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Degrees of Relative Provability

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Abstract There are many classical connections between the proof-theoretic strength of systems of arithmetic and the provable totality of recursive functions. In this paper we study the provability strength of the totality of recursive functions by investigating the degree structure induced by the relative provability order of recursive algorithms. We prove several results about this proof-theoretic degree structure using recursion-theoretic techniques such as diagonalization and the Recursion Theorem.

1 Introduction

It is well known that it is sometimes difficult to prove the totality of a recursive function, especially if it is fast-growing. For example, Goodstein's Theorem, which is a Π^0_2-sentence in arithmetic, can be naturally interpreted as the totality of a recursive function; and it is known (see Kirby and Paris [5]) that we cannot prove Goodstein's Theorem in Peano Arithmetic (**PA**). Another similar example is the Modified Ramsey Theorem of Paris and Harrington (see [10]).

Note that the totality of a partial recursive function (with a fixed algorithm) is a Π^0_2-sentence in arithmetic, and conversely a Π^0_2-sentence in arithmetic can be naturally interpreted as the totality of a partial recursive function. Therefore the study of the totality of recursive functions can be viewed as the study of the Π^0_2-fragments of theories.

There are a lot of classical results on this subject (see, e.g., Fairtlough and Wainer [4] and Pohlers [11]), and the aim of this paper is to provide some ideas from a new viewpoint. We consider the degree structure of recursive algorithms ordered by the relative provability strength of their totality over some base theory (which we will discuss in the next section) and try to study some basic properties of this degree structure.

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There has also been a long history of studying different notions of "subrecursive reducibility," for example, primitive recursive degrees (see Kleene [6] and Axt [1], but restricted to recursive functions) and degrees of honest functions (see Kristiansen [7], [8] and Kristiansen, Schlage-Puchta, and Weiermann [9]). The main difference between these subrecursive degrees and our degree structure is that we try to order algorithms instead of functions (see the discussion after Definition 3.1), and it will be very interesting to know if our degree structure (with possibly different base theories) has some natural correspondence with any of the known subrecursive degrees.

One can think of our degree structure as Lindenbaum's algebra on true Π^0_2-formulae, and some basic operations we introduce (such as the join and the meet) are the corresponding operations (conjunction and disjunction) in Lindenbaum's algebra. An interesting point here is that, for example, by representing these true Π^0_2-formulae as algorithms, we have more freedom controlling the join and meet, and some small modification allows us to prove Proposition 7.1 without much effort. In addition, the theorems and proof techniques in this paper, for example, the incomparable pair theorem (Theorem 5.2), reveal some similarities between this proof-theoretic degree structure and the Turing degrees. Therefore we expect that this representation will allow us to utilize more complicated construction methods from recursion theory (e.g., the injury arguments) to produce results on provability.

More interestingly, we are trying to study the proof-theoretic strength of recursive functions by looking at their degree spectra in this degree structure (see Section 7). Our original intuition was that the degree spectrum of a recursive function may have a least element, which represents the "standard" or "most natural" algorithm for computing the recursive function. However, it is quite surprising that this attempt fails dramatically; that is, the only such degree spectrum is the whole set (see Theorem 7.2).

There are two notions of proofs: one is the social version, which is written in natural language and can be used in communication among mathematicians; the other is the formal version, which is defined as a sequence of formal sentences which satisfy certain properties and which is usually not used in communication (see more discussions in Buss [2]). In recursion theory, we have a similar phenomenon for computations: there are social and formal versions, and we mainly use the social version in communication and in the proofs we write. In fact, recursion theory has taken great advantage of using the social or informal version of computations so that we can write out much more complicated constructions such as injury arguments and tree constructions, which are almost impossible to write out in formal Turing machine language. In this paper, we will use social proofs when we argue for proofs in formal systems, but we shall keep in mind that, when we argue formally over some theory T, the natural numbers we use may be nonstandard, and the argument has to work with nonstandard numbers. If we do not specify that we are working in a formal system, then we always assume that we are arguing in the standard model of the natural numbers.

2 Base Theory

Let $\varphi_i(x)$ be a total recursive function with a fixed algorithm indexed by i. By the *totality* of φ_i we mean the sentence stating that φ_i is total, that is, $\forall x \exists s \varphi_{i,s}(x) \downarrow$. (Here we can use Kleene's T-predicate, which is primitive recursive, to express $\varphi_{i,s}(x)\downarrow$; i.e., the computation with an index e and an input x converges within s steps.) For convenience we use $\mathrm{tot}(\varphi)$ to denote this sentence.

In reverse mathematics, we choose a weak base theory ($\mathbf{RCA_0}$) which is powerful enough to develop the basics of mathematics: pairs, strings, formulae, sentences, proofs, and so on; then we can discuss the relative provability of sentences and theories over it.

Similarly, here we also want to fix a base theory T. However we are not specific about which theory T we use here: it just needs to be strong enough in a sense which we will explain, and in particular, it could be very strong and does not have to be in a fixed language. All of our discussions and all the theorems will work for any possible theory satisfying the requirements we list below.

First, we require T to be axiomatizable, that is, to have a recursive list of axioms. This is to say, given a proof (coded as some number s), we can effectively tell whether it is a valid proof in T. In the theorems we are going to prove, we will construct recursive functions by diagonalization, for example, diagonalize against all possible functions whose totality is provable in the theory T; therefore it is crucial that we can recursively decide whether a proof is a proof in T. It is convenient here to assume that the coding of the recursive list of T-axioms is a provably total function in T (e.g., we do not need to run a Goodstein sequence computation to decide whether a sentence is an axiom in T), but this is not crucial, since we can modify the definition of a proof and require each axiom to have an affiliated computation which confirms that it is an axiom.

We require that T be consistent with true arithmetic (**TA**) in the following sense: In T one has a fixed interpretation of arithmetic; that is, one can define the domain of natural numbers, zero, and the successor operation, together with plus, times, and the order of natural numbers. Then each sentence ψ in arithmetic has a translation $\ulcorner\psi\urcorner$ in the language of T, and we require that if $T\vdash\ulcorner\psi\urcorner$, then $\mathbf{TA}\vdash\psi$. So if we can prove in T that some partial recursive function φ_e is total (i.e., $T\vdash\ulcorner\mathrm{tot}(\varphi_e)\urcorner$), then it is total. (For convenience, we will omit $\ulcorner\urcorner$ in the notation and simply write $T\vdash\mathrm{tot}(\varphi_e)$.)

In addition, like $\mathbf{RCA_0}$, with the interpretation of arithmetic as above, T is powerful enough to develop the basic notions of pairs, strings, sentences, and proofs; therefore we can write out formal sentences such as $T\vdash\varphi$ for some sentence φ in the language of T. It can also express the notion of Turing machines and the computation sequence of Turing machines. In particular, we can talk about φ_e, the eth partial recursive function.

Finally, we want to impose a convention on our partial recursive functions. We automatically convert them into partial recursive functions satisfying the convention. We need $\mathbf{I\Sigma_1}$ to show that we can do such a conversion, and the new function is "equivalent" to the old one as far as we are concerned. We will discuss the details in the next section.

Therefore $\mathbf{I\Sigma_1}$ is a good candidate for our base theory T, and any stronger theory will also work. Note that some of the theorems we will prove may not require the convention we impose, and so they may work in even weaker base theories. It might also be interesting to discuss our degree structure when the base theory is weaker than $\mathbf{I\Sigma_1}$, but this is beyond the scope of this paper. Similarly, we only consider total functions/algorithms in this paper, and it might be quite interesting to consider degrees of partial functions, in particular these functions which are consistently total but are not actually total.

3 Basics

We start with the definition of the order.

Definition 3.1 Given total functions φ_i and φ_j, we say that φ_i is *provably reducible* to φ_j if $T + \mathrm{tot}(\varphi_j) \vdash \mathrm{tot}(\varphi_i)$. We denote this by $\varphi_i \leq_p \varphi_j$.

It is easy to see that $\leq_p$ is reflexive and transitive, so it naturally induces a degree structure: φ_i and φ_j are *provably equivalent* ($\varphi_i \equiv_p \varphi_j$) if they are provably reducible to each other. Given φ_i, we use $[\varphi_i]$ to denote the collection of φ_j's that are provably equivalent to φ_i, that is, the equivalence class of φ_i. We will call such $[\varphi_i]$ a *provability degree*, and we use $\mathcal{P}$ to denote the class of all provability degrees, ordered by the induced provability reducibility.

It is very important to note here that, strictly speaking, this order is defined on recursive *algorithms* instead of recursive *functions*. As we will discuss in Section 7, each function f has many *representations* φ_i; each is the same function as f, but these representations might have different provability degree. In this paper, we will use Greek letters φ, ψ, θ for (possibly partial) recursive functions with fixed algorithms, and use f, g, h for recursive functions as functions. So when we write φ, it is assumed that it corresponds to a fixed algorithm φ_i or that we are constructing such a φ and we refer to the algorithm we are defining.

We use 0 to denote the constant zero function computed in the easiest way, that is, output 0 regardless of input, so [0] is the bottom degree in $\mathcal{P}$. It is also the collection of all functions whose totality is provable in T.

To simplify some arguments, we will impose the following convention, which is commonly used in recursion theory: for a recursive function φ, if $\varphi(x)$ converges in s steps, then all $\varphi(y)$ for $y < x$ converge in less than s steps. More precisely, given any recursive function φ, we first convert φ to another function $\tilde{\varphi}$: for each x, we wait for all $\tilde{\varphi}(y)$ for $y < x$ to converge before we start computing $\varphi(x)$, then output the value $\varphi(x)$ as $\tilde{\varphi}(x)$. We need to show that $[\varphi] = [\tilde{\varphi}]$ (and so we can always assume that every function satisfies this convention). One direction $\varphi \leq_p \tilde{\varphi}$ is trivial. For the other direction $\tilde{\varphi} \leq_p \varphi$, we assume that $\tilde{\varphi}$ is not total. Then we can find the least x where $\tilde{\varphi}(x)$ does not converge (using $\mathbf{I\Sigma}_1$), and so get a contradiction with the construction of $\tilde{\varphi}$ and the fact that $\varphi(x)$ converges. One important fact based on our convention is that if φ converges at infinitely many inputs, then φ is total.

Next we show that in $\mathcal{P}$ one can define a join and a meet operator, and in fact $\mathcal{P}$ is a distributive lattice.

Definition 3.2 Given two recursive functions φ and ψ, the join of φ and ψ, denoted as $\varphi \boxplus \psi$, is defined as the following recursive function: for each input n, we compute $\varphi(n)$ and $\psi(n)$ simultaneously, and output $(\varphi \boxplus \psi)(n) = 0$ only if both converge. The meet of φ and ψ, denoted as $\varphi \boxtimes \psi$, is defined in a very similar way, except that we output $(\varphi \boxtimes \psi)(n) = 0$ if either $\varphi(n)$ or $\psi(n)$ converges.

Note that by these two definitions we really mean that we find two recursive functions $k_{\boxplus}$ and $k_{\boxtimes}$ (which are provably total by the s-m-n theorem) such that $\varphi_i \boxplus \varphi_j = \varphi_{k_{\boxplus}(i,j)}$ and $\varphi_i \boxtimes \varphi_j = \varphi_{k_{\boxtimes}(i,j)}$.

Also note that in the above definitions, one can change the output value quite arbitrarily: for $(\varphi \boxplus \psi)(n)$, we can also use either $\varphi(n)$ or $\psi(n)$ as the output value, since we need to wait for both to converge; for $(\varphi \boxtimes \psi)(n)$, we may use the value of $\varphi(n)$ or $\psi(n)$, whichever converges first.

Recall that in a partial order, $[\varphi] \vee [\psi]$ denotes the join of two degrees and $[\varphi] \wedge [\psi]$ denotes the meet. One can easily show the following.

Proposition 3.3 $[\varphi \boxplus \psi] = [\varphi] \vee [\psi]$, *and* $[\varphi \boxtimes \psi] = [\varphi] \wedge [\psi]$.

Proof For the first claim, first note that $[\varphi] \leq [\varphi \boxplus \psi]$ and $[\psi] \leq [\varphi \boxplus \psi]$ follows directly from the definition. Given $[\theta]$, which is above both $[\varphi]$ and $[\psi]$, $[\theta]$ is also above $[\varphi \boxplus \psi]$ since we can prove the totality of $\varphi \boxplus \psi$ from the totality of both φ and ψ, which we can get from the totality of θ.

For the second claim, it is easy to see that φ being total guarantees that $\varphi \boxtimes \psi$ is total, and similarly ψ being total also proves that $\varphi \boxtimes \psi$ is total. Now given some θ whose degree is below both $[\varphi]$ and $[\psi]$, we want to show that $\theta \leq_p \varphi \boxtimes \psi$. Note that the totality of $\varphi \boxtimes \psi$ shows that either φ is total or ψ is total by our convention. Then since we have proofs that $\theta \leq_p \varphi$ and $\theta \leq_p \psi$, we know that θ has to be total in either case. □

Thus $\mathscr{P}$ is a lattice with these two operations, and distributivity easily follows from the distributivity of "and" and "or" in Definition 3.2.

Here is a lemma which will be used later but which is convenient to make explicit now.

Lemma 3.4 *If* $T \vdash \varphi = \psi$, *then* $[\varphi] = [\psi]$.

By $\varphi = \psi$ we mean that, for each n, if $\varphi(n)$ converges, then $\psi(n)$ has to converge to the same value, and vice versa. (Therefore they are the same as partial functions.) The proof is almost obvious.

4 Jump Operator

As in the Turing degrees, we can also define a natural jump operator in the provability degrees.

Definition 4.1 Given a recursive function φ, the jump of φ, denoted as φ^*, is defined inductively as follows: at each stage s, we check whether s is (the Gödel number of) a proof witnessing $T + \mathrm{tot}(\varphi) \vdash \mathrm{tot}(\varphi_e)$ for some recursive function φ_e. If so, we let $\varphi^*(s) = \varphi_e(s) + 1$; if not, we let $\varphi^*(s) = 0$. Then we proceed to the next stage $s + 1$.

By the s-m-n theorem, this definition also corresponds to a recursive function k_* such that $\varphi_{k_*(i)} = \varphi_i^*$. Before we show that this is a good definition of a jump operator, we will need the following Padding Lemma (and that it is provable in T).

Lemma 4.2 (Padding Lemma) *Suppose that s is (a code of) a proof which witnesses $\Sigma \vdash \psi$; then there exists an infinite recursive list $s_0, s_1, \ldots, s_n, \ldots$ of proofs witnessing the same result. Moreover, s_i as a function of i is provably total in T.*

The proof is almost obvious since we can add arbitrary redundancy into proofs. This lemma works for proofs in the same way as the Padding Lemma in recursion theory does for algorithms.

Proposition 4.3 *We have $T + \mathrm{tot}(\varphi^*) \vdash$ "if $T + \mathrm{tot}(\varphi) \vdash \mathrm{tot}(\varphi_e)$, then φ_e is total," and in fact for every ψ, $[\psi] \geq [\varphi^*]$ if and only if $T + \mathrm{tot}(\psi) \vdash$ "if $T + \mathrm{tot}(\varphi) \vdash \mathrm{tot}(\varphi_e)$, then φ_e is total."*

Proof We argue in $T + \mathrm{tot}(\varphi^*)$ for the first claim. If we have a proof s witnessing $T + \mathrm{tot}(\varphi) \vdash \mathrm{tot}(\varphi_e)$, then by the padding lemma we have infinitely many proofs $s_0, s_1, \ldots$ witnessing the same result. Therefore in the construction of φ^* at each of these stages s_i we make $\varphi^*(s_i) = \varphi_e(s_i) + 1$. Then it is easy to see that the totality of φ^* guarantees the totality of φ_e (by our convention).

If $T + \mathrm{tot}(\psi)$ proves the assertion that every $T + \mathrm{tot}(\varphi)$-provably total function is total, then it is easy to see that $T + \mathrm{tot}(\psi)$ proves the totality of φ^*, and so $[\psi] \geq [\varphi^*]$. □

This proposition directly shows that $[\varphi] \leq [\varphi^*]$ since $T + \mathrm{tot}(\varphi) \vdash \mathrm{tot}(\varphi)$. It is direct from the diagonalization that $[\varphi^*] \not\leq [\varphi]$, therefore $[\varphi] < [\varphi^*]$, that is, $[\varphi^*]$ is strictly above $[\varphi]$.

Corollary 4.4 *There are no maximal degrees in $\mathcal{P}$.*

More importantly, the jump operator preserves $\leq_p$.

Proposition 4.5 *If $\varphi \leq_p \psi$, then $\varphi^* \leq_p \psi^*$.*

Proof We prove this by contradiction (in $T + \mathrm{tot}(\psi^*)$). Assume that φ^* is not total; then there exists a least s where $\varphi^*(s)$ diverges (by $\mathbf{I\Sigma}_1$), so s is a proof witnessing $T + \mathrm{tot}(\varphi) \vdash \mathrm{tot}(\varphi_e)$, and $\varphi_e(s)$ diverges. Since we have a proof witnessing $T + \mathrm{tot}(\psi) \vdash \mathrm{tot}(\varphi)$, from s we can find an s' which is a proof witnessing $T + \mathrm{tot}(\psi) \vdash \mathrm{tot}(\varphi_e)$. Then by the padding lemma we can find an s'' which proves the same result as s' and $s'' > s$. The totality of ψ^* shows that $\varphi_e(s'')$ converges, which contradicts the divergence of $\varphi_e(s)$. □

This proposition shows that the jump operator is well defined on the degrees, and we can write $[\varphi]^*$ for $[\varphi^*]$ since it is independent of the choice of the function φ in $[\varphi]$.

5 Incomparable Degrees

Before we prove our next theorem, we want to mention a technique for constructing recursive functions using the Recursion Theorem. Briefly, we can assume that we know the index of the function we are constructing prior to our construction. This may sound very strange to readers who are not familiar with recursion theory, and so we shall explain it in detail. The Recursion Theorem is stated below, and a proof can be found in Rogers [12] or any other standard textbook in classical recursion theory.

Theorem 5.1 (Recursion Theorem) *For every recursive function f, there is an index e such that $\varphi_e = \varphi_{f(e)}$.*

We will use the Recursion Theorem in the following way. We first give an explicit construction of a partial recursive function with a parameter i; then by the s-m-n theorem, there is a recursive function h such that $\varphi_{h(i)}$ is the function constructed from parameter i; we apply the Recursion Theorem to h and get an index e such that $\varphi_e = \varphi_{h(e)}$ and let this function be ψ. That is, when we describe the construction of ψ we can assume that we already know its index e.

There are two points to make. First, our base theory T is powerful enough to prove the Recursion Theorem, and by the proof of the Recursion Theorem, T actually proves $\varphi_e = \varphi_{h(e)}$ for one index e assuming that $h(e)$ converges. Although in this paper we only need h's that are provably total in T, this argument actually works for any total h regardless of its provability degree. By Lemma 3.4, $[\varphi_e] = [\varphi_{h(e)}]$, and so we can identify φ_e with $\varphi_{h(e)} = \psi$ in our argument.

Second, in the Recursion Theorem it is possible that the φ_e we get is partial, and so we need to make sure that, in the construction of $\varphi_{h(i)}$, regardless of the parameter i we use in the construction, $\varphi_{h(i)}$ is always total. Then by applying the Recursion Theorem, we always get a total function ψ.

Here is the theorem we want to prove. It directly implies that there are incomparable degrees in $\mathcal{P}$.

Theorem 5.2 *For every* $[\varphi] \neq [0]$*, there is a* $[\psi] \leq [\varphi]^*$ *which is incomparable with* $[\varphi]$*, that is,* $[\varphi] \nleq [\psi]$ *and* $[\psi] \nleq [\varphi]$.

Proof Our construction of ψ is divided into even and odd stages. At even stages we try to satisfy $[\psi] \nleq [\varphi]$. This is handled in the same way as the construction of the jump: at stage $2s$ we check whether s is a proof witnessing $T + \mathrm{tot}(\varphi) \vdash \mathrm{tot}(\varphi_e)$ for some φ_e; if so we compute $\varphi_e(s)$ and let $\psi(s) = \varphi_e(s) + 1$; if not we simply let $\psi(s) = 0$.

At odd stages we try to satisfy $[\varphi] \nleq [\psi]$; that is, we cannot prove the totality of φ from the totality of ψ. At stage $2s + 1$, we check whether s is a proof witnessing $T + \mathrm{tot}(\psi) \vdash \mathrm{tot}(\varphi)$ (by the Recursion Theorem). If not, do nothing; if so, we terminate the whole construction and let ψ be the constant zero function afterward, that is, we output $\psi(t) = 0$ for all $t > s$. (In this case we say that we apply *annihilation* to ψ.)

First of all, it is easy to check that, no matter which φ_i we use, or whether annihilation happens or not, $\varphi_{h(i)}$ is always total. So the Recursion Theorem gives us a total ψ.

Next we need to show that $[\varphi] \nleq [\psi]$. If $[\varphi] \leq [\psi]$, then we can pick the least proof s_0 witnessing $T + \mathrm{tot}(\psi) \vdash \mathrm{tot}(\varphi)$. We can then show that the annihilation happens in the construction and ψ is then eventually constant 0, and in fact we can prove its totality in T. We write down the first such proof s_0 and verify that it is the proof we want; then for each $s < s_0$ we check that s is not a proof witnessing $T + \mathrm{tot}(\psi) \vdash \mathrm{tot}(\varphi)$, so annihilation does not happen before stage $2s_0 + 1$; also for each $s \leq s_0$ we write down the complete computation sequence of $\varphi_{e_s}(s)$ at even stages such that s is a proof witnessing $T + \mathrm{tot}(\varphi) \vdash \mathrm{tot}(\varphi_{e_s})$. Our base theory T is consistent with true arithmetic, and φ is total; therefore if we can prove that φ_{e_s} is total, then $\varphi_{e_s}(s)$ does converge and we can write out the complete computation sequence to prove that it converges (without proving the totality of φ_{e_s}).

So combining the sentences we write down, we can show that at stage $2s_0 + 1$ the annihilation happens, that is, we get a proof witnessing $T \vdash \mathrm{tot}(\psi)$. Together with the proof s_0 witnessing $T + \mathrm{tot}(\psi) \vdash \mathrm{tot}(\varphi)$, we would get $T \vdash \mathrm{tot}(\varphi)$, which contradicts the assumption that $[\varphi] \neq [0]$.

Therefore annihilation never happens and $[\varphi] \nleq [\psi]$. By the diagonalization at even stages we have also satisfied $[\psi] \nleq [\varphi]$. So $[\psi]$ is incomparable with $[\varphi]$.

Finally, we check that $[\psi] \leq [\varphi]^*$. Arguing in $T + \mathrm{tot}(\varphi^*)$, if annihilation happens, then ψ is total. If not, then ψ is the same function as φ^* by the construction at even stages. Since φ^* is total, ψ is also total in this case. $\square$

It is interesting to note here that this ψ, as a function, is the same as the jump φ^* but is strictly weaker in the degrees (as an algorithm). This shows that $[\varphi^*]$ is not the lowest degree which contains an algorithm that computes the function φ^*. Later in Theorem 7.2 we will show that this phenomenon actually holds for every nonzero degree.

6 Minimal Pair

As in the Turing degrees, we can also find minimal pairs in the provability degrees.

Theorem 6.1 *There are two nonzero degrees* $[\varphi]$ *and* $[\psi]$ *such that* $[\varphi] \wedge [\psi] = [0]$.

Proof We construct φ and ψ simultaneously by induction. Suppose that at stage s we have already defined φ and ψ up to x_s and y_s, respectively.

Let $s = 2m$ be an even stage. We check whether m is a proof witnessing $T \vdash \mathrm{tot}(\varphi_e)$ for some φ_e. If so, we define φ and ψ as follows:

$$\begin{cases} \varphi(x_s) = \varphi_e(x_s) + 1, & \\ \psi(y) = 0, & \text{for } y \in [y_s, y_s + t), \end{cases}$$

where t is the number of steps needed to compute $\varphi_e(x_s)$. This is to say, we try to compute $\varphi_e(x_s)$ and extend ψ by letting $\psi(y) = 0$ for $y \geq y_s$ until $\varphi_e(x_s)$ converges. So if $\varphi_e(x_s)$ diverges, then φ is partial and ψ is total (and eventually 0). If $\varphi_e(x_s)$ converges at step t, then we have made $\psi(y) = 0$ for $y \in [y_s, y_s + t)$, and we let $\varphi(x_s) = \varphi_e(x_s) + 1$ for diagonalization. Then we go to stage $s + 1$ with $x_{s+1} = x_s + 1$ and $y_{s+1} = y_s + t$.

If m is not a proof witnessing $T \vdash \mathrm{tot}(\varphi_e)$ for any e, then we simply let $\varphi(x_s) = \psi(y_s) = 0$ and go to stage $s + 1$ with $x_{s+1} = x_s + 1$ and $y_{s+1} = y_s + 1$.

At odd stages, we basically do the same thing, except that we switch the roles of φ and ψ, that is, we try to diagonalize with ψ and extend φ by 0 until the corresponding computation converges.

Since T is consistent with true arithmetic, it is easy to see that φ and ψ are both total and both have nonzero degree by diagonalization. Then we prove in T that $\varphi \boxtimes \psi$ is total. If we ever met a divergent $\varphi_e(x_s)$ or $\psi_e(y_s)$ in the construction, then we would make φ or ψ eventually constant 0, and so $\varphi \boxtimes \psi$ would still be total. Otherwise both φ and ψ are total, and $\varphi \boxtimes \psi$ is obviously total. In either case $\varphi \boxtimes \psi$ is total, so $[\varphi] \wedge [\psi] = [0]$. □

It is not difficult to show that in fact $[\varphi] \vee [\psi] = [0]^*$: first it is easy to check that they are both below $[0]^*$; conversely $\mathrm{tot}(\varphi)$ and $\mathrm{tot}(\psi)$ together prove that every φ_e whose totality is provable in T is total, and so by Proposition 4.3 we know that they also prove the totality of 0^*. This shows that $[0]^*$ is the top of a "diamond."

7 Degree Spectrum and Minimal Degrees

A given recursive function f has many *representations* $\varphi_{e_0}, \varphi_{e_1}, \ldots$; that is, each $\varphi_{e_i} = f$ as functions, and they may have different provability degrees. So it is natural to give the following definition.

Given a recursive function f, we define the *degree spectrum* of f, denoted as (f), to be the collection of provability degrees that contain a function which is the same as f as functions, that is, $(f) = \{[\varphi_i] : f = \varphi_i\}$.

It is not difficult to show that the degree spectrum is closed upward and closed under meet.

Proposition 7.1

(1) *If* $[\psi] > [\varphi]$*, then there is a* θ *such that* $\theta = \varphi$ *as functions and* $[\theta] = [\psi]$.

(2) *If* $\varphi = \psi$ *as functions then there is a* θ *such that* $\varphi = \psi = \theta$ *as functions and* $[\theta] = [\varphi \boxtimes \psi]$.

Proof In the definition of the join and the meet, we noted that the output values for $\varphi \boxplus \psi$ or $\varphi \boxtimes \psi$ can be quite arbitrary. For the first claim, we can change the output value of $(\varphi \boxplus \psi)(n)$ to be $\varphi(n)$, and this gives a θ we want. For the second claim, we change the output value of $(\varphi \boxtimes \psi)(n)$ to be either $\varphi(n)$ or $\psi(n)$, whichever converges first, and this also gives a desired θ. □

For example, (0), the degree spectrum of the constant zero function (as a function), is the collection of all degrees. In contrast, (0*), the degree spectrum of 0* (as a function), does not contain the bottom degree [0]. One might naturally ask whether (0*) has a minimum element, or whether there are other degree spectra which are principal ones (i.e., contain a minimum element).

Interestingly, the answer is no by the following theorem; that is, the only principal degree spectrum is (0).

Theorem 7.2 *Given* $[\varphi] \neq [0]$*, then there is a* ψ *such that* $\psi = \varphi$ *as functions and* $[\psi] < [\varphi]$.

Proof We again divide the construction of ψ into even and odd stages. At stage $2s$ we let $\psi(s) = \varphi(s)$, that is, follow the same algorithm and output the same value. At stage $2s + 1$ we check whether s is a proof witnessing $T + \mathrm{tot}(\psi) \vdash \mathrm{tot}(\varphi)$ (by the Recursion Theorem). If not, we do nothing; if so, we let ψ be a constant 0-function afterward. (Similarly we call such a process the *annihilation* of ψ.)

By the same argument as in the proof of Theorem 5.2 we can show that $[\varphi] \not\leq [\psi]$ and so $\psi = \varphi$ as functions. It is also easy to argue that $[\psi] \leq [\varphi]$: if annihilation happens, then ψ is total; if not, then by the totality of φ and our construction at even stages we also know that ψ is total. □

One might have some intuition that the provability degree of a recursive function φ corresponds to the growth rate of the computing time function $\bar{\varphi}$ (i.e., a function which outputs the number of steps in the computation) of φ,[1] or that the functions which need more computing time have higher provability degree. However, the above construction shows that such an intuition is not true: our new function ψ needs more steps in the computation than φ, but has strictly lower degree. It is true, in contrast, that if T proves that $\bar{\varphi}$ dominates $\bar{\psi}$, then $[\varphi] \geq [\psi]$.

This theorem also gives strict conditions on minimal degrees (though we do not know whether they exist). Recall that a nonzero degree is *minimal* if there is no nonzero degree strictly below it.

Corollary 7.3 *If* $[\varphi]$ *is minimal, then for any* $\theta \in [\varphi]$*,* θ *has a representation which is provably total. In addition, all minimal degrees (if they exist) are below* $[0]^*$.

Proof The first claim directly follows from the construction in Theorem 7.2: since ψ has degree strictly below $[\varphi]$, it must be the case that $[\psi] = [0]$. For the second claim, we follow the same construction and get a ψ from φ. By the same reason, $T \vdash \text{tot}(\psi)$, and then we can prove $\text{tot}(\varphi)$ from $\text{tot}(0^*)$ as follows.

Suppose that φ is not total (at s). Since ψ is total, it must be the case that annihilation happens at some stage before $2s$. Therefore we have a proof witnessing $T + \text{tot}(\psi) \vdash \text{tot}(\varphi)$, and combining it with $T \vdash \text{tot}(\psi)$ we get a proof witnessing $T \vdash \text{tot}(\varphi)$. By Proposition 4.3, we know that φ is total (since we are arguing in $T + \text{tot}(0^*)$). This contradicts the assumption that φ is not total; therefore φ is total. □

8 Open Questions

We end with some open questions. First, we want to know whether we can "control the jump" as in various theorems in the Turing degrees. In particular we can ask whether a jump inversion theorem holds in $\mathcal{P}$, that is, whether every $[\varphi] \geq [0]^*$ is the jump of a degree $[\psi] < [\varphi]$. We are also interested in characterizations of degrees below $[0]^*$ (e.g., whether there is an analogous version of the Limit Lemma).

One can also ask questions about various notions from Turing degree theory, for example, the cupping property, the join property, high/low hierarchy, diamond bounding (see Theorem 6.1), and so on. Continuing the discussion of Section 7, we want to know whether minimal degrees exist. It might also be interesting to consider different embedding problems, such as partial orders or distributive lattices, especially if we want to study the decidability or even the degree of the theory of $\mathcal{P}$.

In addition to these *degree-theoretic properties*, we are also interested in *combinatorial properties*, that is, the combinatorial aspects of the functions in each degree, which might be useful in studying specific number-theoretic or combinatorial examples, such as the function associated with Laver Tables (see Dehornoy [3]). For example, one can define a function f to be *diagonally nonprovable* (DNP) if $f(s) \neq \varphi_{e_s}(s)$ for every s which is a proof witnessing $T \vdash \text{tot}(\varphi_{e_s})$ for some φ_{e_s}, and say that a degree is **DNP** if it contains a DNP function. (This is motivated by the definition of **DNR** degrees in recursion theory.) It is easy to see that **DNP** degrees are not zero, and one may ask whether there are nonzero non-**DNP** degrees.

Another important class of combinatorial properties is the class of domination properties. A function f *dominates* g if $f(x) \geq g(x)$ for cofinitely many x. For example, say a degree is $[0]$-*dominated* (or *hyperimmune-free*, using classical terminology from recursion theory) if every function in it is dominated by a function in $[0]$. We can also ask whether there are nonzero $[0]$-dominated degrees. (Note that Corollary 7.3 shows that minimal degrees, if they exist, are $[0]$-dominated and non-**DNP**.) In particular, the notions of domination properties may be more closely related to some old research about the fast-growing hierarchy (see [4]) or the degree theory about the honest functions (see [7]).

As we mentioned in the introduction, the attempt to find a least element in a nontrivial degree spectrum always fails (see Theorem 7.2), so we can also try to find natural variations of this degree structure where one can find least elements in degree spectra.

Note

1. Note that $\bar{\varphi}$ and φ have the same provability degree, since their computations are almost the same, and the only difference is the outputs.

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