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Modeling Mechanisms with Nonholonomic Joints Using the Boltzmann-Hamel Equations

Jonathan M. Cameron   Wayne J. Book

School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

Abstract

This paper describes a new technique for deriving dynamic equations of motion for serial chain and tree topology mechanisms with common nonholonomic constraints. For each type of nonholonomic constraint, the Boltzmann-Hamel equations produce a concise set of dynamic equations. These equations are similar to Lagrange's equations and can be applied to mechanisms which incorporate that type of constraint. A small library of these equations can be used to efficiently analyze many different types of mechanisms.

Nonholonomic constraints are usually included in a Lagrangian setting by adding Lagrange multipliers and then eliminating them from the final set of equations. The approach described in this paper automatically produces a minimum set of equations of motion which do not include Lagrange multipliers.

1 Introduction

The purpose of this paper is to describe a new technique for deriving dynamic equations of motion for serial chain and tree topology mechanisms which incorporate nonholonomic constraints in a Lagrangian setting.

Nonholonomic constraints are nonintegrable motion constraints which typically occur in rolling motion. A common characteristic of such systems is that the number of system coordinates needed to identify the system's configuration is usually greater than the number of instantaneous degrees of freedom of motion. For instance, it requires three variables to locate a car (two for its position and one for its angle). But a car only has two degrees of freedom at any instant (acceleration/deceleration and steering) yet it can reach any configuration in the plane by judicious maneuvers.

The term "nonholonomic" is also sometimes used to describe certain types of quasi-velocities. Quasi-velocities are velocity variables which are functions of the derivatives of the system coordinates. If the relationship is nonintegrable, the resulting quasi-velocities are called nonholonomic. (Nonholonomic being almost synonymous with nonintegrable.) In such cases, the coordinates associated with the quasi-velocities (which are called quasi-coordinates) don't exist, except instantaneously. Quasi-velocities of this type are often used to describe the motion of bodies with three degrees of rotational freedom. A popular choice of such quasi-velocities measure the angular velocities about body-fixed roll, pitch, and yaw axes. For further information on quasi-coordinates and quasi-velocities, consult [Whittaker 1904] or [Meirovitch 1970].

There are numerous ways to derive equations of motion. Although the final equations are equivalent, approaches derived from analytical mechanics often offer greater simplicity and additional insight. For instance, Lagrange's equations are easy to apply and lead simply to integrals of motion in appropriate cases. Unfortunately, the classical form of Lagrange's equations are not applicable to systems which incorporate nonholonomic constraints. This is sometimes overcome by introducing Lagrangian multipliers which represent additional reaction forces necessary to enforce the constraint. Unfortunately, it is then necessary to eliminate these unknown multipliers from the equations of motion. The Boltzmann-Hamel (BH) equations can be used since they are extensions of Lagrange's equations which are suitable for nonholonomic systems even when described with quasi-velocities. The BH equations automatically eliminate any unknown reaction forces. See [Neimark 1972, p. 120-123] for further background on the BH equations.

The basic idea of this paper is to treat the motion allowed by each nonholonomic constraint as a joint and pre-apply (once) the BH equations for that type of joint. The result is a small set of equations which look like Lagrange's equations but are suitable for deriving equations of motion associated with the degrees of freedom for that type of joint. These equations resemble Lagrange's equations with a few additional terms but are simpler and easier to apply than the BH equations or Lagrange's equations with multipliers. Applying them to a specific problem produces a minimum set of equations which do not involve Lagrange multipliers or unknown reaction forces. The additional terms produce the same effect as eliminating the Lagrange multipliers from the final set of equations.

This paper has 4 main sections. In Section 2, related work is briefly overviewed. In Section 3, the BH equations are derived. In Section 4, the nature of nonholonomic joints are examined in context of the BH equations and then the BH equations are applied to several type of joints to illustrate this. In Section 5, a specific problem is given to show how this approach works with a mechanism which includes a nonholonomic joint.

2 Related Work

The primary inspiration for this work is a section in Whittaker's book titled "The Lagrangian equations for quasi-coordinates" [Whittaker 1904, p. 41]. Whittaker derived Boltzmann-Hamel–like equations and applied them to a freely rotating rigid body. However, in his derivation and a similar one in [Meirovitch 1970], there is an implicit assumption that the system is holonomic. Even in Quinn's more recent paper [Quinn 1990], a holonomic system is assumed. Quinn's paper includes a more comprehensive overview of other related work.

The work herein differs from these past works in two ways. First, the approach described in this paper can deal with nonholonomic constraints. Second, the work in these past references anticipates applying the BH equations from scratch for each specific problem. By dealing with each type of nonholonomic constraint once, the approach described in this paper produces a new form of equations that can be applied to any mechanism (of appropriate topology) which incorporates this type of constraint. This procedure offers a considerable advantage since these equations are simpler and easier to apply than the BH equations.

For more background material on holonomic and nonholonomic systems, constraints, and Boltzmann-Hamel equations, consult [Neimark 1972].

3 The Boltzmann-Hamel (BH) equations

The Boltzmann-Hamel equations are suitable for systems with nonholonomic constraints and/or quasi-coordinates. It is useful to rederive them here since this derivation differs a little from some in the literature for reasons that will be explained later.

Start with the d'Alembert-Lagrange equation [Neimark 1972, p. 91], which is an expression of the principle of virtual work:

$$\sum_{k=1}^{n}\left(\frac{d}{dt}\frac{\partial \mathrm{T}}{\partial \dot{q}_k}-\frac{\partial \mathrm{T}}{\partial q_k}-Q_k\right)\delta q_k=0 \tag{1}$$

where q_k are n generalized coordinates, T is the kinetic energy formed with the derivatives of the generalized coordinates, Q_k is the generalized force for the kth generalized coordinate, and δq_k is the virtual variation of q_k. This equation is valid for holonomic or nonholonomic systems. If the variations δq_k are all independent, the term in parentheses in Equation (1) must vanish for each k and therefore the equation simplifies immediately to Lagrange's equations. In our case this simplification cannot occur because of the constraints imposed by the nonholonomic nature of the system—which renders some of the δq_k dependent on others.

This equation can be put into matrix format:

$$\{\delta \boldsymbol{q}\}^T\left(\frac{d}{dt}\left\{\frac{\partial \mathrm{T}}{\partial \dot{\boldsymbol{q}}}\right\}-\left\{\frac{\partial \mathrm{T}}{\partial \boldsymbol{q}}\right\}-\{\mathbf{Q}\}\right)=0 \tag{2}$$

where the vectors involved have the obvious meanings:

$$\left\{\frac{\partial \mathrm{T}}{\partial \dot{\boldsymbol{q}}}\right\}=\left\{\begin{array}{c}\partial \mathrm{T}/\partial \dot{q}_1\\ \partial \mathrm{T}/\partial \dot{q}_2\\ \vdots\\ \partial \mathrm{T}/\partial \dot{q}_n\end{array}\right\}\qquad\left\{\frac{\partial \mathrm{T}}{\partial \boldsymbol{q}}\right\}=\left\{\begin{array}{c}\partial \mathrm{T}/\partial q_1\\ \partial \mathrm{T}/\partial q_2\\ \vdots\\ \partial \mathrm{T}/\partial q_n\end{array}\right\}\qquad\{\mathbf{Q}\}=\left\{\begin{array}{c}Q_1\\ Q_2\\ \vdots\\ Q_n\end{array}\right\} \tag{3}$$

$$\{\boldsymbol{q}\} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix} \qquad \{\dot{\boldsymbol{q}}\} = \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{Bmatrix} \qquad \{\delta \boldsymbol{q}\} = \begin{Bmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_n \end{Bmatrix} \tag{4}$$

The goal is to modify Equation (2) so that it is suitable for nonholonomic systems in which quasi-velocities replace derivatives of generalized coordinates. In general, quasi-velocities are functions of the derivatives of the generalized coordinates. In the systems under consideration, quasi-velocities have a linear relationship with derivatives of the generalized coordinates which has the form:

$$\{\boldsymbol{\omega}\} = [\boldsymbol{\alpha}]^T \{\dot{\boldsymbol{q}}\} \tag{5}$$

where matrix $[\boldsymbol{\alpha}]^T$ is a function of the generalized coordinates, q_k. Assume that this is an invertible relationship:

$$\{\dot{\boldsymbol{q}}\} = [\boldsymbol{\beta}]\{\boldsymbol{\omega}\} \tag{6}$$

where $[\boldsymbol{\alpha}]^T[\boldsymbol{\beta}] = [\boldsymbol{\beta}]^T[\boldsymbol{\alpha}] = [\mathbf{I}]$, the identity matrix. The nature of the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ will be explored in later sections of this paper.

Replace $\dot{\boldsymbol{q}}$ by $\boldsymbol{\omega}$ in the system kinetic energy and call this new expression of the kinetic energy, $\overline{\mathrm{T}}$:

$$\mathrm{T}(\boldsymbol{q};\dot{\boldsymbol{q}}) \longrightarrow \overline{\mathrm{T}}(\boldsymbol{q};\boldsymbol{\omega}) \tag{7}$$

Following [Meirovitch 1970, p. 158] and [Whittaker 1904, p. 41], examine each of the terms involved in Equation (1) and (2) to see the effects of converting from T to $\overline{\mathrm{T}}$. Start with $\partial \mathrm{T}/\partial \dot{q}_k$.

$$\frac{\partial \mathrm{T}}{\partial \dot{q}_k} = \sum_{i=1}^{n} \frac{\partial \overline{\mathrm{T}}}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{q}_k} = \sum_{i=1}^{n} \frac{\partial \overline{\mathrm{T}}}{\partial \omega_i}\alpha_{ki} \tag{8}$$

Put this into matrix form:

$$\left\{\frac{\partial \mathrm{T}}{\partial \dot{\boldsymbol{q}}}\right\} = [\boldsymbol{\alpha}]\left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} \tag{9}$$

Take the total time derivative of this term:

$$\frac{d}{dt}\left\{\frac{\partial \mathrm{T}}{\partial \dot{\boldsymbol{q}}}\right\} = [\boldsymbol{\alpha}]\frac{d}{dt}\left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} + [\dot{\boldsymbol{\alpha}}]\left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} \tag{10}$$

Now consider the term $\partial \mathrm{T}/\partial q_k$:

$$\begin{aligned} \frac{\partial \mathrm{T}}{\partial q_k} &= \frac{\partial \overline{\mathrm{T}}}{\partial q_k} + \sum_{i=1}^{n} \frac{\partial \overline{\mathrm{T}}}{\partial \omega_i}\frac{\partial \omega_i}{\partial q_k} && (11) \\ &= \frac{\partial \overline{\mathrm{T}}}{\partial q_k} + \sum_{i=1}^{n} \frac{\partial \overline{\mathrm{T}}}{\partial \omega_i}\sum_{\ell=1}^{n} \frac{\partial \alpha_{\ell i}}{\partial q_k}\dot{q}_\ell && (12) \\ &= \frac{\partial \overline{\mathrm{T}}}{\partial q_k} + \{\dot{\boldsymbol{q}}\}^T\left[\frac{\partial \boldsymbol{\alpha}}{\partial q_k}\right]\left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} && (13) \end{aligned}$$

However, we would like to express this in terms of the quasi-velocities, $\boldsymbol{\omega}$, instead of the derivatives of the coordinates, $\dot{\boldsymbol{q}}$. Substitute Equation (6) into Equation (13):

$$\frac{\partial \mathrm{T}}{\partial q_k} = \frac{\partial \overline{\mathrm{T}}}{\partial q_k} + \{\boldsymbol{\omega}\}^T [\boldsymbol{\beta}]^T \left[\frac{\partial \boldsymbol{\alpha}}{\partial q_k}\right] \left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} \tag{14}$$

This can be put into matrix form by constructing a special matrix, $[\boldsymbol{\eta}]$, whose rows are of the form $\{\boldsymbol{\omega}\}^T [\boldsymbol{\beta}]^T [\partial \boldsymbol{\alpha}/\partial q_k]$:

$$\left\{\frac{\partial \mathrm{T}}{\partial \boldsymbol{q}}\right\} = \left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{q}}\right\} + \underbrace{\begin{bmatrix} \{\boldsymbol{\omega}\}^T [\boldsymbol{\beta}]^T [\partial \boldsymbol{\alpha}'/\partial q_1] \\ \vdots \\ \{\boldsymbol{\omega}\}^T [\boldsymbol{\beta}]^T [\partial \boldsymbol{\alpha}'/\partial q_n] \end{bmatrix}}_{[\boldsymbol{\eta}]} \left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} \tag{15}$$

Substitute Equations (10) and (15) into Equation (2):

$$\{\delta \boldsymbol{q}\}^T \left([\boldsymbol{\alpha}] \frac{d}{dt}\left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} + [\dot{\boldsymbol{\alpha}}]\left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} - \left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{q}}\right\} - [\boldsymbol{\eta}]\left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} - \{\mathbf{Q}\}\right) = 0 \tag{16}$$

or

$$\{\delta \boldsymbol{\theta}\}^T \left(\frac{d}{dt}\left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} + [\boldsymbol{\beta}]^T [\boldsymbol{\gamma}]\left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}}\right\} - [\boldsymbol{\beta}]^T \left\{\frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{q}}\right\} - \{\mathbf{N}\}\right) = 0 \tag{17}$$

where

$$\begin{aligned} \{\delta \boldsymbol{\theta}\}^T &= \{\delta \boldsymbol{q}\}^T [\boldsymbol{\alpha}] && (18) \\ [\boldsymbol{\gamma}] &= [\dot{\boldsymbol{\alpha}}] - [\boldsymbol{\eta}] && (19) \\ [\mathbf{N}] &= [\boldsymbol{\beta}]^T \{\mathbf{Q}\} && (20) \end{aligned}$$

Notice that $\{\delta \boldsymbol{\theta}\}$ is the vector of variations of the quasi-coordinates, θ_i. In the case of coordinates not involved with the nonholonomic joint, the quasi-coordinates are just the generalized coordinates.

If a joint is "nonholonomic" because it uses nonholonomic quasi-velocities but does not involve any constraint on the motion, then all of the joint's degrees of freedom will be independent. This is the case when quasi-velocities are used to describe the motion of a body with three degrees of rotational freedom. If a joint is nonholonomic because it incorporates a nonholonomic constraint, it's degrees of freedom will not all be independent. This occurs when the nonholonomicity is due to rolling. But if the quasi-velocities are chosen cleverly (by choosing $[\boldsymbol{\alpha}]$ correctly), it is usually possible to separate the degrees of freedom of the joint into a set of independent degrees of freedom (which are arbitrary) and and a set of dependent ones (which are all zero). Assume that the joint's degrees of freedom are all independent or that the quasi-velocities have been carefully chosen to be independent or zero (as just described). This means that the quasi-coordinates are either independent or constant. The variations of the independent quasi-coordinates are not necessarily zero while the variations of the constant quasi-coordinates are definitely zero. For the product in Equation (17) to be zero, the rows of the term in parentheses which correspond to independent quasi-coordinates ($\delta\theta_I$) must be zero and the BH equations are produced for the independent quasi-coordinates:

$$\frac{d}{dt}\frac{\partial \overline{\mathrm{T}}}{\partial \omega_I} + \sum_{j=1}^{n}\sum_{r=1}^{n} \beta_{rI}\gamma_{rj}\frac{\partial \overline{\mathrm{T}}}{\partial \omega_j} - \sum_{j=1}^{n} \beta_{jI}\frac{\partial \overline{\mathrm{T}}}{\partial q_j} = \mathrm{N}_I \tag{21}$$

where the index I is used to denote independent quasi-coordinates (or related quantities). This equation is valid for systems described in terms of quasi-velocities. It is valid for all of the degrees of freedom of a holonomic system or for the independent degrees of freedom for nonholonomic systems.

This derivation is different from typical ones in two ways. The first is that the resulting equations are kept explicitly in terms of the generalized coordinates, q_k. The second difference is that the form of the second term is different. It is regrouped slightly since the form of matrix γ can be determined and used to advantage, as will be shown in the next section.

4 Nonholonomic joints

The previous derivation was general. Now consider a mechanism consisting of a series of bodies connected by joints which allow some predefined type of relative motion between the bodies they connect. In other words, this is an serial chain (or tree) of bodies connected by joints. As before, assume that n generalized coordinates are required to describe the configuration of the mechanism. In this type of mechanism, groups of generalized coordinates are used to describe the relative motion allowed by each joint.

Using the terminology of "joint" to describe nonholonomic constraints on the motion between two bodies is only a little different than the typical meaning of the term "joint". Joints between bodies prescribe the type of motion allowed between the bodies. In typical joints such as revolute and prismatic joints, the constraint involved is holonomic. For nonholonomic constraints, the joint describes more complicated kinematics between the two bodies it connects. The main distinction is that nonholonomic constraints (and joints) have fewer degrees of freedom locally than they do globally. In other words, it takes more variables to describe the global configuration of a nonholonomic joint than it has local degrees of freedom at any instant.

Assume that quasi-velocities are introduced for one joint. This may be because the motion allowed by the joint is subject to some nonholonomic constraint. Or the motion allowed by the joint may be holonomic but nonholonomic quasi-velocities may be introduced to simplify the form of the kinetic energy or equations of motion. Nonholonomic quasi-velocities are quantities which are not derivatives of any generalized coordinates. In either case, this joint will be called a nonholonomic joint.

The number of constraints involved will not be detailed here since it could be from 0 (in the case of using quasi-velocities for a holonomic joint) to m. Usually the number of constraints will be less than m since having m constraints means that there are no independent coordinates for the joint.

Without loss of generality, assume that only one nonholonomic joint is present in the mechanism. There could be more nonholonomic joints, but the arguments that follow can easily be extended to more nonholonomic joints. This means that there is only one group of generalized coordinates which is involved in a nonholonomic joint. Say that m generalized coordinates are involved starting at the pth generalized coordinate. Now suppose that quasi-velocities are introduced to describe the relative velocities allowed by the nonholonomic joint. In order to be useful, the quasi-velocities are constructed (by judicious choice of $[\boldsymbol{\alpha}']$) in such a way that they are either independent or zero. (Most choices of $[\boldsymbol{\alpha}']$ will produce coupled quasi-velocities which are neither independent nor zero. Correct construction of $[\boldsymbol{\alpha}']$ is an important part of this work but must be done on a case-by-case basis.) Assume that the rest of the velocities are not affected. Then the structure of Equation (5) can be shown more explicitly:

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_p \\ \vdots \\ \omega_n \end{Bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \begin{bmatrix} & \boldsymbol{\alpha}'^T & \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_p \\ \vdots \\ \dot{q}_n \end{Bmatrix} \tag{22}$$

The matrix $[\boldsymbol{\alpha}]^T$ is block diagonal as shown. (The submatrix block $[\boldsymbol{\alpha}']^T$ can always be put into the lower right hand corner of $[\boldsymbol{\alpha}]^T$ by appropriate renumbering of coordinates, without loss of generality.) The submatrix $[\boldsymbol{\alpha}']^T$ is the linear relationship between the quasi-velocities $(\omega_p, \omega_{p+1}, \ldots, \omega_n)$ and the (possibly) constrained derivatives of the generalized coordinates $(\dot{q}_p, \dot{q}_{p+1}, \ldots, \dot{q}_n)$ associated with the nonholonomic joint.

Throughout this paper, a prime ($'$) will be used to mean that the vector or matrix indicated is restricted to the quantities associated with the nonholonomic joint. For instance, $\boldsymbol{q}' = \{q_p, q_{p+1}, \ldots, q_n\}^T$.

In the context of multibody mechanisms, it is clear that the elements of the matrix $[\boldsymbol{\alpha}']$ generally involve only the coordinates associated with the nonholonomic joint itself. In other words:

$$\alpha'_{ij} = \alpha'_{ij}(q_p, \ldots, q_n) \tag{23}$$

This will always be true if the descriptions identify the relative motion of the joint. In other words, the variables $q_p, \ldots, q_n$ must describe the relative motion between the bodies connected by this nonholonomic joint.

Since we have assumed that $[\boldsymbol{\alpha}]$ is invertible, it follows that $[\boldsymbol{\alpha}']^T$ is invertible. Therefore the relationship between the derivatives of the generalized coordinates and the quasi-velocities can be inverted and has the explicit form:

$$\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_p \\ \vdots \\ \dot{q}_n \end{Bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \begin{bmatrix} & \boldsymbol{\beta}' & \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_p \\ \vdots \\ \omega_n \end{Bmatrix} \tag{24}$$

where $[\boldsymbol{\beta}'] [\boldsymbol{\alpha}']^T = [\mathbf{I}]$. Notice that $\dot{q}_k \equiv \omega_k$ except for the $\dot{q}_p \ldots \dot{q}_n$ involved in the nonholonomic joint.

Now we would like to examine the structure of each of the vectors inside the parentheses of Equation (17). The structure of the first term, $\frac{d}{dt}\{\partial \mathrm{T}/\partial \boldsymbol{\omega}\}$, is apparent. Consider the second term, $[\boldsymbol{\beta}]^T [\gamma] \left\{\partial \overline{\mathrm{T}}/\partial \boldsymbol{\omega}\right\}$. Recall that $[\gamma] = [\dot{\boldsymbol{\alpha}}] - [\boldsymbol{\eta}]$. Examining Equation (22), the structure of $[\dot{\boldsymbol{\alpha}}]$ is straightforward:

$$[\dot{\boldsymbol{\alpha}}] = \begin{bmatrix} 0 & & & \\ & 0 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \begin{bmatrix} & \dot{\boldsymbol{\alpha}}' & \end{bmatrix} \end{bmatrix} \tag{25}$$

In order to make the structure of $[\gamma]$ explicit, we now need to examine the structure of $[\boldsymbol{\eta}]$. Recall that the kth row of $[\boldsymbol{\eta}]$ has the form $\{\boldsymbol{\omega}\}^T [\boldsymbol{\beta}]^T [\partial \boldsymbol{\alpha}/\partial q_k]$. It is clear from the structure of $[\boldsymbol{\alpha}]$ and the assumption stated in Equation (23) that each matrix $[\partial \boldsymbol{\alpha}/\partial q_k]$ is completely zero for each k except those associated with the nonholonomic joint. Therefore, all the rows of $[\boldsymbol{\eta}]$ are zero except for those associated with the nonholonomic joint.

Given the assumption in Equation (23), it is clear that for ℓ associated with the nonholonomic joint, that:

$$[\partial \boldsymbol{\alpha}/\partial q_\ell] = \begin{bmatrix} 0 & & & \\ & 0 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \begin{bmatrix} & \frac{\partial \boldsymbol{\alpha}'}{\partial q_\ell} & \end{bmatrix} \end{bmatrix} \tag{26}$$

Therefore, considering the structure of $[\partial \boldsymbol{\alpha}'/\partial q_\ell]$ and the structure of $[\boldsymbol{\beta}]$, the rows of $[\boldsymbol{\eta}]$ associated with the nonholonomic joint have the form:

$$\{\boldsymbol{\omega}\}^T [\boldsymbol{\beta}]^T \left[\frac{\partial \boldsymbol{\alpha}}{\partial q_\ell}\right] = \{\boldsymbol{\omega}\}^T \begin{bmatrix} 0 & & & \\ & 0 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \begin{bmatrix} & [\boldsymbol{\beta}']^T \left[\frac{\partial \boldsymbol{\alpha}'}{\partial q_\ell}\right] & \end{bmatrix} \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & \{\boldsymbol{\omega}'\}^T [\boldsymbol{\beta}']^T \left[\frac{\partial \boldsymbol{\alpha}'}{\partial q_\ell}\right] \end{bmatrix} \tag{28}$$

So the structure of matrix $[\boldsymbol{\eta}]$ has the form:

$$[\boldsymbol{\eta}] = \begin{bmatrix} 0 & & & \\ & 0 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \begin{bmatrix} & \boldsymbol{\eta}' & \end{bmatrix} \end{bmatrix} \tag{29}$$

where

$$[\boldsymbol{\eta}'] = \begin{bmatrix} \{\boldsymbol{\omega}'\}^T [\boldsymbol{\beta}']^T [\partial\boldsymbol{\alpha}'/\partial q_p] \\ \vdots \\ \{\boldsymbol{\omega}'\}^T [\boldsymbol{\beta}']^T [\partial\boldsymbol{\alpha}'/\partial q_n] \end{bmatrix} \tag{30}$$

In light of Equations (25) and (29), the structure of $[\boldsymbol{\gamma}]$ is finally clear:

$$[\boldsymbol{\gamma}] = \begin{bmatrix} 0 & & & \\ & 0 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \begin{bmatrix} & \boldsymbol{\gamma}' & \end{bmatrix} \end{bmatrix} \tag{31}$$

where $[\boldsymbol{\gamma}'] = [\dot{\boldsymbol{\alpha}}'] - [\boldsymbol{\eta}']$. Given this structure of $[\boldsymbol{\gamma}]$, the structure of $[\boldsymbol{\beta}]^T [\boldsymbol{\gamma}]$ can be determined:

$$[\boldsymbol{\beta}]^T [\boldsymbol{\gamma}] = \begin{bmatrix} 0 & & & \\ & 0 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \begin{bmatrix} & [\boldsymbol{\beta}']^T [\boldsymbol{\gamma}'] & \end{bmatrix} \end{bmatrix} \tag{32}$$

And now our goal of understanding the structure of the second term of Equation (17) can be realized:

$$[\boldsymbol{\beta}]^T [\boldsymbol{\gamma}] \left\{ \frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{\omega}} \right\} = \left\{ \begin{array}{c} 0 \\ \vdots \\ 0 \\ [\boldsymbol{\beta}']^T [\boldsymbol{\gamma}'] \, \partial\overline{\mathrm{T}}/\partial\boldsymbol{\omega}' \end{array} \right\} \tag{33}$$

Now consider the third term of Equation (17):

$$[\boldsymbol{\beta}]^T \left\{ \frac{\partial \overline{\mathrm{T}}}{\partial \boldsymbol{q}} \right\} = \begin{bmatrix} 1 & & & \\ & 1 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \begin{bmatrix} & \boldsymbol{\beta}'^T & \end{bmatrix} \end{bmatrix} \left\{ \begin{array}{c} \partial\overline{\mathrm{T}}/\partial q_1 \\ \partial\overline{\mathrm{T}}/\partial q_2 \\ \vdots \\ \partial\overline{\mathrm{T}}/\partial q_p \\ \vdots \\ \partial\overline{\mathrm{T}}/\partial q_n \end{array} \right\} \tag{34}$$

$$= \left\{ \begin{array}{c} \partial\overline{\mathrm{T}}/\partial q_1 \\ \vdots \\ \partial\overline{\mathrm{T}}/\partial q_{p-1} \\ \left[\boldsymbol{\beta}'^T\right] \left\{\partial\overline{\mathrm{T}}/\partial \boldsymbol{q}'\right\} \end{array} \right\} \tag{35}$$

The final term of Equation (17) is also simple:

$$\{\mathbf{N}\} = [\boldsymbol{\beta}]^T \{\mathbf{Q}\} = \left\{ \begin{array}{c} Q_1 \\ \vdots \\ Q_{p-1} \\ [\boldsymbol{\beta}']^T \mathbf{Q}' \end{array} \right\} \tag{36}$$

Substitute in Equations (33), (35), and (36) into Equation (17):

$$\{\delta\boldsymbol{\theta}\}^T \left(\left\{ \begin{array}{c} \frac{d}{dt}\partial\overline{\mathrm{T}}/\partial\omega_1 \\ \vdots \\ \frac{d}{dt}\partial\overline{\mathrm{T}}/\partial\omega_{p-1} \\ \frac{d}{dt}\left\{\partial\overline{\mathrm{T}}/\partial\boldsymbol{\omega}'\right\} \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ \vdots \\ 0 \\ [\boldsymbol{\beta}']^T [\gamma'] \frac{\partial\overline{\mathrm{T}}}{\partial\boldsymbol{\omega}'} \end{array} \right\} - \left\{ \begin{array}{c} \partial\overline{\mathrm{T}}/\partial q_1 \\ \vdots \\ \partial\overline{\mathrm{T}}/\partial q_{p-1} \\ \left[\boldsymbol{\beta}'^T\right] \left\{\frac{\partial\overline{\mathrm{T}}}{\partial\boldsymbol{q}'}\right\} \end{array} \right\} - \left\{ \begin{array}{c} Q_1 \\ \vdots \\ Q_{p-1} \\ [\boldsymbol{\beta}']^T \mathbf{Q}' \end{array} \right\} \right) = 0 \tag{37}$$

Since all the quasi-coordinates not associated with the nonholonomic joint are independent, the equations for those degrees of freedom are simply Lagrange's equations:

$$\frac{d}{dt}\frac{\partial\overline{\mathrm{T}}}{\partial\dot{q}_i} - \frac{\partial\overline{\mathrm{T}}}{\partial q_i} = Q_i \qquad \text{(for } i \text{ not associated with the nonholonomic joint)} \tag{38}$$

Notice that we have taken the liberty of replacing ω_i by $\dot{q}_i$ here since they are the same for the coordinates not associated with the nonholonomic joint.

After removing the terms due to the generalized coordinates not associated with the nonholonomic joint, the equation has the the same form as that for the entire system:

$$\{\delta\boldsymbol{\theta}'\}^T \left(\frac{d}{dt}\left\{\frac{\partial\overline{\mathrm{T}}}{\partial\boldsymbol{\omega}'}\right\} + [\boldsymbol{\beta}']^T [\gamma'] \frac{\partial\overline{\mathrm{T}}}{\partial\boldsymbol{\omega}'} - \left[\boldsymbol{\beta}'^T\right] \left\{\frac{\partial\overline{\mathrm{T}}}{\partial\boldsymbol{q}'}\right\} - [\mathbf{N}'] \right) = 0 \tag{39}$$

Before analyzing these equations further, a few comments on quasi-velocities are in order. When quasi-velocities are introduced to describe nonholonomic mechanisms, they have two related purposes. The independent quasi-velocities are typically used to simplify the kinetic energy and equations of motion. The dependent quasi-velocities are zero and used to enforce the constraints involved. Some of the quasi-coordinates for the nonholonomic joint are independent and the variation for these quasi-coordinates is not necessarily zero, therefore the term in the parentheses above must vanish. This produces the BH equations for the independent coordinates, q'_I.

$$\frac{d}{dt}\frac{\partial\overline{\mathrm{T}}}{\partial\omega'_I} + \sum_{j=1}^{n}\sum_{r=1}^{n} \beta'_{rI}\gamma'_{rj}\frac{\partial\overline{\mathrm{T}}}{\partial\omega'_j} - \sum_{j=1}^{n} \beta'_{jI}\frac{\partial\overline{\mathrm{T}}}{\partial q'_j} = \mathbf{N}'_I \tag{40}$$

Notice that the kinetic energy, T, can be replaced by the Lagrangian (L = T − V where V is the potential energy) in all of the previous manipulations since the potential energy never involves the quasi-velocities.

The only remaining question is how to compute the entries of the matrix $[\gamma']$. Recall that $[\gamma'] = [\dot{\boldsymbol{\alpha}}'] - [\boldsymbol{\eta}']$. The elements of the matrix $[\dot{\boldsymbol{\alpha}}']$ have the form:

$$\dot{\alpha}'_{ij} = \sum_{k=1}^{m} \frac{\partial \alpha'_{ij}}{\partial q_k} \dot{q}_k \tag{41}$$

Now substitute $\dot{q}_k = \sum_{r=1}^{m} \beta'_{kr} \omega'_r$:

$$\begin{aligned} \dot{\alpha}'_{ij} &= \sum_{k=1}^{m} \frac{\partial \alpha'_{ij}}{\partial q_k} \left(\sum_{r=1}^{m} \beta'_{kr} \omega'_r \right) && (42) \\ &= \sum_{k=1}^{m} \sum_{r=1}^{m} \omega'_r \beta'_{kr} \frac{\partial \alpha'_{ij}}{\partial q_k} && (43) \end{aligned}$$

The second part of γ'_{ij} is due to η'_{ij}. Recall that

$$[\boldsymbol{\eta}'] = \begin{bmatrix} \{\boldsymbol{\omega}'\}^T [\boldsymbol{\beta}']^T [\partial \boldsymbol{\alpha}' / \partial q_p] \\ \vdots \\ \{\boldsymbol{\omega}'\}^T [\boldsymbol{\beta}']^T [\partial \boldsymbol{\alpha}' / \partial q_n] \end{bmatrix} \tag{44}$$

So the i^{th} row of matrix $[\boldsymbol{\eta}']$ is:

$$\text{Row}_i \left([\boldsymbol{\eta}'] \right) = \{\boldsymbol{\omega}'\}^T [\boldsymbol{\beta}']^T [\partial \boldsymbol{\alpha}' / \partial q_i] \tag{45}$$

Therefore, element η'_{ij} is:

$$\begin{aligned} \eta'_{ij} &= \sum_{r=1}^{m} \omega'_r \left([\boldsymbol{\beta}']^T [\partial \boldsymbol{\alpha}' / \partial q_i] \right)_{rj} && (46) \\ &= \sum_{r=1}^{m} \omega'_r \sum_{k=1}^{m} \beta'^T_{rk} \frac{\partial \alpha'_{kj}}{\partial q_i} && (47) \\ &= \sum_{k=1}^{m} \sum_{r=1}^{m} \omega'_r \beta'_{kr} \frac{\partial \alpha'_{kj}}{\partial q_i} && (48) \end{aligned}$$

Assembling these pieces gives the desired result in a useful form:

$$\gamma'_{ij} = \dot{\alpha}'_{ij} - \eta'_{ij} = \sum_{k=1}^{m} \sum_{r=1}^{m} \omega'_r \beta'_{kr} \left(\frac{\partial \alpha'_{ij}}{\partial q_k} - \frac{\partial \alpha'_{kj}}{\partial q_i} \right) \tag{49}$$

In the following sections, these results are applied to various types of joints.

4.1 Application to a knife joint

For the first specific type of joint, consider the mechanism called a knife. It is one of the simplest types of nonholonomic constraints and is very common in examples dealing with

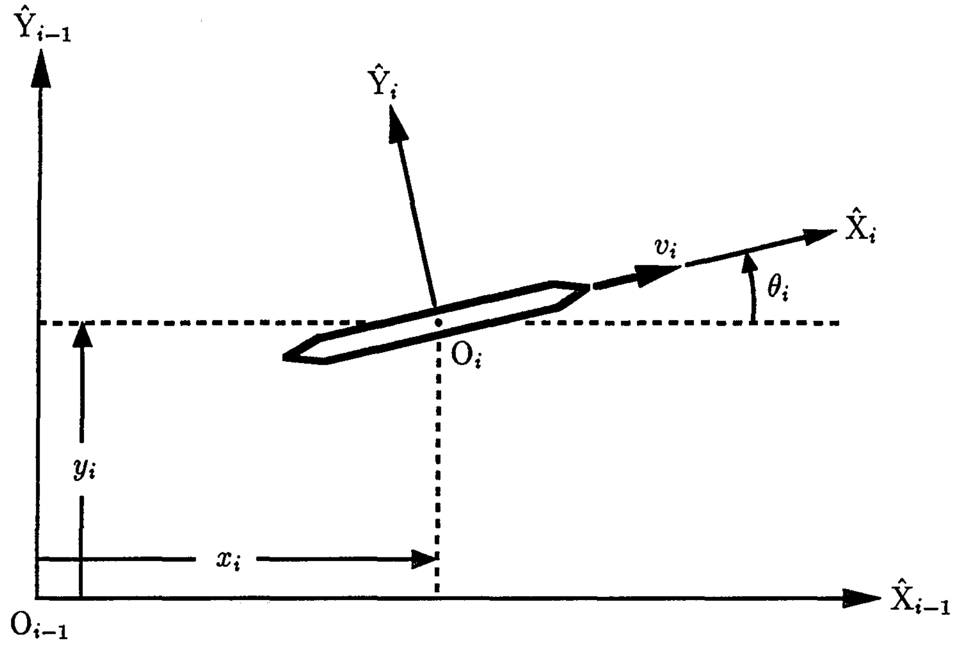

Figure 1: Knife joint geometry and conventions. Note that the point of contact is at O_i

nonholonomic motion. A knife actually represents any system whose motion is constrained in the same way as a knife cutting through some material. For instance, wheels which roll without slipping, racing sailboats, and pizza cutters all allow similar motions. Figure 1 shows the geometry involved and the selected conventions.

The knife is a nonholonomic joint. The constraint involved is that the velocity of the center of the blade is strictly in the direction that the blade is pointing. This can be expressed in the following relationships:

$$\dot{x}_i = v_i \cos\theta_i \tag{50}$$

$$\dot{y}_i = v_i \sin\theta_i \tag{51}$$

In other words, the velocity orthogonal to the direction that the blade is pointing is zero. This can be expressed in the following single constraint equation:

$$0 = -\dot{x}_i \sin\theta_i + \dot{y}_i \cos\theta_i \tag{52}$$

Notice that the velocity of the blade, v_i, is a quasi-velocity: It is not the derivative of any generalized coordinate. Introduce the following quasi-velocities for this ith "joint" and rename them appropriately:

$$\begin{Bmatrix} \omega_{i1} \\ \omega_{i2} \\ \omega_{i3} \end{Bmatrix} = \begin{Bmatrix} v_i \\ \omega_{i2} \\ \dot{\theta}_i \end{Bmatrix} \tag{53}$$

Equations (50) and (51) can be put into an invertible relationship using the quasi-velocities:

$$\begin{Bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{\beta}'} \begin{Bmatrix} v_i \\ \omega_{i2} \\ \dot{\theta}_i \end{Bmatrix} \tag{54}$$

When the matrix $[\boldsymbol{\beta}']$ is first laid out, the top two entries of its middle column are arbitrary since $\omega_{i2} = 0$. It is critical to choose these entries so the constraints are satisfied in the inverse relationship. The procedure to do this will be described below. Notice the matrix $[\boldsymbol{\beta}']$ which was defined earlier in Equation (24). When this velocity relationship is inverted, the result is:

$$\begin{Bmatrix} v_i \\ \omega_{i2} \\ \dot{\theta}_i \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{\alpha}'^T} \begin{Bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{Bmatrix} \tag{55}$$

where the matrix $\boldsymbol{\alpha}'$ is identified; it was defined in Equation (22). Notice that $\boldsymbol{\alpha}' = \boldsymbol{\beta}'$ for this joint; this is not true in general.

The procedure for determining the top two entries of the second column of matrix $[\boldsymbol{\beta}']$ of Equation (54) is as follows. Insert two symbolic unknowns into those two positions and invert the matrix $[\boldsymbol{\beta}']$ symbolically. This produces matrix $[\boldsymbol{\alpha}']^T$ as shown in Equation (55) (except that some of its entries are symbolic unknowns). Compare the middle line of the resulting equation with the constraint equation, Equation (52), and solve for the two unknowns.

Once the two arbitrary entries of $[\boldsymbol{\beta}']$ are chosen properly, the middle line of Equation (55) expresses the motion constraint. In other words, dependent quasi-velocity ω_{i2} is used to enforce the kinematic constraint by forcing it to zero:

$$\omega_{i2} = 0 = -\dot{x}_i \sin\theta_i + \dot{y}_i \cos\theta_i \tag{56}$$

When this joint is included in a mechanism, the equations of motion associated with the joint's degrees of freedom can be found using the Boltzmann–Hamel equations. However, the Lagrangian, $\overline{\mathrm{L}}$, must be formed using the quasi-velocity ω_{i2}. The constraint $\omega_{i2} = 0$ must not be enforced until the equations of motion are determined.

The next step is to apply Equation (40) for this joint. The main piece needed first is $\boldsymbol{\gamma}'$. Using Equation (49), the result is:

$$\boldsymbol{\gamma}' = \begin{bmatrix} -\dot{\theta}_i \sin\theta_i & -\dot{\theta}\cos\theta_i & 0 \\ \dot{\theta}_i \cos\theta_i & -\dot{\theta}\sin\theta_i & 0 \\ -\omega_{i2} & v_i & 0 \end{bmatrix} \tag{57}$$

There are two independent quasi-velocities, v_i and $\dot{\theta}_i$. Applying Equation (40) for each produces the following two equations of motion (which can be applied like Lagrange's equations):

$$\frac{d}{dt}\left(\frac{\partial \overline{\mathrm{L}}}{\partial v_i}\right) - \cos\theta_i \frac{\partial \overline{\mathrm{L}}}{\partial x_i} - \sin\theta_i \frac{\partial \overline{\mathrm{L}}}{\partial y_i} - \dot{\theta}_i P_{i2} = F_{v_i} \tag{58}$$

$$\frac{d}{dt}\left(\frac{\partial \overline{\mathrm{L}}}{\partial \dot{\theta}_i}\right) - \frac{\partial \overline{\mathrm{L}}}{\partial \theta_i} + v_i P_{i2} = \tau_i \tag{59}$$

where F_{v_i} is the resultant external force acting on the knife in the direction the blade is pointing, τ_i is the external torque turning the blade, and P_{i2} is the quasi-momentum associated with ω_{i2}:

$$P_{i2} = \frac{\partial \overline{\mathrm{L}}}{\partial \omega_{i2}} \tag{60}$$

In practice, once P_{i2} has been determined, the constraint can be enforced by letting $\omega_{i2} = 0$ in $\overline{\mathrm{L}}$ before Equations (58) and (59) are applied.

4.2 Application to a tricycle joint

As a second type of joint, consider the common tricycle. This type of mechanism can also describe the idealized motion of cars. The tricycle joint allows motion in the plane subject to two nonholonomic constraints. The first constraint is that the velocity of the front wheel is in the direction it is pointing and the second is that the rear wheels do not slide from side to side. There are four degrees of freedom involved. Three describe the position of the tricycle body (x_i, y_i, and θ_i). The other degree of freedom is the steering angle ϕ_i. Figure 2 shows the geometry involved and the selected conventions.

The tricycle joint is also nonholonomic and is similar to the knife joint. The main difference is that there is an additional constraint: the speed, v_i, and the steering angle, ϕ_i, determine the body turning rate, $\dot{\theta}_i$. The constraints are described by several equations:

$$\dot{x}_i = v_i \cos\phi_i \cos\theta_i \tag{61}$$

$$\dot{y}_i = v_i \cos\phi_i \sin\theta_i \tag{62}$$

$$\dot{\theta}_i = \frac{v_i}{\ell_i} \sin\phi_i \tag{63}$$

These equations can be put into an invertible relationship using the quasi-velocities and including $\dot{\phi}_i$:

$$\left\{\begin{array}{c} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \\ \dot{\phi}_i \end{array}\right\} = \underbrace{\left[\begin{array}{cccc} \cos\theta_i \cos\phi_i & -\sin\theta & 0 & 0 \\ \sin\theta_i \cos\phi_i & \cos\theta & 0 & 0 \\ \sin\phi_i/\ell_i & -\tan\theta_i \tan\phi_i/\ell_i & 1/(\ell_i \cos\theta_i \cos\phi_i) & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{\boldsymbol{\beta}'} \left\{\begin{array}{c} v_i \\ \omega_{i2} \\ \omega_{i3} \\ \dot{\phi}_i \end{array}\right\} \tag{64}$$

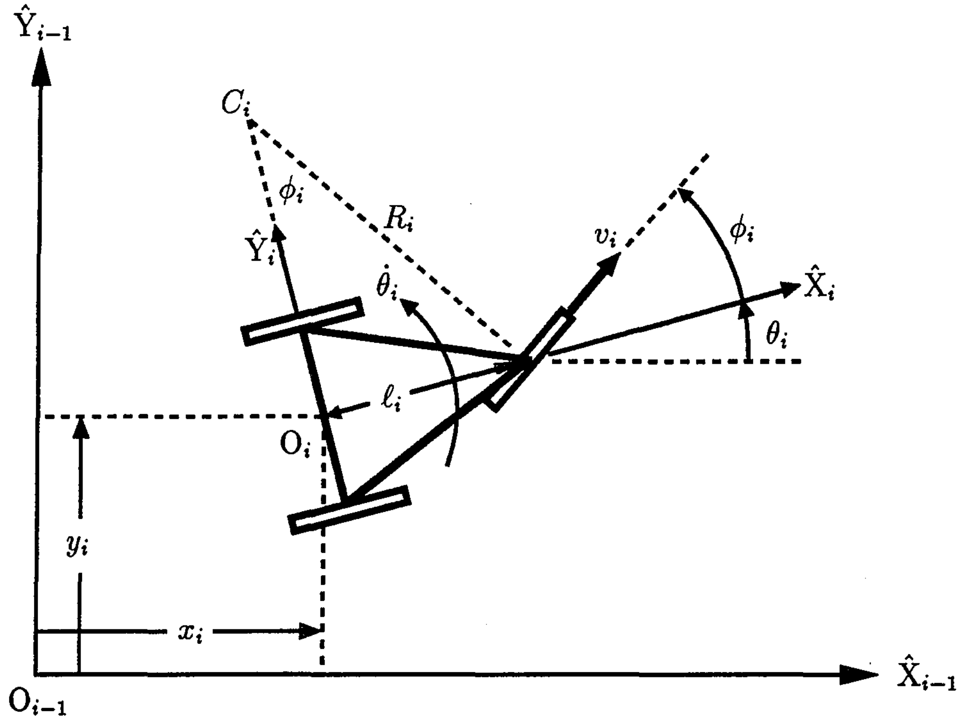

Figure 2: Tricycle joint geometry and conventions

where the (zero) quasi-velocities ω_{i2} and ω_{i3} have been introduced. In this case, the upper three entries in the second and third columns of $[\boldsymbol{\beta}']$ are arbitrary and have been chosen so that the motion constraints are embedded in $[\boldsymbol{\alpha}']$. Inverting this relationship gives:

$$\left\{ \begin{array}{c} v_i \\ \omega_{i2} \\ \omega_{i3} \\ \dot{\phi}_i \end{array} \right\} = \underbrace{\left[\begin{array}{cccc} \cos\theta_i/\cos\phi_i & \sin\theta_i/\cos\phi_i & 0 & 0 \\ -\sin\theta_i & \cos\theta_i & 0 & 0 \\ -\sin\phi_i & 0 & \ell_i\cos\theta_i\cos\phi_i & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]}_{\boldsymbol{\alpha}'^T} \left\{ \begin{array}{c} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \\ \dot{\phi}_i \end{array} \right\} \tag{65}$$

The two dependent quasi-velocities, ω_{i2} and ω_{i3}, are used to enforce the two constraints by letting them go to zero:

$$\omega_{i2} = 0 = -\dot{x}_i \sin\theta_i + \dot{y}_i \cos\theta_i \tag{66}$$

$$\omega_{i3} = 0 = -\dot{x}_i \sin\phi_i + \ell_i\dot{\theta}_i \cos\theta_i \cos\phi_i \tag{67}$$

As before, when this joint is included in a mechanism, the equations of motion associated with the joint's degrees of freedom can be found using the Boltzmann–Hamel equations. However, the Lagrangian, $\overline{\mathrm{L}}$, must be formed using the quasi-velocities ω_{i2} and ω_{i3}. The constraints that $\omega_{i2} = 0$ and $\omega_{i3} = 0$ must not be enforced until the equations of motion are determined (or at least until the quasi-momenta have been formed).

After determining γ', Equation (40) is applied for the two independent quasi-velocities: v_i and $\dot{\theta}_i$ to produce the joint-specific equations (after letting $\omega_{i2} = 0$ and $\omega_{i3} = 0$) are:

$$\frac{d}{dt}\left(\frac{\partial \overline{\mathrm{L}}}{\partial v_i}\right) - \cos\theta_i \cos\phi_i \frac{\partial \overline{\mathrm{L}}}{\partial x_i} - \sin\theta_i \cos\phi_i \frac{\partial \overline{\mathrm{L}}}{\partial y_i} - \frac{\sin\phi_i}{\ell_i}\frac{\partial \overline{\mathrm{L}}}{\partial \theta_i} + \tag{68}$$

$$\dot{\phi}_i \tan\phi_i \frac{\partial \overline{\mathrm{L}}}{\partial v_i} - \dot{\phi}_i \cos\theta_i P_{i3} = \mathrm{F}_{v_i} \tag{69}$$

$$\frac{d}{dt}\left(\frac{\partial \overline{\mathrm{L}}}{\partial \dot{\phi}_i}\right) - \frac{\partial \overline{\mathrm{L}}}{\partial \phi_i} - v_i \tan\phi_i \frac{\partial \overline{\mathrm{L}}}{\partial v_i} + v_i \cos\theta_i P_{i3} = \tau_i \tag{70}$$

where P_{i3} is the quasi-momentum associated with ω_{i3}:

$$P_{i3} = \frac{\partial \overline{\mathrm{L}}}{\partial \omega_{i3}} \tag{71}$$

Notice that P_{i2} is not needed.

4.3 Application to a roll-pitch-yaw rotational joint

As a final type of joint, consider a rotational joint which allows free rotation about a point in all three possible directions. The rotation involved can be described in a number of ways. In this case, Z-Y-X body-fixed Euler angles are used. Rotations about axes fixed in the body can be thought of as Roll (about $\hat{\mathrm{X}}_i$), Pitch (about $\hat{\mathrm{Y}}_i$), and Yaw (about $\hat{\mathrm{Z}}_i$). This is achieved by first rotating through angle ϕ_i about the body's Z–axis, then rotating through θ_i about the body's Y–axis, and finally rotating ψ_i about the body's X–axis.

This example is different than the previous two examples because there are no constraints on the rotational motion. However, it is advantageous to introduce roll-pitch-yaw quasi-velocities to simplify the equations of motion. For instance, ω_{xi} is the roll rate (or the instantaneous angular velocity about the roll axis, $\hat{\mathrm{X}}_i$). Similarly ω_{yi} is the pitch rate and ω_{zi} is the yaw rate. These quasi-velocities are not the derivatives of any generalized coordinates and therefore the joint is considered nonholonomic in that sense. Figure 3 shows the geometry involved and the selected conventions. The relationships between the quasi-velocities and the derivatives of the Euler angles follow:

$$\begin{Bmatrix} \dot{\phi}_i \\ \dot{\theta}_i \\ \dot{\psi}_i \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & \sin\phi_i \tan\theta_i & \cos\phi_i \tan\theta_i \\ 0 & \cos\phi_i & -\sin\phi_i \\ 0 & \sin\phi_i / \cos\theta_i & \cos\phi_i / \cos\theta_i \end{bmatrix}}_{\boldsymbol{\beta}'} \begin{Bmatrix} \omega_{xi} \\ \omega_{yi} \\ \omega_{zi} \end{Bmatrix} \tag{72}$$

and

$$\begin{Bmatrix} \omega_{xi} \\ \omega_{yi} \\ \omega_{zi} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -\sin\theta_i \\ 0 & \cos\phi_i & \cos\theta_i \sin\phi_i \\ 0 & -\sin\phi_i & \cos\theta_i \cos\phi_i \end{bmatrix}}_{\boldsymbol{\alpha}'^T} \begin{Bmatrix} \dot{\phi}_i \\ \dot{\theta}_i \\ \dot{\psi}_i \end{Bmatrix} \tag{73}$$

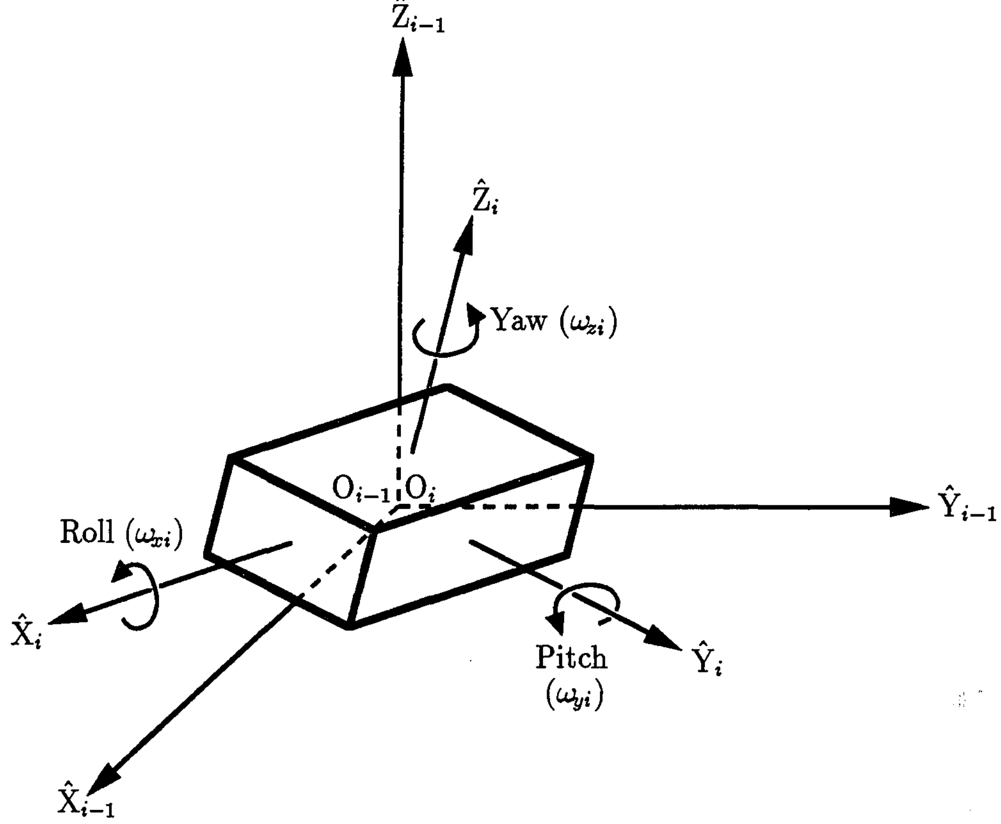

Figure 3: 3D Rotational joint with z-y-x Euler angles—geometry and conventions

The BH equations for this joint's three degrees of freedom have the following form (where $\overline{\mathrm{L}}$ is the Lagrangian formed with the quasi-velocities, ω_{xi}, ω_{yi}, and ω_{zi}):

$$\frac{d}{dt}\left(\frac{\partial\overline{\mathrm{L}}}{\partial\omega_{xi}}\right) - \frac{\partial\overline{\mathrm{L}}}{\partial\phi_i} - \omega_{zi}\frac{\partial\overline{\mathrm{L}}}{\partial\omega_{yi}} + \omega_{yi}\frac{\partial\overline{\mathrm{L}}}{\partial\omega_{zi}} = \tau_{xi} \tag{74}$$

$$\frac{d}{dt}\left(\frac{\partial\overline{\mathrm{L}}}{\partial\omega_{yi}}\right) - \sin\phi_i\tan\theta_i\frac{\partial\overline{\mathrm{L}}}{\partial\phi_i} - \cos\phi_i\frac{\partial\overline{\mathrm{L}}}{\partial\theta_i} - \frac{\sin\phi_i}{\cos\theta_i}\frac{\partial\overline{\mathrm{L}}}{\partial\psi_i} + \omega_{zi}\frac{\partial\overline{\mathrm{L}}}{\partial\omega_{xi}} - \omega_{xi}\frac{\partial\overline{\mathrm{L}}}{\partial\omega_{zi}} = \tau_{yi} \tag{75}$$

$$\frac{d}{dt}\left(\frac{\partial\overline{\mathrm{L}}}{\partial\omega_{yi}}\right) - \cos\phi_i\tan\theta_i\frac{\partial\overline{\mathrm{L}}}{\partial\phi_i} + \sin\phi_i\frac{\partial\overline{\mathrm{L}}}{\partial\theta_i} - \frac{\cos\phi_i}{\cos\theta_i}\frac{\partial\overline{\mathrm{L}}}{\partial\psi_i} - \omega_{yi}\frac{\partial\overline{\mathrm{L}}}{\partial\omega_{xi}} + \omega_{xi}\frac{\partial\overline{\mathrm{L}}}{\partial\omega_{yi}} = \tau_{zi} \tag{76}$$

5 Example

The following example illustrates how to use these results with a specific problem. Consider a "sled" of mass m_1 moving on a horizontal plane. See Figure 4.

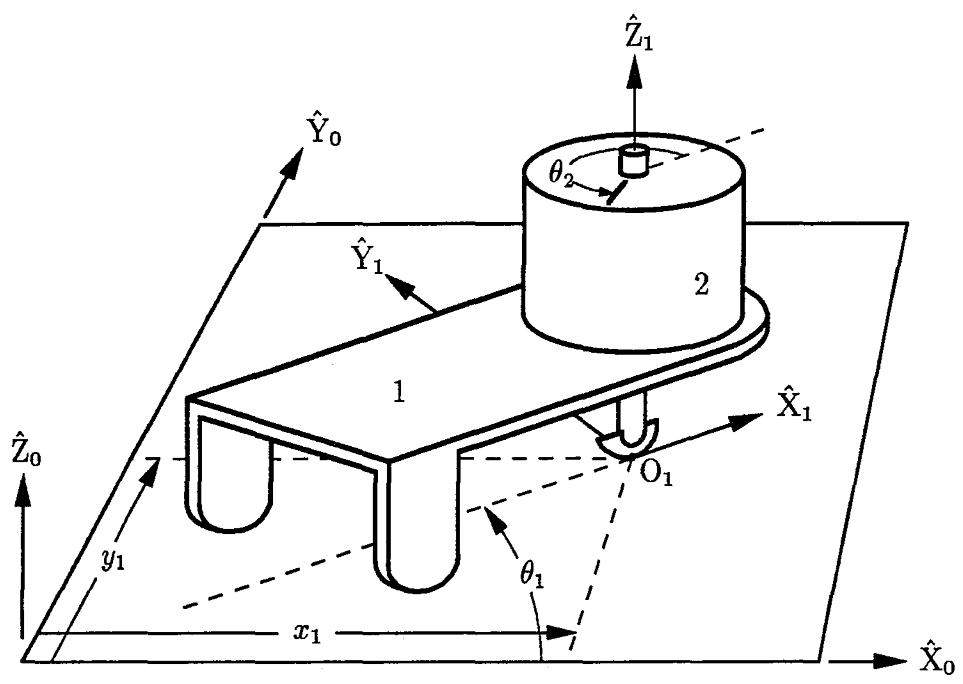

Figure 4: Sled (body 1) with rotational mass (body 2). The knife contacts the surface at O_i. The rear skids slide freely. The blade is rigidly attached to the sled (body 1).

The sled rides on a blade which touches the surface at one small spot at O_1. Its two rear skids slide freely on the surface but the blade on the front of the sled does not slide sideways. The blade is rigidly attached to the sled. On top of the sled (over the point of contact) is a rotational mass. The moment of inertia of the sled (body 1) about a vertical line through the point of contact, O_1, is I_1. The center of mass is ℓ away from the point of contact along a line through the point of contact and parallel to the blade. The Lagrangian, $\overline{L}$, (in terms of the quasi-velocities) for the sled (body 1) is:

$$\begin{aligned}
\overline{L}_1 &= \frac{1}{2}m_1 v_{CM1}^2 + \frac{1}{2}I_{CM1}\dot{\theta}_1^2 && (77)\\
&= \frac{1}{2}m_1(v_1^2 + (\omega_{12} - \ell\dot{\theta})^2) + \frac{1}{2}(I_1 - m_1\ell^2)\dot{\theta}_1^2 && (78)\\
&= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}m_1\omega_{12}^2 - m_1\ell\dot{\theta}_1\omega_{12} && (79)
\end{aligned}$$

where v_{CM1} is the velocity of the center of mass of the sled (body 1)—not the point of contact. Similarly, I_{CM1} is the moment of inertia about a vertical axis through the center of mass of the sled (body 1)—not the point of contact. Notice that ω_{12}, the velocity of the point of contact in the $\hat{Y}_1$ direction, is not (and must not be) assumed to be zero yet. In other words, the constraint (that the blade does not slip sideways) has not been enforced yet. This leads to the last two terms in Equation (79).

The moment of inertia of the rotational mass (body 2) about a vertical axis through the point of contact is I_2. The mass of body 2 is m_2. The Lagrangian for the rotational mass (body 2) is:

$$\overline{\mathrm{L}}_2 = \frac{1}{2}m_2 v_1^2 + \frac{1}{2}\mathrm{I}_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \tag{80}$$

The Lagrangian of the entire system is the sum of the two Lagrangians:

$$\overline{\mathrm{L}} = \overline{\mathrm{L}}_1 + \overline{\mathrm{L}}_2 \tag{81}$$

$$= \frac{1}{2}(m_1 + m_2)v_1^2 + \frac{1}{2}m_1\omega_{12}^2 - m_1\ell\dot{\theta}_1\omega_{12} + \frac{1}{2}\mathrm{I}_1\dot{\theta}_1^2 + \frac{1}{2}\mathrm{I}_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \tag{82}$$

In the terminology of this paper, this system has two "joints". The first is the knife joint—which is nonholonomic. The second joint is the revolute joint between the sled and the rotational mass—which is holonomic. The knife joint has two degrees of freedom and the revolute joint has one degree of freedom. To derive the equations of motion for the system, apply Equations (58) and (59) to produce two of the equations of motion. Apply Lagrange's equation for the revolute joint to generate the third equation of motion.

In preparation for applying Equations (58) and (59), the quasi-momentum P_{12} is needed:

$$P_{12} = \frac{\partial \overline{\mathrm{L}}}{\partial \omega_{i2}} = m_1\omega_{12} - m_1\ell\dot{\theta}_1 \tag{83}$$

Now the constraint can be enforced so that $\omega_{12} = 0$ (in P_{12} and in the Lagrangian):

$$P_{12} = -m_1\ell\dot{\theta}_1 \tag{84}$$

Now apply Equation (58) to derive the equation of motion for the knife's translational degree of freedom:

$$F_{v_1} = \frac{d}{dt}\left(\frac{\partial \overline{\mathrm{L}}}{\partial v_1}\right) - \cos\theta_1 \frac{\partial \overline{\mathrm{L}}}{\partial x_1} - \sin\theta_1 \frac{\partial \overline{\mathrm{L}}}{\partial y_1} - \dot{\theta}_1 P_{12} \tag{85}$$

$$= \frac{d}{dt}((m_1 + m_2)v_1) + m_1\ell\dot{\theta}_1^2 \tag{86}$$

$$= (m_1 + m_2)\dot{v}_1 + m_1\ell\dot{\theta}_1^2 \tag{87}$$

Similarly, apply Equation (59) to derive the equation of motion for the knife's rotational degree of freedom:

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial \overline{\mathrm{L}}}{\partial \dot{\theta}_1}\right) - \frac{\partial \overline{\mathrm{L}}}{\partial \theta_1} + v_1 P_{12} \tag{88}$$

$$= \frac{d}{dt}\left(\mathrm{I}_1\dot{\theta}_1 + \mathrm{I}_2(\dot{\theta}_1 + \dot{\theta}_2)\right) - m_1\ell\dot{\theta}_1 v_1 \tag{89}$$

$$= (\mathrm{I}_1 + \mathrm{I}_2)\ddot{\theta}_1 + \mathrm{I}_2\ddot{\theta}_2 - m_1\ell\dot{\theta}_1 v_1 \tag{90}$$

where τ_1 is an external torque turning the sled.

Finally, apply Lagrange's equation for the holonomic revolute joint:

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial \overline{\mathrm{L}}}{\partial \dot{\theta}_2}\right) - \frac{\partial \overline{\mathrm{L}}}{\partial \theta_2} \tag{91}$$

$$= \frac{d}{dt}\left(\mathrm{I}_2(\dot{\theta}_1 + \dot{\theta}_2)\right) \tag{92}$$

$$= \mathrm{I}_2(\ddot{\theta}_1 + \ddot{\theta}_2) \tag{93}$$

where τ_2 is the torque between the sled and the rotational mass.

Equations (87), (90), and (93) are the three equations of motion for this system.

Notice that this system is nonholonomic. The results of this paper lets the analyst treat the system in the same style as if it were holonomic. Using the predetermined Lagrange-style Equations (58) and (59) for the nonholonomic degrees of freedom and Lagrange's equations for the holonomic degrees of freedom allows relatively simple derivation of the system's reactionless equations of motion.

Summary and conclusions

This paper describes a new technique based on the Boltzmann-Hamel equations for deriving equations of motion for mechanisms which involve nonholonomic constraints. In this technique the BH equations are pre-applied for each particular type of joint to produce a small set of equations which are similar to Lagrange's equations. This set of equations can be efficiently applied to mechanisms which incorporate that type of joint to derive a minimum set of equations of motion which do not involve unknown reaction forces.

These equations can be applied to multibody mechanisms in several ways. They can be used to find equations of motion (as in the example in Section 5). These equations can be used to check results from other techniques for finding equations of motion. They can also be used to gain insights into the dynamics of such mechanisms. For instance, these pre-applied equations could be used to find steady motions for mechanisms incorporating nonholonomic constraints. (Steady motions resemble states of equilibrium even though some velocities are not zero.)

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