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Grigorieff Forcing on Uncountable Cardinals Does Not Add a Generic of Minimal Degree

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Abstract Grigorieff showed that forcing to add a subset of ω using partial functions with suitably chosen domains can add a generic real of minimal degree. We show that forcing with partial functions to add a subset of an uncountable κ without adding a real never adds a generic of minimal degree. This is in contrast to forcing using branching conditions, as shown by Brown and Groszek.

1 Introduction

Two classic ways of adding a generic real to a model of ZFC are Cohen forcing, whose conditions are partial functions from ω to 2 with finite domain, and Silver forcing, whose conditions are partial functions from ω to 2 with coinfinite domain. A Cohen real is not of minimal real degree over the ground model, while a Silver real is.

Grigorieff [2] generalized these two notions of forcing as follows: Given any ideal on ω, the forcing conditions of the associated generalized Cohen, or Silver, forcing are partial functions from ω to 2 whose domains are in the ideal, or not in the dual filter, respectively. Grigorieff found necessary and sufficient conditions on the ideal to guarantee the generalized Cohen generic is of minimal degree. (Generalized Silver forcing reduces to generalized Cohen forcing if the ideal is maximal, and otherwise factors so that the second factor is a generalized Cohen forcing in the intermediate model.)

Groszek [3] extended Grigorieff's ideas to tree forcings, generalizations of Laver and Miller forcings defined by requiring branching sets to be in some filter or, alternatively, not to be in the dual ideal. Groszek obtained conditions on the filter sufficient to guarantee the generic real is of minimal degree. Much later, Brown and Groszek [1] generalized these ideas to uncountable cardinals, obtaining analogous

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conditions on a filter on κ sufficient to guarantee that a tree forcing adds a generic subset of κ of minimal degree.

One can attempt to do the same for Grigorieff's original forcing generalized to uncountable κ: Consider a forcing whose conditions are partial functions from κ to 2 whose domains are in some ideal on κ, or whose domains are not in the dual filter. Do Grigorieff's results generalize in a straightforward way, so that analogous conditions on the ideal are necessary and sufficient to guarantee the generic is of minimal degree? If not, what conditions on the ideal will suffice?

The answer, as we show here, is that no conditions will suffice. This is in stark contrast to the situation with tree forcings. Specifically, a consequence of our main theorem, Theorem 3.3, is the following.

Theorem 1.1 *Let $\mathcal{I}$ be any κ-complete ideal on an uncountable cardinal κ, and $\mathcal{F}$ the dual filter. Define forcing partial orders by*

$$\mathbb{P} = \{p : X \to 2 \mid X \in \mathcal{I}\}$$
$$\mathbb{P}' = \{p : X \to 2 \mid X \notin \mathcal{F})\},$$

where in both cases p is stronger than q if p extends q as a partial function,

$$p \leq q \iff p \supseteq q.$$

Then each of $\mathbb{P}$ and $\mathbb{P}'$ adds a generic function from κ to 2 that is not of minimal degree over the ground model.

It follows from the proof that the generic is not even of minimal κ-degree, where a κ-degree is a degree containing a subset of κ. This does not answer the question of whether there is some other set in the generic extension of minimal degree, or of minimal κ-degree.

Dorais has made the following observation: From the set f we will show to be of intermediate degree, one can construct the tree of binary $< \kappa$-sequences that differ from the generic κ-sequence g at only finitely many places. Provided κ is of uncountable cofinality, any branch through this tree constructs g. Since $g \notin M[f]$, it follows that κ is not weakly compact in the intermediate model $M[f]$. This holds even if, for example, the measurability of κ is indestructible by κ-closed forcing, so that (in the case of $\mathbb{P}$ above) κ is measurable in both M and $M[G]$.

In Section 2 we discuss what it means to be of minimal degree over some model M of ZFC. The reader familiar with these concepts can skip this section. In Section 3 we state and prove our main theorem.

2 Inner Models and Minimal Degrees

Definition 2.1 An inner model of ZFC is a class M that contains all the ordinals and satisfies all the axioms of ZFC.

In the context of a model V of ZFC a class M is generally taken to be definable in V with parameters. We may broaden that definition to include an externally defined submodel M of V, provided that V still satisfies the axioms of ZFC (specifically the replacement and separation schemes) when a predicate for M is added to the language. For instance, if V is a generic extension of M, it is not obvious that M is definable in V, but M is still a class in this broader sense.

Definition 2.2 Let M be an inner model of ZFC and X a set of ordinals. Then $M[X]$ is the smallest inner model N of ZF with $M \subseteq N$ and $X \in N$.

Proposition 2.4 below gives a precise characterization of $M[X]$ and shows that $M[X]$ satisfies the axiom of choice. For the skeptical, it also shows the existence of $M[X]$.

Definition 2.3 If M is an inner model of ZFC and X and Y are sets of ordinals, we define

$$X \leq_M Y \iff X \in M[Y] \quad \text{and}$$
$$X \equiv_M Y \iff X \leq_M Y \,\&\, Y \leq_M X.$$

Then the degree of X over M is the collection of all sets Y such that $Y \equiv_M X$.

In particular, X is of minimal degree over M just in case

$$(\forall Y \subset \mathrm{ORD})\,[Y \in M[X] \implies (Y \in M \ \vee\ X \in M[Y])]$$

and $X \subset \kappa$ is of minimal κ-degree over M just in case

$$(\forall Y \subset \kappa)\,[Y \in M[X] \implies (Y \in M \ \vee\ X \in M[Y])].$$

By extension, we may talk of the degree over M of any set naturally coded by a set of ordinals (for example, a set of pairs of ordinals), or in fact of any subset of M.

Proposition 2.4 *Let M be any inner model of* ZFC *and X any set of ordinals. Let*

$$N = \bigcup_{Y \subset \mathrm{ORD}\ \&\ Y \in M} L[X,\, Y].$$

Then $N = M[X]$.

Proof Clearly $M[X]$ must contain every $L[X, Y]$ with $Y \in M$, so must contain N. Therefore, it suffices to show that N is in fact a model of ZF. By a theorem of Hajnal [4] (see Jech [5], pp. 97–99), it suffices to show that N is closed under the Gödel operations and satisfies the universal cover property, namely, if Z is any set and $Z \subseteq N$, then there is some $W \in N$ with $Z \subseteq W$.

Because N is a directed union of models of the form $L[X, Y]$, which are closed under Gödel operations, N itself is closed under Gödel operations. (Also, as each $L[X, Y]$ satisfies the axiom of choice, so does N.) To show the universal cover property, suppose $Z \subset N$, and consider the function f on Z mapping z to the least ordinal α such that

$$(\exists Y \subseteq \alpha)\,[Y \in M \,\&\, z \in L[X, Y]].$$

Let β be a bound on the range of f. Because M satisfies the axiom of choice, there is a set of ordinals $\mathcal{Y} \in M$ coding every subset of β in M:

$$(\forall Y \in M)\,[Y \subseteq \beta \implies Y \in L[X, \mathcal{Y}]].$$

Now, if $z \in Z$, then $z \in L[X, Y]$ for some $Y \in M$ with $Y \subseteq \alpha \leq \beta$, from which $Y \in L[X, \mathcal{Y}]$ and $z \in L[X, \mathcal{Y}]$. Thus $Z \subset L[X, \mathcal{Y}]$, and because the ranks of elements of Z are bounded, $L_\lambda[X, \mathcal{Y}]$, for any sufficiently large λ, is an element of N containing Z. □

The theorem of Hajnal on which this proof depends applies to classes in our sense. We must require that adding a parameter for N to the language preserves the truth of the replacement and separation schemes in V. Given a countable, standard transitive model of ZFC, one can produce by forcing a model V and an externally defined

$N \subset V$ that contains the ordinals of V, is closed under Gödel operations, and satisfies the universal cover property relative to V, yet fails to be a model of ZF.

3 The Main Theorem

Definition 3.1 Let $\mathcal{S}$ be any collection of sets of ordinals. The forcing $\mathbb{P}_{\mathcal{S}}$ consists of functions from elements of $\mathcal{S}$ to 2, ordered by inclusion:

$$\mathbb{P}_{\mathcal{S}} = \{p : X \to \{0, 1\} \mid X \in \mathcal{S}\}$$
$$p \leq q \iff p \supseteq q.$$

Definition 3.2 Let K be any infinite set of ordinals. A forcing to add a subset of K by partial functions is any forcing of the form $\mathbb{P}_{\mathcal{S}}$ where $\mathcal{S} \subseteq \mathcal{P}(K)$ and $\mathcal{S}$ has the following properties:

1. $X \in \mathcal{S}$ & $Y \subseteq X \implies Y \in \mathcal{S}$;
2. $\varnothing \in \mathcal{S}$ & $K \notin \mathcal{S}$;
3. $X \in \mathcal{S}$ & $\alpha \in K \implies X \cup \{\alpha\} \in \mathcal{S}$;
4. $X \in S$ & $Y \subseteq K$ & $|Y| < |K| \implies X \cup Y \in \mathcal{S}$.

Note that a forcing to add a subset of K by partial functions does in fact add a generic characteristic function $g : K \to 2$, defined by $g = \bigcup G$.

Theorem 3.3 *Let M be a model of* ZFC*, and let K be an uncountable set in M. No forcing in M to add a subset of K by partial functions adds a generic function g of minimal degree over M.*

In fact, if $|K| = \kappa$, there is a set f such that

$$M \subset M[f] \subset M[g]$$

and f is equivalent over M to a subset of κ.

Proof We exhibit a set of intermediate degree, that is, a set f such that $M \subset M[f] \subset M[g]$. As M is a model of choice, we can identify K with an uncountable cardinal κ. In M, for each $\alpha < \kappa$, define an equivalence relation on functions from α to 2 by

$$h \equiv^* g \iff |\{\beta \mid h(\beta) \neq g(\beta)\}| < \omega.$$

Then choose a representative for each equivalence class, and for $h : \alpha \to 2$, let h^* denote the chosen representative of the equivalence class of h.

By condition (4) of Definition 3.2, if g is the generic function from κ to 2 added by forcing with $\mathbb{P}_S$, then for each $\alpha < \kappa$, we have that $g \restriction \alpha \in M$. Define a function f on κ by

$$f(\alpha) = (g \restriction \alpha)^*.$$

Let $\dot{f}$ be a term in the forcing language representing f. We will show that $f \notin M$ and $g \notin M[f]$.

First, suppose that $p \Vdash \dot{f} = k$ for some function $k \in M$. The properties of S guarantee that $\kappa - \mathrm{dom}(p)$ is uncountable, hence there is some $\alpha < \kappa$ for which $\alpha - \mathrm{dom}(p)$ is infinite. Also, $\mathrm{dom}(p) \cup \alpha \in S$. Therefore, we can extend p to q such that $\alpha \subset \mathrm{dom}(q)$ and $q \restriction \alpha \not\equiv^* k(\alpha)$. Thus

$$q \Vdash \dot{f}(\alpha) = (g \restriction \alpha)^* = (q \restriction \alpha)^* \neq k(\alpha),$$

contradicting our assumption on p. Hence $f \notin M$.

Second, suppose that $\dot{\tau}_f$ is some term in the forcing language for an element of $M[f]$ and that $p \Vdash g = \dot{\tau}_f$. Let β be any element of κ not in the domain of p, and define a "bit-switching" automorphism of the forcing partial order, φ, by

$$\mathrm{dom}(\varphi(q)) = \mathrm{dom}(q)$$

$$\varphi(q)(\gamma) = \begin{cases} q(\gamma) & \gamma \neq \beta; \\ 1 - q(\beta) & \gamma = \beta. \end{cases}$$

Note that $\varphi(p) = p$.

Let G be a $\mathbb{P}_S$-generic containing p, and $\overline{G}$ be the image of G under φ. Both G and $\overline{G}$ are generics containing p, their respective generic functions g and $\overline{g}$ differ at exactly one point (β), and therefore their respective interpretations of the term $\dot{f}$, namely, f and $\overline{f}$, are equal. Hence

$$\dot{\tau}_f = \dot{\tau}_{\overline{f}}.$$

However, since both generics contain p, we have

$$g = \dot{\tau}_f \quad \& \quad \overline{g} = \dot{\tau}_{\overline{f}}.$$

As $g \neq \overline{g}$, this is a contradiction; hence $g \notin M[f]$. □

The proof of Theorem 3.3 implicitly assumes that any element of $M[f]$ can be named by a term $\dot{\tau}_f$ in the forcing language whose interpretation in $M[G]$ depends only on f. This follows from Proposition 2.4; the terms in question are terms $\dot{\tau}_f$ such that, for some $Y \in M$ and some ordinal γ, it is forced that $\dot{\tau}_f$ is the γth element in the canonical well-ordering of $L[f, Y]$ (or, literally, of $L[X, Y]$ where X is a set of ordinals coding f in some predetermined fashion).

Our remaining remarks concern the desirability of the conditions in Definition 3.2 and the possibility of stating Theorem 3.3 in greater generality.

First, we consider the conditions in Definition 3.2. Condition (1) is innocuous: If $\mathcal{S}$ does not satisfy (1), we define

$$\mathcal{T} = \{Y \mid (\exists X \in \mathcal{S})\ (Y \subseteq X)\}.$$

Then $\mathcal{T}$ does satisfy (1), and the forcings $\mathbb{P}_{\mathcal{S}}$ and $\mathbb{P}_{\mathcal{T}}$ are equivalent; $\mathbb{P}_S$ is a dense subset of $\mathbb{P}_T$. Given condition (1), conditions (2) and (3) are necessary to guarantee that if G is generic over M for $\mathbb{P}_{\mathcal{S}}$, and $g = \bigcup G$, then g is in fact a function from all of K to 2, and

$$M[g] \neq M.$$

Condition (4) implies that if $Y \subset K$ is of smaller cardinality than K, then $g \restriction Y$ is in the ground model; that is, $\mathbb{P}_S$ is essentially a forcing to add a subset of K, and not a subset of something smaller. This is intended to rule out examples such as the following: Suppose $K = \omega_1$ and $\mathcal{I}$ is some ideal on ω and

$$S = \{X \subset \omega_1 \mid \omega \cap X \in \mathcal{I}\}.$$

Then $\mathbb{P}_S$ is equivalent to a Grigorieff forcing on ω, which we know may add a generic real of minimal degree.

In the statement of condition (4), $|Y| < |K|$ can be replaced by "Y is countable" and the first part of the theorem still holds. The second holds as well provided $\kappa^\omega = \kappa$.

Condition (4) can be shown to be irrelevant in the following sense: If S satisfies (1)–(3), g is a generic function on K added by $\mathbb{P}_S$, and g is of minimal degree, then for some $Y \subset K$, the restriction $g \restriction Y$ is a generic function added by $\mathbb{P}_T$, where T is a collection of subsets of Y that satisfies (1)–(4) (with K replaced by Y). Thus, by the theorem, Y is in fact countable and $g \restriction Y$ is (essentially) a real added by forcing with partial functions. By the minimality of g, we have that g is equivalent to $g \restriction Y$, so g is essentially the result of adding a real by forcing with partial functions.

At the cost of adding undesirable complexity, Theorem 3.3 can be extended to a wider class of forcings. For example, we can allow only partial functions with certain properties, or we can add to a condition p a side condition determining which elements of S can be the domain of an extension of p. The important features (again identifying K with κ) are that $g \restriction \alpha \in M$ for $\alpha < \kappa$ (so that f can be defined), that for any condition p and sufficiently large $\alpha < \kappa$, there are extensions q and r of p whose domains contain α and such that $q \restriction \alpha \not\equiv^* r \restriction \alpha$ (so that one can prove $f \notin M$), and that the bit-flipping function defined in the proof is in fact an automorphism of the forcing partial order (so that one can prove $g \notin M[f]$).

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