and is inevitable since the consequences of actions are uncertain at the time when a decision is made. Hence, risk has to be taken into account by the decision-maker, consciously or unconsciously. An economically rational decision-making rule, which is *risk-neutral*, is to select the alternative with the highest expected reward. In the context of sequential or multistage

*Accepted for publication in *Neural Computation*.

decision-making problems, *reinforcement learning* (RL, Sutton and Barto, 1998) follows this line of thought. It describes how an agent ought to take actions that maximize expected cumulative rewards in an environment typically described by a *Markov decision process* (MDP, Puterman, 1994). RL is a well-developed model not only for human decision-making, but also for models of free choice in non-humans, because similar computational structures, such as dopaminergically mediated reward prediction errors, have been identified across species (Schultz et al., 1997; Schultz, 2002).

Besides risk-neutral policies, *risk-averse* policies, which accept a choice with a more certain but possibly lower expected reward, are also considered economically rational (Gollier, 2004). For example, a risk-averse investor might choose to put money into a bank account with a low but guaranteed interest rate, rather than into a stock with possibly high expected returns but also a chance of high losses. Conversely, *risk-seeking* policies, which prefer a choice with less certain but possibly high reward, are considered economically irrational. Human agents are, however, not always economically rational (Gilboa, 2009). Behavioral studies show that human can be risk-seeking in one situation while risk-averse in another situation (Kahneman and Tversky, 1979). RL algorithms developed so far cannot effectively model these complicated risk-preferences.

Risk-sensitive decision-making problems, in the context of MDPs, have been investigated in various fields, e.g., in machine learning (Heger, 1994; Mihatsch and Neuneier, 2002), optimal control (Hernández-Hernández and Marcus, 1996), operations research (Howard and Matheson, 1972; Borkar, 2002), finance (Ruszczynski, 2010), as well as neuroscience (Nagengast et al., 2010; Braun et al., 2011; Niv et al., 2012). Note that the core of MDPs consists of two sets of *objective* quantities describing the environment: immediate *rewards* obtained at states by executing actions, and *transition probabilities* for switching states when performing actions. Facing the same environment, however, different agents might have different policies, which indicates that risk is taken into account differently by different agents. Hence, to incorporate risk, which is derived from both quantities, all existing literature applies a nonlinear transformation to either the experienced reward values or to the transition probabilities, or to both. The former is the canonical approach in classical economics, as in expected utility theory (Gollier, 2004), while the latter originates from behavioral economics, as in *subjective probability* (Savage, 1972), but is also derived from a rather recent development in mathematical finance, *convex/coherent risk measures* (Artzner et al., 1999; Föllmer and Schied, 2002). For modeling human behaviors, prospect theory (Kahneman and Tversky, 1979) suggests that we should combine both approaches, i.e., human beings have different perceptions not only for the same objective amount of rewards but also the same value of the true probability. Recently, Niv et al. (2012) combined both approaches by applying piecewise linear functions (an approximation of a nonlinear transformation) to reward prediction errors that contain the information of rewards directly and the information of transition probabilities indirectly. Importantly, the reward prediction errors that incorporated experienced risk were strongly coupled to activity in the nucleus accumbens of the ventral striatum, providing a biologically based plausibility to this combined approach. In this work we show (in Section 2.1) that the risk-sensitive algorithm proposed by Niv and colleagues is a special case of our general risk-sensitive RL framework.

Most of the literature in economics or engineering fields focuses on economically rational risk-averse/-neutral strategies, which are not always adopted by humans. The models proposed in behavioral economics, despite allowing economic irrationality, require knowledge of the true probability, which usually is not available at the outset of a learning task. In neuroscience, on the one hand, several works (e.g., Wu et al., 2009; Preuschoff et al., 2008) follow the same line as in behavioral economics and require knowledge of the true probability. On the other hand, though different modified RL algorithms (e.g., Glimcher et al., 2008; Symmonds et al., 2011) are applied to model human behaviors in learning tasks, the algorithms often fail to generalize across different tasks. In our previous work (Shen et al., 2013), we described a general framework for incorporating risk into MDPs by introducing nonlinear transformations to both rewards and transition probabilities. A risk-sensitive objective was derived and optimized by value iteration or dynamic programming. This solution, hence, does not work in learning tasks where the true transition probabilities are unknown to learning agents. For this purpose, a model-free framework for RL algorithms is to be derived in this paper, where, similar to Q-learning, the knowledge of the transition and reward model is not needed.

This paper is organized as follows. Section 2 starts with a mathematical introduction into *valuation functions* for measuring risk. We then specify a sufficiently rich class of valuation functions in Section 2.1 and provide the intuition behind our approach by applying this class to a simple example in Section 2.2. We also show that key features of prospect theory can be captured by this class of valuation functions. Restricted to the same class, we derive a general framework for risk-sensitive Q-learning algorithms and prove its convergence in Section 3. Finally, in Section 4, we apply this framework to quantify human behavior. We show that the risk-sensitive variant provides a significantly better fit to the behavioral data and significant correlations are found between sequences generated by the proposed framework and changes of fMRI BOLD signals.

2 Valuation Functions and Risk Sensitivities

Suppose that we are facing choices. Each *choice* might yield different outcomes when events are generated by a random process. Hence, to keep generality, we model the outcome of each choice by a real-valued random variable $\{X(i), \mu(i)\}_{i \in I}$, where I denotes an *event space* with a finite cardinality $|I|$ and $X(i) \in \mathbb{R}$ is the outcome of i th event with probability $\mu(i)$. We say two vectors $X \leq Y$ if $X(i) \leq Y(i)$ for all $i \in I$. Let $\mathbf{1}$ (resp. $\mathbf{0}$) denote the vector with all elements equal 1 (resp. 0). Let \mathcal{P} denote the space of all possible distributions μ .

Choices are made according to their outcomes. Hence, we assume that there exists a mapping $\rho : \mathbb{R}^{|I|} \times \mathcal{P} \rightarrow \mathbb{R}$ such that one prefers (X, μ) to (Y, ν) whenever $\rho(X, \mu) \geq \rho(Y, \nu)$. We assume further that ρ satisfies the following axioms inspired by the *risk measure theory* applied in mathematical finance (Artzner et al., 1999; Föllmer and Schied, 2002). A mapping $\rho : \mathbb{R}^{|I|} \times \mathcal{P} \rightarrow \mathbb{R}$ is called a **valuation function**, if it satisfies for each $\mu \in \mathcal{P}$,

$$\text{I (monotonicity)} \quad \rho(X, \mu) \leq \rho(Y, \mu), \text{ whenever } X \leq Y \in \mathbb{R}^{|I|};$$

II (translation invariance) $\rho(X + y\mathbf{1}, \mu) = \rho(X, \mu) + y$, for any $y \in \mathbb{R}$.

Within the economic context, X and Y are outcomes of two choices. Monotonicity reflects the intuition that given the same event distribution μ , if the outcome of one choice is *always* (for all events) higher than the outcome of another choice, the *valuation* of the choice must be also higher. Under the axiom of translation invariance, the sure outcome $y\mathbf{1}$ (equal outcome for every event) after executing decisions, is considered as a sure outcome before making decision. This also reflects the intuition that there is no risk if there is no uncertainty.

In our setting, valuation functions are not necessarily centralized, i.e. $\rho(\mathbf{0}, \mu)$ is not necessarily 0, since $\rho(\mathbf{0}, \mu)$ in fact sets a reference point, which can differ for different agents. However, we can centralize any valuation function by $\tilde{\rho}(X, \mu) := \rho(X, \mu) - \rho(\mathbf{0}, \mu)$. From the two axioms, it follows that (for the proof see Lemma A.1 in Appendix)

$$\min_{i \in I} X_i =: \underline{X} \leq \tilde{\rho}(X, \mu) \leq \overline{X} := \max_{i \in I} X_i, \forall \mu \in \mathcal{P}, X \in \mathbb{R}^{|I|}. \quad (1)$$

\overline{X} is the possibly largest outcome, which represents the most optimistic prediction of the future, while \underline{X} is the possibly smallest outcome and the most pessimistic estimation. The centralized valuation function $\tilde{\rho}(X, \mu)$ satisfying $\tilde{\rho}(\mathbf{0}, \mu) = 0$ can be in fact viewed as a subjective mean of the random variable X , which varies from the best scenario \overline{X} to the worst scenario \underline{X} , covering the objective mean as a special case.

To judge the risk-preference induced by a certain type of valuation functions, we follow the rule that *diversification* should be preferred if the agent is *risk-averse*. More specifically, suppose an agent has two possible choices, one of which leads to the future reward (X, μ) while the other one leads to the future reward (Y, ν) . For simplicity we assume $\mu = \nu$. If the agent *diversifies*, i.e., if one spends only a fraction α of the resources on the first and the remaining amount on the second alternative, the future reward is given by $\alpha X + (1 - \alpha)Y$. If the applied valuation function is concave, i.e.,

$$\rho(\alpha X + (1 - \alpha)Y, \mu) \geq \alpha \rho(X, \mu) + (1 - \alpha) \rho(Y, \mu),$$

for all $\alpha \in [0, 1]$ and $X, Y \in \mathbb{R}^{|I|}$, then the diversification should increase the (subjective) valuation. Thus, we call the agent's behavior *risk-averse*. Conversely, if the applied valuation function is *convex*, the induced risk-preference should be *risk-seeking*.

2.1 Utility-based Shortfall

We now introduce a class of valuation functions, the utility-based shortfall, which generalizes many important special valuation functions in literature. Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a *utility function*, which is continuous and strictly increasing. The shortfall $\rho_{x_0}^u$ induced by u and an *acceptance level* x_0 is then defined as

$$\rho_{x_0}^u(X, \mu) := \sup \left\{ m \in \mathbb{R} \mid \sum_{i \in I} u(X(i) - m) \mu(i) \geq x_0 \right\}, \quad (2)$$

It can be shown (cf. Föllmer and Schied, 2004) that $\rho_{x_0}^u$ is a valid valuation function satisfying the axioms. The utility-based shortfall was first introduced in the mathematical finance literature (Föllmer and Schied, 2004). The class of utility functions considered here will, however, be more general than the class of utility functions typically used in finance.

Comparing with the expected utility theory, the utility function in Eq. (2) is applied to the relative value $X(i) - m$ rather than to the absolute outcome $X(i)$. This reflects the intuition that human beings judge utilities usually by comparing those outcome with a reference value which may not be zero. The property of u being convex or concave determines the risk sensitivity of $\rho_{x_0}^u$: given a concave function u , ρ is also concave and hence risk-averse (see Theorem 4.61, Föllmer and Schied, 2004). Vice versa, ρ is convex (hence risk-seeking) for convex u .

Utility-based shortfalls cover a large family of valuation functions, which have been proposed in literature of various fields.

- (a) For $u(x) = x$ and $x_0 = 0$, one obtains the standard expected reward $\rho(X, \mu) = \sum_i X(i)\mu(i)$.
- (b) For $u(x) = e^{\lambda x}$ and $x_0 = 1$, one obtains $\rho(X, \mu) = \frac{1}{\lambda} \log [\sum_i \mu(i)e^{\lambda X(i)}]$ (the so called *entropic map*, see e.g. Cavazos-Cadena, 2010 and references therein). Expansion w.r.t. λ leads to

$$\rho(X, \mu) = \mathbb{E}^\mu[X] + \lambda \text{Var}^\mu[X] + O(\lambda^2)$$

where $\text{Var}^\mu[X]$ denotes the variance of X under the distribution μ . Hence, the entropic map is risk-averse if $\lambda < 0$ and risk-seeking if $\lambda > 0$. In neuroscience, Nagengast et al. (2010) and Braun et al. (2011) applied this type of valuation function to test risk-sensitivity in human sensorimotor control.

- (c) Mihatsch and Neuneier (2002) proposed the following setting

$$u(x) = \begin{cases} (1 - \kappa)x & \text{if } x > 0 \\ (1 + \kappa)x & \text{if } x \leq 0 \end{cases} ,$$

where $\kappa \in (-1, 1)$ controls the degree of risk sensitivity. Its sign determines the property of the utility function u being convex vs. concave and, therefore, the risk-preference of ρ . In a recent study, Niv et al. (2012) applied this type of valuation function to quantify risk-sensitive behavior of human subjects and to interpret the measured neural signals.

When quantifying human behavior, combined convex/concave utility functions, e.g.,

$$u_p(x) = \begin{cases} k_+ x^{l_+} & x \geq 0 \\ -k_- (-x)^{l_-} & x < 0 \end{cases} , \quad (3)$$

are of special interest, since people tend to treat gains and losses differently and, therefore, have different risk preferences on gain and loss sides. In fact, the polynomial function in Eq. (3) was used in the prospect theory (Kahneman and Tversky, 1979) to model human risk preferences and the results show that l_+ is usually below 1, i.e., $u_p(x)$ is concave and thus risk-averse on gains, while l_- is also below 1 and $u_p(x)$ is therefore convex and risk-seeking on losses.

2.2 Utility-based Shortfall and Prospect Theory

To illustrate the risk-preferences induced by different utility functions, we consider a simple example with two events. The first event has outcome x_1 with probability p , while the other event has smaller outcome $x_2 < x_1$ with $1 - p$. Note that $p = \frac{\mathbb{E}X - x_2}{x_1 - x_2}$, where $\mathbb{E}X = px_1 + (1 - p)x_2$ denotes the risk-neutral mean.

Replacing $\mathbb{E}X$ with the *subjective mean* $\tilde{\rho}(X, p) = \rho(X, p) - \rho(0, p)$ defined in Eq. (1), we can define a *subjective probability* (cf. Tversky and Kahneman (1992)) as

$$w(p) := \frac{\tilde{\rho}(X, p) - x_2}{x_1 - x_2}, \quad (4)$$

which measures agents' subjective perception of the true probability p .

In risk-neutral cases, $\tilde{\rho}(X, p)$ is simply the mean and $w(p) = p$. In risk-averse cases, the balance moves towards the worst scenario. Hence, the probability of the first event (with larger outcome x_1) is always underestimated. On the contrary, in risk-seeking cases, the probability of the first event is always overestimated. Behavioral studies show that human subjects usually overestimate low probabilities and underestimate high probabilities (Tversky and Kahneman, 1992). This can be quantified by applying mixed valuation functions ρ . If we apply utility-based shortfalls, it can be quantified by using mixed utility function u .

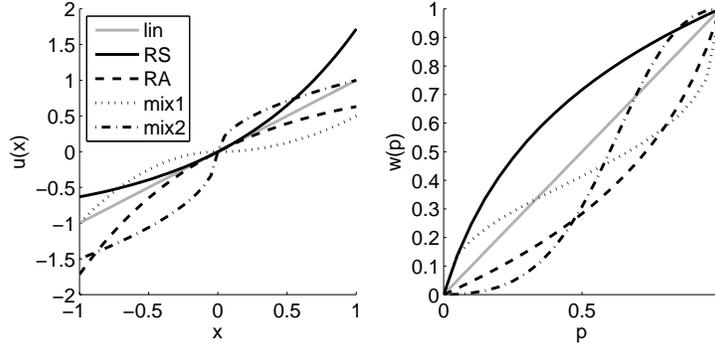


Figure 1: Shortfalls with different utility functions and induced subjective probabilities. (Left) utility functions defined as follows: lin : x ; RS : $e^x - 1$; RA : $1 - e^{-x}$; mix1: $u_p(x)$ as defined in Eq. (3) with $k_+ = 0.5$, $l_+ = 2$, $k_- = 1$ and $l_- = 2$; mix2: same as mix1 but with $k_+ = 1$, $l_+ = 0.5$, $k_- = 1.5$ and $l_- = 0.5$. (Right) subjective probability functions calculated according to Eq. (4).

Let $x_1 = 1$, $x_2 = -1$ and the acceptance level $x_0 = 0$. Fig. 1 (left) shows five different utility functions, one linear function “lin”, one convex function “RS”, one concave function “RA”, and two mixed functions “mix1” and “mix2” (for details see caption). The corresponding subjective probabilities are shown in Fig. 1 (right). Since the function “RA” is concave, the corresponding valuation function is risk-averse and therefore the probability of high-reward event is always underestimated. For the case of the convex function “RS”, the probability of high-reward event

is always overestimated. However, since the “mix1” function is convex on $[0, \infty)$ but concave on $(-\infty, 0]$, high probabilities are underestimated while low probabilities are overestimated, which replicates very well the probability weighting function applied in prospect theory for gains (cf. Fig. 1, Tversky and Kahneman, 1992). Conversely, the “mix2” function, which is concave on $[0, \infty)$ and convex on $(-\infty, 0]$, corresponds to the overestimation of high probabilities and the underestimation of low probabilities. This corresponds to the weighting function used for losses in prospect theory (cf. Fig. 2, Tversky and Kahneman, 1992).

We will see in the following section that the advantage of using the utility-based shortfall is that we can derive iterating learning algorithms for the estimation of the subjective valuations, whereas it is difficult to derive such algorithms in the framework of prospect theory.

3 Risk-sensitive Reinforcement Learning

A Markov decision process (see e.g. Puterman 1994)

$$\mathcal{M} = \{\mathbf{S}, (\mathbf{A}, \mathbf{A}(s), s \in \mathbf{S}), \mathcal{P}, (r, \mathcal{P}_r)\},$$

consists of a state space \mathbf{S} , admissible action spaces $\mathbf{A}(s) \subset \mathbf{A}$ at $s \in \mathbf{S}$, a transition kernel $\mathcal{P}(s'|s, a)$, which denotes the transition probability moving from one state s to another state s' by executing action a , and a reward function r with its distribution \mathcal{P}_r . In order to model random rewards, we assume that the reward function has the form¹

$$r(s, a, \varepsilon) : \mathbf{S} \times \mathbf{A} \times \mathbf{E} \rightarrow \mathbb{R}.$$

\mathbf{E} denotes the noise space with distribution $\mathcal{P}_r(\varepsilon|s, a)$, i.e., given (s, a) , $r(s, a, \varepsilon)$ is a random variable with values drawn from $\mathcal{P}_r(\cdot|s, a)$. Let $R(s, a)$ be the *random* reward gained at (s, a) , which follows the distribution $\mathcal{P}_r(\cdot|s, a)$. The random state (resp. action) at time t is denoted by S_t (resp. A_t). Finally, we assume that all sets \mathbf{S} , \mathbf{A} , \mathbf{E} are finite.

A *Markov policy* $\pi = [\pi_0, \pi_1, \dots]$ consists of a sequence of single-step Markov policies at times $t = 0, 1, \dots$, where $\pi_t(A_t = a|S_t = s)$ denotes the probability of choosing action a at state s . Let Π be the set of all Markov policies. The optimal policy within a time horizon T is obtained by maximizing the expectation of the discounted cumulative rewards,

$$J_T(\pi, s) := \max_{\pi \in \Pi} \mathbb{E} \left[\sum_{t=0}^{T-1} \gamma^t R(S_t, A_t) | S_0 = s, \pi \right]. \quad (5)$$

where $s \in \mathbf{S}$ denotes the initial state and $\gamma \in [0, 1)$ the discount factor. Expanding the sum leads to

$$J_T(\pi, s) = \mathbb{E}_{S_0=s}^{\pi_0} [R(S_0, A_0) + \gamma \mathbb{E}_{S_1}^{\pi_1} [R(S_1, A_1) + \dots + \gamma \mathbb{E}_{S_T}^{\pi_T} [R(S_T, A_T)] \dots]]. \quad (6)$$

¹In standard MDPs, it is sufficient (Puterman, 1994) to consider the *deterministic* reward function $\bar{r}(s, a) := \sum_{\varepsilon \in \mathbf{E}} r(s, a, \varepsilon) \mathcal{P}_r(\varepsilon|s, a)$, i.e., the mean reward at each (s, a) -pair. In risk-sensitive cases, random rewards cause also risk and uncertainties. Hence, we keep the generality by using random rewards.

We now generalize the conditional expectation \mathbb{E}_s^π to represent the valuation functions considered in Section 2. Let $\mathbf{K} := \{(s, a) | s \in \mathbf{S}, a \in \mathbf{A}(s)\}$ be the set of all admissible state-action pairs. Let

$$I = \mathbf{S} \times \mathbf{E} \quad \text{and} \quad \mu_{s,a}(s', \varepsilon) = \mathcal{P}(s'|s, a)\mathcal{P}_r(\varepsilon|s, a). \quad (7)$$

A mapping $\mathcal{U}(X, \mu|s, a) : \mathbb{R}^{|I|} \times \mathcal{P} \times \mathbf{K} \rightarrow \mathbb{R}$ is called a **valuation map**, if for each $(s, a) \in \mathbf{K}$, $\mathcal{U}(\cdot|s, a)$ is a valuation function on $\mathbb{R}^{|I|} \times \mathcal{P}$. Let $\mathcal{U}_{s,a}(X, \mu)$ be a short notation of $\mathcal{U}(X, \mu|s, a)$ and let

$$\mathcal{U}_s^\pi(X, \mu) := \sum_{a \in \mathbf{A}(s)} \pi(a|s) \mathcal{U}(X, \mu|s, a)$$

be the valuation map averaged over all actions. Since $\mu \equiv \mu_{s,a}$ for each $(s, a) \in \mathbf{K}$, we will omit μ in \mathcal{U} in the following. Replacing the conditional expectation \mathbb{E}_s^π with \mathcal{U}_s^π in Eq. (6), the risk-sensitive objective becomes

$$\tilde{J}_T(\boldsymbol{\pi}, s) := \mathcal{U}_{S_0=s}^{\pi_0} [R(S_0, A_0) + \gamma \mathcal{U}_{S_1}^{\pi_1} [R(S_1, A_1) + \dots + \gamma \mathcal{U}_{S_T}^{\pi_T} [R(S_T, A_T)] \dots]]. \quad (8)$$

The optimal policy is then given by $\max_{\boldsymbol{\pi} \in \Pi} \tilde{J}_T(\boldsymbol{\pi}, s)$. For infinite-horizon problem, we obtain

$$\max_{\boldsymbol{\pi} \in \Pi} \tilde{J}(\boldsymbol{\pi}, s) := \lim_{T \rightarrow \infty} \tilde{J}_T(\boldsymbol{\pi}, s), \quad (9)$$

using the same line of argument.

The optimization problem for finite-stage objective function \tilde{J}_T can be solved by a generalized *dynamic programming* (Bertsekas and Tsitsiklis, 1996), while the one defined in Eq. (9) requires the solution to the *risk-sensitive Bellman equation*:

$$V^*(s) = \max_{a \in \mathbf{A}(s)} \mathcal{U}_{s,a}(R(s, a) + \gamma V^*). \quad (10)$$

The latter is a consequence of the following theorem.

Theorem 3.1 (Theorem 5.5, Shen et al., 2013). *$V^*(s) = \max_{\boldsymbol{\pi}} \tilde{J}(\boldsymbol{\pi}, s)$ holds for all $s \in \mathbf{S}$, whenever V^* satisfies the equation (10). Furthermore, a deterministic policy π^* is optimal, if $\pi^*(s) = \arg \max_{a \in \mathbf{A}(s)} \mathcal{U}_{s,a}(R + \gamma V^*)$.*

Define $Q^*(s, a) := \mathcal{U}_{s,a}(R + \gamma V^*)$. Then Eq. (10) becomes

$$Q^*(s, a) = \mathcal{U}_{s,a} \left(R(s, a) + \gamma \max_{a' \in \mathbf{A}(s')} Q^*(s', a') \right), \forall (s, a) \in \mathbf{K}. \quad (11)$$

To carry out value iteration algorithms, the MDP \mathcal{M} must be known *a priori*. In many real-life situations, however, the transition probabilities are unknown as well as the outcome of an action before its execution. Therefore, an agent has to explore the environment while gradually improving its policy. We now derive RL-type algorithms for estimating Q-values of general valuation maps based on the utility-based shortfall, which do not require knowledge of the reward and transition model.

Proposition 3.1 (cf. Proposition 4.104, Föllmer and Schied, 2004). *Let $\rho_{x_0}^u$ be a shortfall defined in Eq. (2), where u is continuous and strictly increasing. Then the following statements are equivalent: (i) $\rho_{x_0}^u(X) = m^*$ and (ii) $\mathbb{E}^\mu[u(X - m^*)] = x_0$.*

For proof see Appendix A.

Consider the valuation map induced by the utility-based shortfall²

$$\mathcal{U}_{s,a}(X) = \sup\{m \in \mathbb{R} \mid \mathbb{E}^{\mu_{s,a}}[u(X - m)] \geq x_0\},$$

where $\mu_{s,a}$ is defined in Eq. (7). If $\mathcal{U}_{s,a}(X) = m^*(s, a)$ exists, Proposition 3.1 assures that $m^*(s, a)$ is the unique solution to equation

$$\mathbb{E}^{\mu_{s,a}}[u(X - m^*(s, a))] = x_0.$$

Let $X = R + \gamma V^*$. Then $m^*(s, a)$ corresponds to the optimal Q-value $Q^*(s, a)$ defined in Eq. (11), which is equivalent to

$$\begin{aligned} \sum_{s' \in \mathbf{S}, \varepsilon \in \mathbf{E}} \mathcal{P}(s'|s, a) \mathcal{P}_r(\varepsilon|s, a) u \left(r(s, a, \varepsilon) + \gamma \max_{a' \in \mathbf{A}(s')} Q^*(s', a') - Q^*(s, a) \right) \\ = x_0, \forall (s, a) \in \mathbf{K}. \end{aligned} \quad (12)$$

Let $\{s_t, a_t, s_{t+1}, r_t\}$ be the sequence of states, chosen actions, successive states and received rewards. Analogous to the standard Q-learning algorithm, we consider the following iterative procedure

$$Q_{t+1}(s_t, a_t) = Q_t(s_t, a_t) + \alpha_t(s_t, a_t) \left[u \left(r_t + \gamma \max_a Q_t(s_{t+1}, a) - Q_t(s_t, a_t) \right) - x_0 \right], \quad (13)$$

where $\alpha_t \geq 0$ denotes learning rate function that satisfies $\alpha_t(s, a) > 0$ only if (s, a) is updated at time t , i.e., $(s, a) = (s_t, a_t)$. In other words, for all (s, a) that are not visited at time t , $\alpha_t(s, a) = 0$ and their Q-values are not updated. Consider utility functions u with the following properties.

Assumption 3.1. (i) *The utility function u is strictly increasing and there exists some $y_0 \in \mathbb{R}$ such that $u(y_0) = x_0$.* (ii) *There exist positive constants ϵ, L such that $0 < \epsilon \leq \frac{u(x) - u(y)}{x - y} \leq L$, for all $x \neq y \in \mathbb{R}$.*

Then the following theorem holds (for proof see Appendix A.1).

Theorem 3.2. *Suppose Assumption 3.1 holds. Consider the generalized Q-learning algorithm stated in Eq. (13). If the nonnegative learning rates $\alpha_t(s, a)$ satisfy*

$$\sum_{t=0}^{\infty} \alpha_t(s, a) = \infty \quad \text{and} \quad \sum_{t=0}^{\infty} \alpha_t^2(s, a) < \infty, \quad \forall (s, a) \in \mathbf{K}, \quad (14)$$

then $Q_t(s, a)$ converges to $Q^(s, a)$ for all $(s, a) \in \mathbf{K}$ with probability 1.*

²In principle, we can apply different utility functions u and acceptance levels x_0 at different (s, a) -pairs. However, for simplicity, we drop their dependence on (s, a) .

The assumption in Eq. (14) requires in fact that all possible state-action pairs must be visited infinitely often. Otherwise, the first sum in Eq. (14) would be bounded by the setting of the learning rate function $\alpha_t(s, a)$. It means that, similar to the standard Q-learning, the agent has to explore the whole state-action space for gathering sufficient information about the environment. Hence, it can not take a too greedy policy in the learning procedure before the state-action space is well explored. We call a policy **proper** if under such policy every state is visited infinitely often. A typical policy, which is widely applied in RL literature as well as in models of human reward-based learning, is given by

$$a_t \sim p(a_t | s_t) := \frac{e^{\beta Q(s_t, a_t)}}{\sum_a e^{\beta Q(s_t, a)}}, \quad (15)$$

where $\beta \in [0, \infty)$ controls how greedy the policy should be. In Appendix A.4, we prove that under some technical assumptions upon the transition kernel of the underlying MDP, this policy is always proper. A widely used setting satisfying both conditions in Eq. (14) is to let $\alpha_t(s, a) := \frac{1}{N_t(s, a)}$, where $N_t(s, a)$ counts the number of times of visiting the state-action pair (s, a) up to time t and is updated trial-by-trial. This leads to the learning procedure shown in Algorithm 1 (see also Fig. 2).

Algorithm 1 Risk-sensitive Q-learning

initialize $Q(s, a) = 0$ and $N(s, a) = 0$ for all s, a .
for $t = 1$ to T **do**
 at state s_t choose action a_t randomly using a proper policy (e.g. Eq. (15));
 observe date (s_t, a_t, r_t, s_{t+1}) ;
 $N(s_t, a_t) \leftarrow N(s_t, a_t) + 1$ and set learning rate: $\alpha_t := 1/N(s_t, a_t)$;
 update Q as in Eq. (13);
end for

The expression

$$TD_t := r_t + \gamma \max_a Q_t(s_{t+1}, a) - Q_t(s, a)$$

inside the utility function of Eq. (13) corresponds to the standard temporal difference (TD) error. Comparing Eq. (13) with the standard Q-learning algorithm, we find that the nonlinear utility function is applied to the TD error (cf. Fig. 2). This induces nonlinear transformation not only of the true rewards but also of the true transition probabilities, as has been shown in Section 2.1. By applying S-shape utility function, which is partially convex and partially concave, we can therefore replicate key effects of prospect theory without the explicit introduction of a probability-weighting function.

Assumption 3.1 (ii) seems to exclude several important types of utility functions. The exponential function $u(x) = e^x$ and the polynomial function $u(x) = x^p$, $p > 0$, for example, do not satisfy the global Lipschitz condition required in Assumption 3.1 (ii). This problem can be solved by a truncation when x is very large and by an approximation when x is very close to 0. For more details see Appendices A.2 and A.3.

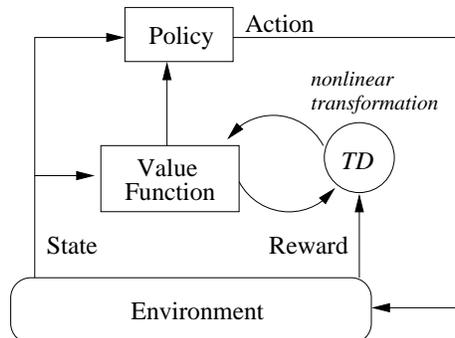


Figure 2: Illustration of risk-sensitive Q-learning (cf. Algorithm 1). The value function $Q(s, a)$ quantifies the current subjective evaluation of each state-action pair (s, a) . The next action is then randomly chosen according to a proper policy (e.g. Eq. (15)) which is based on the current values of Q . After interacting with the environment, the agent obtains the reward r and moves to the successor s' . The value function $Q(s, a)$ is then updated by the rule given in Eq. (13). This procedure continues until some stopping criterion is satisfied.

4 Modeling Human Risk-sensitive Decision Making

4.1 Experiment

Subjects were told that they are influential stock brokers, whose task is to invest into a fictive stock market (cf. Tobia et al., 2013). At every trial (cf. Fig. 3a) subjects had to decide how much ($a = 0, 1, 2$, or 3 EUR) to invest into a particular stock. After the investment, subjects first saw the change of the stock price and then were informed how much money they earned or lost. The received reward was proportional to the investment. The different trials, however, were not independent from each other (cf. Fig. 3b). The sequential investment game consisted of 7 states, each one coming with a different set of contingencies, and subjects were transferred from one state to the next dependent of the amount of money they invested. For high investments, transitions followed the path labeled “risk seeking” (RS in Fig. 3b). For low investments, transitions followed the path labeled “risk averse” (RA in Fig. 3b). After 3 decisions subjects were always transferred back to the initial state, and the reward, which was accumulated during this round, was shown. State information was available to the subjects throughout every trial (cf. Fig. 3a). Altogether, 30 subjects (young healthy adults) experienced 80 rounds of the 3-decision sequence.

Formally, the sequential investment game can be considered as an MDP with 7 states and 4 actions (see Fig. 3b). Depending on the strategy of the subjects, there are 4 possible paths, each of which is composed of 3 states. The total expected return for each path, averaged over all policies consistent with it, are shown in the right panels of Fig. 3b (“EV”). Path 1 provides the largest expected return per round ($EV = 90$), while Path 4 leads to an average loss of -9.75. Hence, to follow the on-average highest rewarded path 1, subjects have to take “risky” actions (investing 2 or 3 EUR at each

state). Always taking conservative actions (investing 0 or 1 EUR) results in Path 4 and a high on-average loss. On the other hand, since the standard deviation of the return R of each state equals $\text{std}(R) = a \times C$, where a denotes the action (investment) the subject takes and C denotes the price change, the higher the investment, the higher the risk. Path 1 has, therefore, the highest standard deviation ($\text{std} = 14.9$) of the total average reward, whereas the standard deviation of Path 4 is smallest ($\text{std} = 6.9$). Path 3 provides a trade-off option: it has slightly lower expected value ($\text{EV} = 52.25$) than Path 1 but comes with a lower risk ($\text{std} = 12.3$). Hence, the paradigm is suitable for observing and quantifying the risk-sensitive behavior of subjects.

4.2 Risk-sensitive Model of Human Behavior

Fig. 4 summarizes the strategies which were chosen by the 30 subjects. 17 subjects mainly chose Path 1, which provided them high rewards. 6 subjects chose Path 4, which gave very low rewards. The remaining 7 subjects show no significant preference among all 4 paths and the rewards they received are on average between the rewards received by the other 2 groups. The optimal policy for maximizing expected reward is the policy that follows Path 1. The results shown in Fig. 4, however, indicate that the standard model fails to explain the behavior of more than 40% of the subjects.

We now quantify subjects' behavior by applying three classes of Q-learning algorithm: (1) standard Q-learning, (2) the risk-sensitive Q-learning (RSQL) method described by Algorithm 1, and (3) an expected utility (EU) algorithm with the following update rule

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha \left(u(r_t) - x_0 + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t) \right), \quad (16)$$

where the nonlinear transformation is applied to the reward r_t directly. The latter one is a straightforward extension of expected utility theory. Risk-sensitivity is implemented via the nonlinear transformation of the true reward r_t . For both risk-sensitive Q-learning methods (RSQL and EU), we set the reference level $x_0 = 0$ and consider the family of polynomial mixed utility functions

$$u(x) = \begin{cases} k_+ x^{l_+} & x \geq 0 \\ -k_- (-x)^{l_-} & x < 0 \end{cases}. \quad (17)$$

The parameters $k_{\pm} > 0$ and $l_{\pm} > 0$ quantify the risk-preferences separately for wins and losses (see Table 1). Hence, there are 4 parameters for u which have to be determined from the data. For all three classes, actions are generated according to the "softmax" policy Eq. (15), which is a proper policy for the paradigm (for proof see Appendix A.4), and the learning rate α is set constant across trials.

For RSQL, the learning rate is absorbed by the coefficients k_{\pm} . Hence, there are 6 parameters $\{\beta, \gamma, k_{\pm}, l_{\pm}\} =: \theta$ which have to be determined. Standard Q-learning corresponds to the choice $l_{\pm} = 1$ and $k_{\pm} = \alpha$. The risk-sensitive model applied by Niv et al. (2012) is also a special case of the RSQL-framework and corresponds $l_{\pm} = 1$. For the EU algorithm, there are 7 parameters, $\{\alpha, \beta, \gamma, k_{\pm}, l_{\pm}\} =: \theta$, which have to be fitted to the data. $l_{\pm} = 1$ and $k_{\pm} = 1$ again corresponds to the standard Q-learning method.

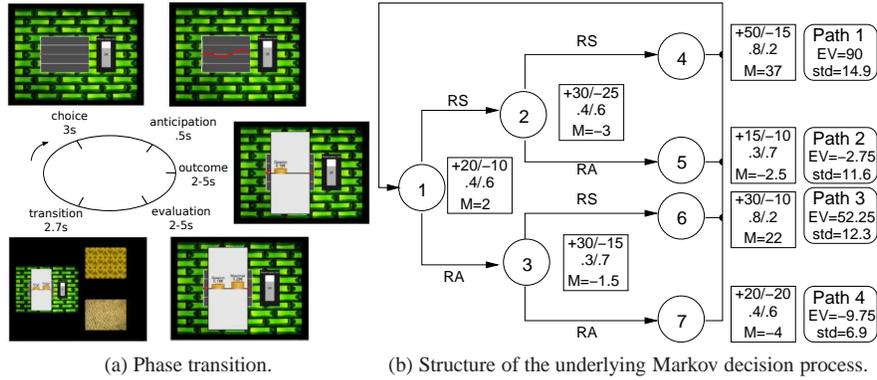


Figure 3: The sequential investment paradigm. The paradigm is an implementation of a Markov decision process with 7 states and 4 possible actions (decisions to take) at every state. (a) Every decision (trial) consists of a choice phase (3s), during which an action (invest 0, 1, 2, or 3 EUR) must be taken by adjusting the scale bar on the screen, an anticipation phase (.5s), an outcome phase (2-5s), where the development of the stock price and the reward (wins and losses) are revealed, an evaluation phase (2-5s), where it reveals the maximal possible reward that could have been obtained for the (in hindsight) best possible action, and a transition phase (2.7s), where subjects are informed about the possible successor states and the specific transition, which will occur. The intervals of the outcome and evaluation phase are jittered for improved fMRI analysis. State information is provided by the colored patterns, the black field provides stock price information during anticipation phase, and the white field provides the reward and the maximal possible reward of this trial. After each round (3 trials), the total reward of this round is shown to subjects. (b) Structure of the underlying Markov decision process. The 7 states are indicated by numbered circles; arrows denote the possible transitions. Labels “RS” and “RA” indicate the transitions caused by the two “risk-seeking” (investment of 2 or 3 EUR) and the two “risk-averse” (investment of 0 or 1 EUR) actions. Bi-Gaussian distributions with a standard deviation of 5 are used to generate the random price changes of the stocks. Panels next to the states provide information about the means (top row) and the probabilities (center row) of ever component. M (bottom row) denotes the mean price change. The reward received equals the price change multiplied by the amount of money the subject invests. The rightmost panels provide the total expected rewards (EV) and the standard deviations (std) for all possible state sequences (Path 1 to Path 4) under the assumption that every sequence of actions consistent with a particular sequence of states is chosen with equal probability.

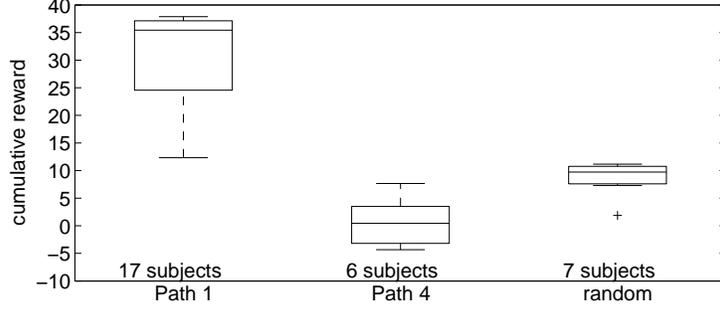


Figure 4: Distribution of “strategies” chosen by the subjects in the sequential investment game and the corresponding cumulative rewards. Subjects are grouped according to the sequence of states (Path 1 to Path 4, cf. Fig. 3b) they chose during the last 60 trials of the game. If a path i is chosen in more than 60% of the trials, the subject is assigned the group “Path i ”. Otherwise, subjects are assigned the group labeled “random”. The vertical axis denotes the cumulative reward obtained during the last 60 trials.

branch $x \geq 0$	shape	risk preference	branch $x < 0$	shape	risk preference
$0 < l_+ < 1$	concave	risk-averse	$0 < l_- < 1$	convex	risk-seeking
$l_+ = 1$	linear	risk-neutral	$l_- = 1$	linear	risk-neutral
$l_+ > 1$	convex	risk-seeking	$l_- > 1$	concave	risk-averse

Table 1: Parameters for the two branches $x \geq 0$ (left) and $x < 0$ (right) of the polynomial utility function $u(x)$ (Eq. (17)), its shape and the induced risk preference.

Parameters were determined subject-wise by maximizing the log-likelihood of the subjects’ action sequences,

$$\max_{\theta} L(\theta) := \sum_{t=1}^T \log p(a_t | s_t, \theta) = \sum_{t=1}^T \log \frac{e^{\beta Q(s_t, a_t | \theta)}}{\sum_a e^{\beta Q(s_t, a | \theta)}} \quad (18)$$

where $Q(s, a | \theta)$ indicates the dependence of the Q-values on the model parameters θ . Since RSQL/EU and the standard Q-learning are nested model classes, we apply the Bayesian information criterion (BIC, see e.g. Ghosh et al., 2006)

$$B := -2L + k \log(n)$$

for model selection. L denotes the log-likelihood, Eq. (18). k and n are the number of parameters and trials respectively.

To compare results, we report relative BIC scores, $\Delta B := B - B_Q$, where B is the BIC score of the candidate model and B_Q is the BIC score of the standard Q-learning model. We obtain

$$\begin{aligned} \Delta B &= -500.14 && \text{for RSQL, and} \\ \Delta B &= -23.10 && \text{for EU.} \end{aligned}$$

The more negative the relative BIC score is, the better the model fits data. Hence, the RSQl algorithm provides a significantly better explanation for the behavioral data than the EU algorithm and standard Q-learning. In the following, we only discuss the results obtained with the RSQl model.

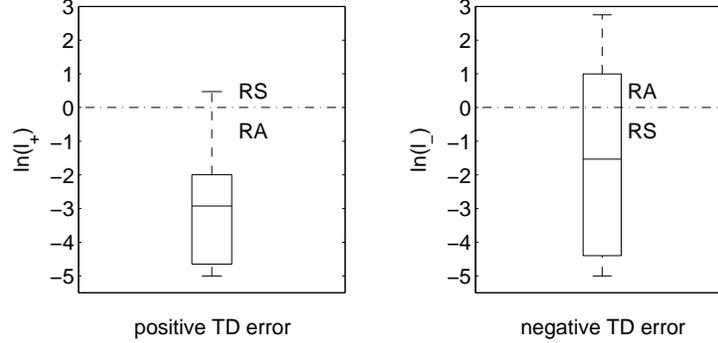


Figure 5: Distribution of values for the shape parameters l_+ (left) and l_- (right) for the RSQl model.

Fig. 5 shows the distribution of best-fitting values for the two parameters l_{\pm} which quantify the risk-preferences of the individual subjects. We conclude (cf. Table 1) that most of the subjects are risk-averse for positive and risk-seeking for negative TD errors. The result is consistent with previous studies from the economics literature (see Tversky and Kahneman, 1992, and references therein).

After determining the parameters $\{k_{\pm}, l_{\pm}\}$ for the utility functions, we perform an analysis similar to the analysis discussed in Section 2.2. Given an observed reward sequence $\{r_i\}_{i=1}^N$, the empirical subjective mean m_{sub} is obtained by solving the following equation

$$\frac{1}{N} \sum_{i=1}^N u(r_i - m_{sub}) = 0.$$

If subjects are risk-neutral, then $u(x) = x$, and $m_{sub} = m_{emp} = \frac{1}{N} \sum_{i=1}^N r_i$ is simply the empirical mean. Following the idea of prospect theory, we define a normalized subjective probability Δp ,

$$\Delta p := \frac{m_{sub} - \min_i r_i}{\max_i r_i - \min_i r_i} - \frac{m_{emp} - \min_i r_i}{\max_i r_i - \min_i r_i} = \frac{m_{sub} - m_{emp}}{\max_i r_i - \min_i r_i}. \quad (19)$$

If Δp is positive, the probability of rewards is overestimated and the induced policy is, therefore, risk-seeking. If Δp is negative, the probability of rewards is underestimated and the policy is risk-averse. Fig. 6 summarizes the distribution of normalized subjective probabilities for subjects from the “Path 1”, “Path 4” and “random” groups of Fig. 4. For subjects within group “Path 1”, $|\Delta p|$ is small and their behaviors are similar to those of risk-neutral agents. This is consistent with their policy, because both risk-seeking and risk-neutral agents should prefer Path 1. For subjects within groups “Path

4” and “random”, the normalized subjective probabilities are on average 10 % lower than those of risk-neutral agents. This explains why subjects in these groups adopt the conservative policies and only infrequently choose Path 1.

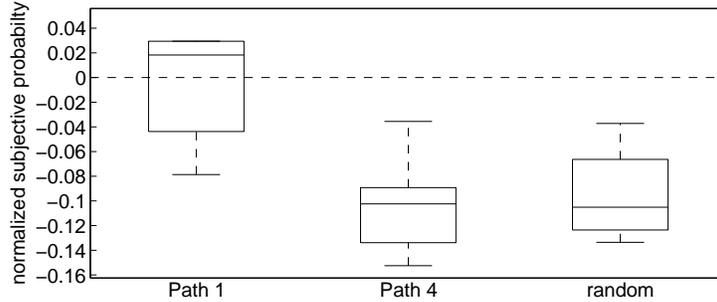


Figure 6: Distribution of normalized subjective probabilities, Δp , Eq. (19), for the different subject groups defined in Fig. 4.

4.3 fMRI Results

Functional magnetic resonance imaging (fMRI) data were simultaneously recorded while subjects played the sequential investment game. The analysis of fMRI data was conducted in SPM8 (Wellcome Department of Cognitive Neurology, London, UK; details of the magnetic resonance protocol and data processing are presented in Appendix B). The sequence of Q-values for the action chosen at each state were used as parametric modulators during the choice phase, and temporal difference (TD) errors were used at the outcome phase (see Fig. 3a).

Fig. 7a shows that the sequence of TD errors for the RSQL model (with best fitting parameters) positively modulated the BOLD signal in the subcallosal gyrus extending into the ventral striatum (-14 8 -16) (marked by the cross in Fig. 7a), the anterior cingulate cortex (8 48 6), and the visual cortex (-8 -92 16; $z = 7.9$). The modulation of the BOLD signal in the ventral striatum is consistent with previous experimental findings (cf. Schultz, 2002; O’Doherty, 2004), and supports the primary assertion of computational models that reward-based learning occurs when expectations (here, expectations of “subjective” quantities) are violated (Sutton and Barto, 1998).

Fig. 7b shows the results for the sequence of Q-values for the RSQL model (with best fitting parameters), which correspond to the subjective (risk-sensitive) expected value of the reward for each discrete choice. Several large clusters of voxels in cortical and subcortical structures were significantly modulated by the Q-values at the moment of choice. The sign of this modulation was negative. The peak of this negative modulation occurred in the left anterior insula (-36 12 -2, $z = 4.6$), with strong modulation also in the bilateral ventral striatum (14 8 -4, marked by the cross in Fig. 7b; -16 4 0) and the cingulate cortex (4 16 28). The reward prediction error processed by the ventral striatum (and other regions noted above) would not be computable in the absence of an expectation, and as such, this activation is important for substantiating the plausibil-

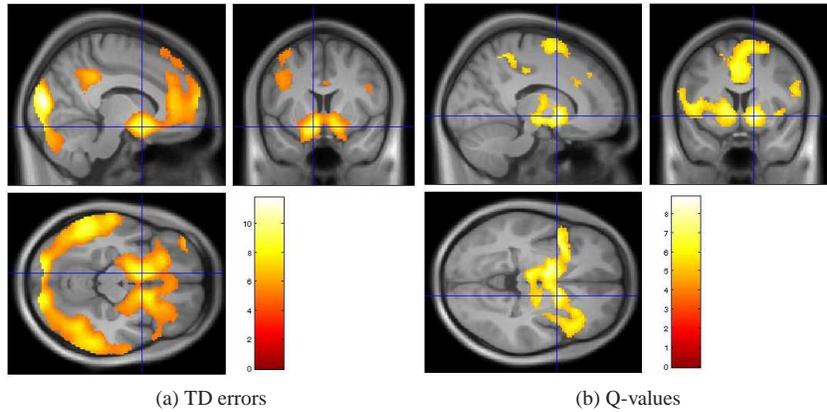


Figure 7: Modulation of the fMRI BOLD signal by TD errors (a) and by Q-values (b) generated by the RSQL model with best fitting parameters. The data is shown whole-brain corrected to $p < .05$ (voxel-wise $p < .001$ and minimum 125 voxels). The color bar indicates the t -value ranging from 0 to the maximal value. The cross indicates location of strongest modulation for TD errors (in the left ventral striatum (-14 8 -16)) and for Q-values (in the right ventral striatum (14 8 -4)). However, it is remarkable that for both TD errors and Q-values, modulations in the left and right ventral striatum are almost equally strong with a slight difference.

ity for the RSQL model of learning and choice. Sequences of Q-values obtained with standard Q-learning (with best fitting parameters), on the other hand, failed to predict any changes in brain activity even at a liberal statistical threshold of $p < .01$ (uncorrected). This lack of neural activity for the standard Q model, in combination with the significant activation for our RSQL, supports the hypothesis that some assessment of risk is induced and influences valuation. Whereas the areas modulated by Q-values differ from what has been reported in other studies (i.e., the ventromedial prefrontal cortex as in Gläscher et al., 2009), it does overlap with the representation of TD errors. Furthermore, the opposing signs of the correlated neural activity suggests that a neural mismatch of signals in the ventral striatum between Q-value and TD errors may underlie the mechanism by which values are learned.

4.4 Discussion

We applied the risk-sensitive Q-learning (RSQL) method to quantify human behavior in a sequential investment game and investigated the correlation of the predicted TD- and Q-values with the neural signals measured by fMRI.

We first showed that the standard Q-learning algorithm cannot explain the behavior of a large number of subjects in the task. Applying RSQL generated a significantly better fit and also outperformed the expected utility algorithm. The risk-sensitivity revealed by the best fitting parameters is consistent with the studies in behavioral economics, that is, subjects are risk-averse for positive while risk-seeking for negative TD

errors. Finally, the relative subjective probabilities provide a good explanation why some subjects take conservative policies: they underestimate the true probabilities of gaining rewards.

The fMRI results showed that TD sequence generated by our model has a significant correlation with the activity in the subcallosal gyrus extending into the ventral striatum. The sequence of Q-values has a significant correlation with the activities in the left anterior insula. Previous studies (see e.g., Chapter 23 of Glimcher et al., 2008 and Symmonds et al., 2011) suggest that higher order statistics of outcomes, e.g., variance (the 2nd order) and skewness (the 3rd order), are encoded in human brains separately and the individual integration of these risk metrics induces the corresponding risk-sensitivity. Our results indicate, however, that the risk-sensitivity can be simply induced (and therefore encoded) by a nonlinear transformation of TD errors and no additional neural representation of higher order statistics is needed.

5 Summary

We applied a family of valuation functions, the utility-based shortfall, to the general framework of risk-sensitive Markov decision processes, and we derived a risk-sensitive Q-learning algorithm. We proved that the proposed algorithm converges to the optimal policy corresponding to the risk-sensitive objective. By applying S-shape utility functions, we show that key features predicted by prospect theory can be replicated using the proposed algorithm. Hence, the novel Q-learning algorithm provides a good candidate model for human risk-sensitive sequential decision-making procedures in learning tasks, where mixed risk-preferences are shown in behavioral studies. We applied the algorithm to model human behaviors in a sequential investment game. The results showed that the new algorithm fitted data significantly better than the standard Q-learning and the expected utility model. The analysis of fMRI data shows a significant correlation of the risk-sensitive TD error with the BOLD signal change in the ventral striatum, and also a significant correlation of the risk-sensitive Q-values with neural activity in the striatum, cingulate cortex and insula, which is not present if standard Q-values are applied.

Some technical extensions are possible within our general risk-sensitive reinforcement learning (RL) framework: (a) The Q-learning algorithm derived in this paper can be regarded a special type of RL algorithms, TD(0). It can be extended to other types of RL algorithms like SARSA and TD(λ) for $\lambda \neq 0$. (b) In our previous work (Shen et al., 2013), we also provided a framework for the average case. Hence, RL algorithms for the average case can also be derived similar to the discounted case considered in this paper. (c) The algorithm in its current form applies to MDPs with finite state spaces only. It can be extended for MDPs with continuous state spaces by applying function approximation technique.

Acknowledgments

Thanks to Wendelin Böhmer, Rong Guo and Maziar Hashemi-Nezhad for useful discussions and suggestions and to the anonymous referee for helpful comments. The

work of Y. Shen and K. Obermayer was supported by the BMBF (Bersteinfokus Lernen TP1), 01GQ0911, and the work of M.J. Tobia and T. Sommer was supported by the BMBF (Bersteinfokus Lernen TP2), 01GQ0912.

A Mathematical Proofs

The sup-norm is defined as $\|X\|_\infty := \max_{i \in I} |X(i)|$, where $X = [X(i)]_{i \in I}$ can be considered as a $|I|$ -dimensional vector.

Lemma A.1. *Let ρ be valuation function on $\mathbb{R}^{|I|} \times \mathcal{P}$ and $\tilde{\rho}(X, \mu) := \rho(X, \mu) - \rho(\mathbf{0}, \mu)$. Then the following inequality holds*

$$\min_{i \in I} X_i =: \underline{X} \leq \tilde{\rho}(X, \mu) \leq \overline{X} := \max_{i \in I} X_i, \forall \mu \in \mathcal{P}, X \in \mathbb{R}^{|I|}.$$

Proof. By $\underline{X} \leq X_i \leq \overline{X}, \forall i \in I$ and monotonicity of valuation functions, we obtain

$$\rho(\underline{X}\mathbf{1}, \mu) \leq \rho(X, \mu) \leq \rho(\overline{X}\mathbf{1}, \mu).$$

Due to the translation invariance, we have then

$$\rho(\overline{X}\mathbf{1}, \mu) = \rho(\mathbf{0}, \mu) + \overline{X}, \text{ and } \rho(\underline{X}\mathbf{1}, \mu) = \rho(\mathbf{0}, \mu) + \underline{X}.$$

which immediately imply that

$$\overline{X} \leq \rho(X, \mu) - \rho(\mathbf{0}, \mu) \leq \underline{X}, \forall \mu \in \mathcal{P}, X \in \mathbb{R}^{|I|}.$$

□

Proof of Proposition 3.1. (ii) \Rightarrow (i). By definition, $m^* \leq \rho_{x_0}^u(X)$. For any $\epsilon > 0$, since u is strictly increasing, we have $u(X(i) - m^* - \epsilon) < u(X(i) - m^*), \forall i \in I$, which implies $\mathbb{E}u(X - m^* - \epsilon) < \mathbb{E}u(X - m^*) = x_0$. Hence, $m^* = \rho_{x_0}^u(X)$.

(i) \Rightarrow (ii). By definition we have $\mathbb{E}u(X - m^*) \geq x_0$. Assume that $\mathbb{E}u(X - m^*) > x_0$. By the continuity of u , there exists an $\epsilon > 0$ such that $\mathbb{E}u(X - m^* - \epsilon) > x_0$, which implies $\rho_{x_0}^u(X) \geq m^* + \epsilon > m^*$ and hence contradicts (i). Thus, (ii) holds. □

A.1 Proofs for Risk-sensitive Q-learning

Before proving the risk-sensitive Q-learning, we consider a more general update rule

$$q_{t+1}(i) = (1 - \alpha_t(i))q_t(i) + \alpha_t(i) [(Hq_t)(i) + w_t(i)]. \quad (20)$$

where $q_t \in \mathbb{R}^d$, $H : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an operator, w_t denotes some random noise term and α_t is learning rate with the understanding that $\alpha_t(i) = 0$ if $q(i)$ is not updated at time t . Denote by \mathcal{F}_t the history of the algorithm up to time t ,

$$\mathcal{F}_t = \{q_0(i), \dots, q_t(i), w_0(i), \dots, w_{t-1}(i), \alpha_0(i), \dots, \alpha_t(i), i = 1, \dots, t\}.$$

We restate the following proposition.

Proposition A.1 (Proposition 4.4, Bertsekas and Tsitsiklis, 1996). *Let q_t be the sequence generated by the iteration (20). We assume the following*

a *The learning rates $\alpha_t(i)$ are nonnegative and satisfy*

$$\sum_{t=0}^{\infty} \alpha_t(i) = \infty, \quad \sum_{t=0}^{\infty} \alpha_t^2(i) = \infty, \quad \forall i$$

b *The noise terms $w_t(i)$ satisfy (i) for every i and t , $\mathbb{E}[w_t(i)|\mathcal{F}_t] = 0$; (ii) Given some norm $\|\cdot\|$ on \mathbb{R}^d , there exist constants A and B such that $\mathbb{E}[w_t^2(i)|\mathcal{F}_t] \leq A + B\|q_t\|^2$.*

c *The mapping H is a contraction under sup-norm.*

Then q_t converges to the unique solution q^ of the equation $Hq^* = q^*$ with probability 1.*

To apply Proposition A.1, we first reformulate the Q-learning rule (13) in a different form

$$q_{t+1}(s, a) = \left(1 - \frac{\alpha_t(s, a)}{\alpha}\right)q_t(s, a) + \frac{\alpha_t(s, a)}{\alpha} [\alpha u(d_t) - x_0 + q_t(s, a)]$$

where α denotes an arbitrary constant such that $\alpha \in (0, \min(L^{-1}, 1)]$. Recall that L is defined in Assumption 3.1. For simplicity, we define $\tilde{u}(x) := u(x) - x_0$, $d_t := r_t + \gamma \max_a q_t(s_{t+1}, a) - q_t(s, a)$ and set

$$(Hq_t)(s, a) = \alpha \mathbb{E}_{s,a} \tilde{u}(r_t + \gamma \max_a q_t(s_{t+1}, a) - q_t(s, a)) + q_t(s, a) \quad (21)$$

$$w_t(s, a) = \alpha \tilde{u}(d_t) - \alpha \mathbb{E}_{s,a} \tilde{u}(r_t + \gamma \max_a q_t(s_{t+1}, a) - q_t(s, a)) \quad (22)$$

More explicitly, Hq is defined as

$$(Hq)(s, a) = \alpha \sum_{s', \varepsilon} \tilde{\mathcal{P}}(s', \varepsilon | s, a) \tilde{u} \left(r(s, a, \varepsilon) + \gamma \max_{a'} q(s', a') - q(s, a) \right) + q(s, a),$$

where $\tilde{\mathcal{P}}(s', \varepsilon | s, a) := \mathcal{P}(s' | s, a) \mathcal{P}_r(\varepsilon | s, a)$. We assume the size of the space \mathbf{K} is d .

Lemma A.2. *Suppose that Assumption 3.1 holds and $0 < \alpha \leq \min(L^{-1}, 1)$. Then there exists a real number $\bar{\alpha} \in [0, 1)$ such that for all $q, q' \in \mathbb{R}^d$, $\|Hq - Hq'\|_{\infty} \leq \bar{\alpha} \|q - q'\|_{\infty}$.*

Proof. Define $v(s) := \max_a q(s, a)$ and $v'(s) := \max_a q'(s, a)$. Thus,

$$|v(s) - v'(s)| \leq \max_{(s,a) \in \mathbf{K}} |q(s, a) - q'(s, a)| = \|q - q'\|_{\infty}.$$

By Assumption 3.1 (ii) and the monotonicity of \tilde{u} , there exists a $\xi_{(x,y)} \in [\epsilon, L]$ such that $\tilde{u}(x) - \tilde{u}(y) = \xi_{(x,y)}(x - y)$. Analogously, we obtain

$$\begin{aligned}
& (Hq)(s, a) - (Hq')(s, a) \\
&= \sum_{s', \epsilon} \tilde{\mathcal{P}}(s', \epsilon | s, a) \{ \alpha \xi_{(s, a, \epsilon, s', q, q')} [\gamma v(s') - \gamma v'(s') - q(s, a) + q'(s, a)] \\
&\quad + (q(s, a) - q'(s, a)) \} \\
&= \alpha \gamma \sum_{s', \epsilon} \tilde{\mathcal{P}}(s', \epsilon | s, a) \xi_{(s, a, \epsilon, s', q, q')} [v(s') - v'(s')] \\
&\quad + (1 - \alpha \sum_{s', \epsilon} \tilde{\mathcal{P}}(s', \epsilon | s, a) \xi_{(s, a, \epsilon, s', q, q')}) [q(s, a) - q'(s, a)] \\
&\leq \left(1 - \alpha(1 - \gamma) \sum_{s', \epsilon} \tilde{\mathcal{P}}(s', \epsilon | s, a) \xi_{(s, a, \epsilon, s', q, q')} \right) \|q - q'\|_\infty \\
&\leq (1 - \alpha(1 - \gamma)\epsilon) \|q - q'\|_\infty
\end{aligned}$$

Hence, $\bar{\alpha} = 1 - \alpha(1 - \gamma)\epsilon$ is the required constant. \square

Proof of Theorem 3.2. Obviously, Condition (a) in Proposition A.1 is satisfied and Condition (c) holds also due to Lemma A.2. It remains to check Condition (b).

$\mathbb{E}[w_t(s, a) | \mathcal{F}_t] = 0$ holds by its definition in (22). Next we prove (ii). In fact,

$$\mathbb{E}[w_t^2(s, a) | \mathcal{F}_t] = \alpha^2 \mathbb{E}[(\tilde{u}(d_t))^2 | \mathcal{F}_t] - \alpha^2 (\mathbb{E}[\tilde{u}(d_t) | \mathcal{F}_t])^2 \leq \alpha^2 \mathbb{E}[(\tilde{u}(d_t))^2 | \mathcal{F}_t]$$

Let \bar{R} be the upper bound for r_t . Then $|d_t| \leq \bar{R} + 2\|q_t\|_\infty$, which implies that $|\tilde{u}(d_t) - \tilde{u}(0)| \leq L(\bar{R} + 2\|q_t\|_\infty)$ due to Assumption 3.1(ii). Hence, $|\tilde{u}(d_t)| \leq |\tilde{u}(0)| + L(\bar{R} + 2\|q_t\|_\infty)$. On the other hand, since

$$(|\tilde{u}(0)| + L\bar{R} + 2L\|q_t\|_\infty)^2 \leq 2(|\tilde{u}(0)| + L\bar{R})^2 + 8L^2\|q_t\|_\infty^2$$

we have $\alpha^2 \mathbb{E}[(\tilde{u}(d_t))^2 | \mathcal{F}_t] \leq 2\alpha^2(|\tilde{u}(0)| + L\bar{R})^2 + 8\alpha^2 L^2\|q_t\|_\infty^2$. Hence, Condition (b) holds. \square

A.2 Truncated Algorithms with Weaker Assumptions

Some functions like $u(x) = e^x$ and $u(x) = x^p$, $p > 0$, do not satisfy the global Lipschitz condition required in Assumption 3.1 (ii). In real applications, however, we can relax the assumption to assume that the Lipschitz condition holds locally within a “sufficiently large” subset. Lemma A.4 states such subset provided the upper bound of absolute value of rewards is known.

Assumption A.1. *The reward function $r(s, a, \epsilon)$ is bounded under sup-norm, i.e.,*

$$\bar{R} := \sup_{(s,a) \in \mathbf{K}, \epsilon \in \mathbf{E}} |r(s, a, \epsilon)| < \infty.$$

Define an operator $\mathcal{T} : \mathbb{R}^{|\mathbf{S}|} \rightarrow \mathbb{R}^{|\mathbf{S}|}$ as

$$\mathcal{T}_s(V) = \max_{a \in \mathbf{A}(s)} \mathcal{U}_{s,a}(R(s,a) + \gamma V).$$

Lemma A.3 (cf. Lemma 5.4, Shen et al., 2013). \mathcal{T} is a contracting map under sup-norm, i.e.,

$$\|\mathcal{T}(V) - \mathcal{T}(V')\|_\infty \leq \gamma \|V - V'\|_\infty, \forall V, V' \in \mathbb{R}^{|\mathbf{S}|}.$$

Lemma A.4. Under Assumption 3.1 (i) and A.1, applying the valuation map in (12), the solution Q^* satisfies $\frac{-\bar{R}-y_0}{1-\gamma} \leq Q^*(s,a) \leq \frac{\bar{R}-y_0}{1-\gamma}, \forall (s,a) \in \mathbf{K}$.

Proof. By assumption, $u^{-1}(x_0)$ exists. Since u is strictly increasing, we have $\mathcal{U}_{s,a}(0) = \sup\{m \in \mathbb{R} | u(-m) \geq x_0\} = -u^{-1}(x_0)$. Hence, together with Eq. (1), we obtain for all $(s,a) \in \mathbf{K}$,

$$-u^{-1}(x_0) - \bar{R} = \mathcal{U}_{s,a}(0) - \bar{R} \leq \mathcal{U}_{s,a}(R) \leq \mathcal{U}_{s,a}(0) + \bar{R} = -u^{-1}(x_0) + \bar{R}$$

Note that Lemma A.3 implies that $V^* = \mathcal{T}^\infty(V_0)$ for any $V_0 \in \mathbb{R}^{|\mathbf{S}|}$. Without loss of generality, we start from $V_0 = 0$. Define $\underline{u} := -u^{-1}(x_0) - \bar{R}$ and $\bar{u} := -u^{-1}(x_0) + \bar{R}$. Hence, we have $\underline{u} \leq \mathcal{T}(0) = \max_a \mathcal{U}_{s,a}(R) \leq \bar{u}$, which implies

$$\begin{aligned} \mathcal{T}^2(0) &= \max_a \mathcal{U}_{s,a}(R + \gamma \mathcal{T}(0)) \leq \max_a \mathcal{U}_{s,a}(R) + \gamma \bar{u} \leq (1 + \gamma) \bar{u} \\ \text{and } \mathcal{T}^2(0) &= \max_a \mathcal{U}_{s,a}(R + \gamma \mathcal{T}(0)) \geq \max_a \mathcal{U}_{s,a}(R) + \gamma \underline{u} \geq (1 + \gamma) \underline{u} \end{aligned}$$

Repeating above procedure, we obtain $(1 + \gamma + \dots + \gamma^{n-1}) \underline{u} \leq \mathcal{T}^n(0) \leq (1 + \gamma + \dots + \gamma^{n-1}) \bar{u}$. Hence, $\frac{\underline{u}}{1-\gamma} \leq V^* = \mathcal{T}^\infty(0) \leq \frac{\bar{u}}{1-\gamma}$. By the definition of Q^* , above inequalities hold for Q^* as well. \square

Define

$$\underline{x} := y_0 - \frac{2\bar{R}}{1-\gamma} \quad \text{and} \quad \bar{x} := y_0 + \frac{2\bar{R}}{1-\gamma} \quad (23)$$

Given Lemma A.4, we can truncate the utility function u outside the interval $[\underline{x}, \bar{x}]$ as

$$u'(x) = \begin{cases} u(\underline{x}) + \epsilon(x - \underline{x}), & x \in (-\infty, \underline{x}) \\ u(x), & x \in [\underline{x}, \bar{x}] \\ u(\bar{x}) + \epsilon(x - \bar{x}), & x \in (\bar{x}, \infty) \end{cases}. \quad (24)$$

Theorem A.1. Suppose that Assumption 3.1 (i) and A.1 hold. Assume further that There exist positive constants $\epsilon, L \in \mathbb{R}^+$ such that $0 < \epsilon \leq \frac{u(x)-u(y)}{x-y} \leq L$, for all $x \neq y \in [\underline{x}, \bar{x}]$, where \underline{x}, \bar{x} are defined in Eq. (23). Then the unique solution Q_1^* to Eq. (12) with u and the unique solution Q_2^* to Eq. (12) with u' are identical.

Proof. Both uniqueness is due to Theorem 3.1 and Proposition 3.1. By Lemma A.4, $\frac{-\bar{R}-y_0}{1-\gamma} \leq Q_i^*(s, a) \leq \frac{\bar{R}-y_0}{1-\gamma}$ hold for all $(s, a) \in \mathbf{K}$ and $i = 1, 2$. Hence, we have for both $i = 1, 2$ and for all $(s, a), (s', a') \in \mathbf{K}, \epsilon \in \mathbf{E}$,

$$y_0 - \frac{2\bar{R}}{1-\gamma} \leq r(s, a, \epsilon) + \gamma Q_i^*(s', a') - Q_i^*(s, a) \leq y_0 + \frac{2\bar{R}}{1-\gamma}.$$

Since u and u' are identical within the set $[\underline{x}, \bar{x}]$, $Q_1^*(s, a) = Q_2^*(s, a)$ for all $(s, a) \in \mathbf{K}$. \square

Now we state the risk-sensitive Q-learning algorithm with truncation.

Algorithm 2 Q-learning with truncation

initialize $Q(s, a) = 0$ and $N(s, a) = 0$ for all s, a .

for $t = 1$ to T **do**

 at state s_t choose action a_t randomly using a proper policy (e.g. Eq. (15));

 observe data (s_t, a_t, r_t, s_{t+1}) ;

$N(s_t, a_t) \leftarrow N(s_t, a_t) + 1$ and set learning rate: $\alpha_t := 1/N(s_t, a_t)$;

 update Q as in Eq. (13);

 truncate Q as in Eq. (24), where \bar{x} and \underline{x} are defined in Eq. (23).

end for

A.3 Heuristics for Polynomial Utility Functions

So far we have relaxed the assumption for utility functions to locally Lipschitz. However, some functions of interest are even not locally Lipschitz. For instance, the function $u(x) = x^p$, $p \in (0, 1)$ is not Lipschitz at the area close to 0. We suggest two types of approximation to avoid this problem.

1. Approximate u by $u^\varphi(x) = (x + \varphi)^p - \varphi^p$ with some positive φ .
2. Approximate u close to 0 by a linear function, i.e.

$$u^\varphi(x) = \begin{cases} u(x) & x \geq \varphi \\ \frac{xu(\varphi)}{\varphi} & x \in [0, \varphi) \end{cases}.$$

In both cases, φ should be set very close to 0.

The assumption in Theorem (A.1) and Assumption 3.1 (ii) requires also the strictly positive lower bound ϵ . This causes problem when applying $u(x) = x^p$, $p > 1$ at the area close to 0. We can again apply above two approximation schemes to overcome the problem by selecting small φ . In Section 4, for both $p > 1$ and $p \in (0, 1)$, we apply the second scheme to ensure Assumption 3.1.

A.4 Softmax Policy

Recall that we call a policy is proper, if under such policy every state is visited infinitely often. In this subsection, we show that under some technical assumptions the softmax policy (cf. Eq. (15)) is proper. A policy $\pi = [\pi_0, \pi_1, \dots]$ is deterministic if for all state s and t , there exists an action $a \in \mathbf{A}(s)$ such that $\pi_t(a|s) = 1$. Under one policy π , the n -step transition probability $P^\pi(S_n = s'|S_0 = s)$ for some $s, s' \in \mathbb{S}$ can be calculated as follows

$$P^\pi(S_n = s'|S_0 = s) = \sum_{S_1, S_2, \dots, S_{n-1}} P^{\pi_0}(S_1|s)P^{\pi_1}(S_2|S_1) \dots P^{\pi_{n-1}}(s'|S_{n-1})$$

where $P^\pi(y|x) := \sum_a \mathcal{P}(y|x, a)\pi(a|x)$ and \mathcal{P} is the transition kernel of the underlying MDP.

Proposition A.2. *Assume that the state and action space are finite and the assumptions required by Theorem 3.2 hold. Assume further that for each $s, s' \in \mathbf{S}$, there exist a deterministic policy π_d , $n \in \mathbb{N}$ and a positive $\epsilon > 0$ such that $P^{\pi_d}(S_n = s'|S_0 = s) > \epsilon$. Then the softmax policy stated in Eq. (15) is proper.*

Proof. Due to the contraction property of Q (see Lemma A.2), $\{Q_t\}$ is uniformly bounded w.r.t. t . Let $\pi_s = [\pi_0, \pi_1, \dots]$ be a softmax policy associated with $\{Q_t\}$. Then, by the definition of softmax policies (see Eq. (15)), there exists a positive $\epsilon_0 > 0$ such that $\pi_t(a|s) \geq \epsilon_0$ holds for each $(s, a) \in \mathbf{K}$ and $t \in \mathbb{N}$. It implies that for each $s, s' \in \mathbf{S}$,

$$P^{\pi_s}(S_n = s'|S_0 = s) \geq \epsilon_0^n P^{\pi_d}(S_n = s'|S_0 = s),$$

for any deterministic policy π_d . Then by the assumption of this proposition, we obtain that for each $s, s' \in \mathbf{S}$, $P^{\pi_s}(S_n = s'|S_0 = s) \geq \epsilon_0^n \epsilon > 0$. It implies that each state will be visited infinitely often. \square

The MDP applied in the behavioral experiment in Section 4 satisfies above assumptions, since for each $s, s' \in \mathbf{S}$, there exists a deterministic policy π_d such that $P^{\pi_d}(S_n = s'|S_0 = s) = 1$, $n \leq 4$, no matter which initial state s we start with.

B Magnetic Resonance Protocol and Data Processing

Magnetic resonance (MR) images were acquired with a 3T whole-body MR system (Magnetom TIM Trio, Siemens Healthcare) using a 32-channel receive-only head coil. Structural MRI were acquired with a T1 weighted magnetization-prepared rapid gradient-echo (MPRAGE) sequence with a voxel resolution of $1 \times 1 \times 1 \text{ mm}^3$, coronal orientation, phase-encoding in left-right direction, FoV = $192 \times 256 \text{ mm}$, 240 slices, 1100 ms inversion time, TE = 2.98 ms, TR = 2300 ms, and 90 flip angle. Functional MRI time series were recorded using a T2* GRAPPA EPI sequence with TR = 2380 ms, TE = 25 ms, anterior-posterior phase encode, 40 slices acquired in descending (non- interleaved) axial plane with $2 \times 2 \times 2 \text{ mm}^3$ voxels ($204 \times 204 \text{ mm}$ FoV; skip factor = .5), with an acquisition time of approximately 8 minutes per scanning run.

Structural and functional magnetic resonance image analyzes were conducted in SPM8 (Wellcome Department of Cognitive Neurology, London, UK). Anatomical images were segmented and transformed to Montreal Neurological Institute (MNI) standard space, and a group average T1 custom anatomical template image was generated using DARTEL. Functional images were corrected for slice-timing acquisition offsets, realigned and corrected for the interaction of motion and distortion using unwarp toolbox, co-registered to anatomical images and transformed to MNI space using DARTEL, and finally smoothed with an 8 mm FWHM isotropic Gaussian kernel.

Functional images were analyzed using the general linear model (GLM) implemented in SPM8. First level analyzes included onset regressors for each stimulus event excluding the anticipation phase (see Fig. 3a), and a set of parametric modulators corresponding to trial-specific task outcome variables and computational model parameters. Trial-specific task outcome variables (and their corresponding stimulus event) include the choice value of the investment (choice phase) and the total value of rewards (gains/losses) over each round (corresponding to multi-trial feedback event). Model derived parametric modulators included the time series of Q values for the selected action (choice phase), TD (outcome phase). Reward value was not modeled as a parametric modulator because the TD error time series and trial-by-trial reward values were strongly correlated (all $r_s > .7$; $p_s < .001$). The configuration of the first-level GLM regressors for the standard Q-learning model was identical to that employed in the risk-sensitive Q-learning model. All regressors were convolved with a canonical hemodynamic response function. Prior to model estimation, coincident parametric modulators were serially orthogonalized as implemented in SPM (i.e., the Q-value regressor was orthogonalized with respect to the choice value regressor). In addition, we included a set of regressors for each participant to censor EPI images with large, head movement related spikes in the global mean. These first level beta values were averaged across participants and tested against zero with a t-test. Monte Carlo simulations determined that a cluster of more than 125 contiguous voxels with a single-voxel threshold of $p < .001$ achieved a corrected p -value of .05.

References

- P. Artzner, F. Delbaen, J.M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.
- D.P. Bertsekas and J.N. Tsitsiklis. *Neuro-Dynamic Programming*. Athena Scientific, 1996.
- V.S. Borkar. Q-learning for risk-sensitive control. *Mathematics of Operations Research*, pages 294–311, 2002.
- D.A. Braun, A.J. Nagengast, and D.M. Wolpert. Risk-sensitivity in sensorimotor control. *Frontiers in human neuroscience*, 5, 2011.
- R. Cavazos-Cadena. Optimality equations and inequalities in a class of risk-sensitive average cost Markov decision chains. *Mathematical Methods of Operations Research*, 71(1):47–84, 2010.

- H. Föllmer and A. Schied. Convex measures of risk and trading constraints. *Finance and Stochastics*, 6(4):429–447, 2002.
- H. Föllmer and A. Schied. *Stochastic Finance*. Walter de Gruyter & Co., Berlin, 2004.
- J.K. Ghosh, M. Delampady, and T. Samanta. *An Introduction to Bayesian Analysis*. Springer, New York, 2006.
- I. Gilboa. *Theory of Decision under Uncertainty*. Cambridge University Press, 2009.
- J. Gläscher, A.N. Hampton, and J.P. O’Doherty. Determining a role for ventromedial prefrontal cortex in encoding action-based value signals during reward-related decision making. *Cerebral Cortex*, 19(2):483–495, 2009.
- P.W. Glimcher, C.F. Camerer, E. Fehr, and R.A. Poldrack. *Neuroeconomics: Decision Making and the Brain*. Academic Press, 2008.
- C. Gollier. *The Economics of Risk and Time*. The MIT Press, 2004.
- M. Heger. Consideration of risk in reinforcement learning. In *Proceedings of the 11th International Conference on Machine Learning*, pages 105–111, 1994.
- D. Hernández-Hernández and S.I. Marcus. Risk sensitive control of Markov processes in countable state space. *Systems & Control Letters*, 29(3):147–155, 1996.
- R.A. Howard and J.E. Matheson. Risk-sensitive markov decision processes. *Management Science*, pages 356–369, 1972.
- D. Kahneman and A. Tversky. Prospect theory: an analysis of decision under risk. *Econometrica: Journal of the Econometric Society*, pages 263–291, 1979.
- O. Mihatsch and R. Neuneier. Risk-sensitive reinforcement learning. *Machine Learning*, 49(2):267–290, 2002.
- A.J. Nagengast, D.A. Braun, and D.M. Wolpert. Risk-sensitive optimal feedback control accounts for sensorimotor behavior under uncertainty. *PLoS Computational Biology*, 6(7):e1000857, 2010.
- Y. Niv, J.A. Edlund, P. Dayan, and J.P. O’Doherty. Neural prediction errors reveal a risk-sensitive reinforcement-learning process in the human brain. *The Journal of Neuroscience*, 32(2):551–562, 2012.
- J.P. O’Doherty. Reward representations and reward-related learning in the human brain: insights from neuroimaging. *Current Opinion in Neurobiology*, 14(6):769–776, 2004.
- K. Preuschoff, S.R. Quartz, and P. Bossaerts. Human insula activation reflects risk prediction errors as well as risk. *The Journal of Neuroscience*, 28(11):2745–2752, 2008.
- M.L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons, Inc., 1994.

- A. Ruszczyński. Risk-averse dynamic programming for Markov decision processes. *Mathematical Programming*, pages 1–27, 2010.
- L.J. Savage. *The Foundations of Statistics*. Dover Publications, 1972.
- W. Schultz. Getting formal with dopamine and reward. *Neuron*, 36(2):241–263, 2002.
- W. Schultz, P. Dayan, and P.R. Montague. A neural substrate of prediction and reward. *Science*, 275(5306):1593–1599, 1997.
- Y. Shen, W. Stannat, and K. Obermayer. Risk-sensitive Markov control processes. *SIAM Journal on Control and Optimization*, (5):3652–3672, 2013.
- R.S. Sutton and A.G. Barto. *Reinforcement Learning*. MIT Press, 1998.
- M. Symmonds, N.D. Wright, D.R. Bach, and R.J. Dolan. Deconstructing risk: Separable encoding of variance and skewness in the brain. *NeuroImage*, 58(4):1139–1149, 2011.
- M.J. Tobia, R. Guo, U. Schwarze, W. Böhmer, J. Gläscher, B. Finckh, A. Marschner, C. Büchel, K. Obermayer, and T. Sommer. Neural systems for choice and valuation with counterfactual learning signals. *To appear in NeuroImage*, 2013.
- A. Tversky and D. Kahneman. Advances in prospect theory: cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5(4):297–323, 1992.
- S.-W. Wu, M.R. Delgado, and L.T. Maloney. Economic decision-making compared with an equivalent motor task. *Proceedings of the National Academy of Sciences*, 106(15):6088–6093, 2009.