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Logic, Mathematics, and the Mind: A Critical Study of Richard Tieszen's *Phenomenology, Logic, and the Philosophy of Mathematics*

Richard Tieszen. *Phenomenology, Logic, and the Philosophy of Mathematics*. Cambridge University Press, Cambridge, 2005. x + 357 pages.

1 Introduction

What has the conscious, intentional human mind[1] to do with the nature and status of logic and mathematics? Are irreducible facts about the human mind essential to logic and mathematics, or are logic and mathematics essentially mindless even if accidentally connected to human minds by means of contingent episodes of thinking? The rejection of idealism and psychologism in the early 20th century origins of analytic philosophy initiated the process of (as Dummett [3] aptly puts it) "extruding" the human mind from the theoretical content of logic and mathematics; the failures of the Hilbert and Brouwer programs of intuitionism in the 1930s hastened the extrusion process; then the messy divorce of analytic philosophy and phenomenology in the late 1940s and early 1950s completed it. Nevertheless, I do think that there are some serious foundational problems for essentially mindless logic and essentially mindless mathematics, and also that there are some correspondingly strong arguments for systematically reintroducing irreducible facts about the conscious, intentional human mind into the foundations of logic and mathematics. I have argued elsewhere for putting facts about the human mind back into the foundations of logic.[2] What I want to do here is to develop a parallel but distinct argument for putting facts about the human mind back into the foundations of mathematics, by means of a critical study of Richard Tieszen's very interesting, informative, and clearly-written book, *Phenomenology, Logic, and the Philosophy of Mathematics* [22], henceforth PLPoM.

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2 The Phenomenology Gap

It is a peculiar fact about philosophy and its history—a fact that sets it methodologically sharply apart from the formal and exact sciences—that sometimes in order to make forward progress in the subject, we must look backward first. Let us then look briefly backward at the intellectual situation in the philosophy of logic and mathematics in the late 1930s, so that we can recapture a sense of the foundational crisis which has largely shaped work in this area for the last seventy years.

At the fin de siécle Russell's discovery of his set-theoretic paradox and the contradiction in Frege's *Basic Laws of Arithmetic* had made logic "totter" (as Frege sadly wrote to Russell), and in the early decades of the century the (re)discovery of the Liar and other semantic or syntactic paradoxes had made all impredicativity—that is, defining or constructing totalities in terms of, or by reference to, those very totalities themselves—seem viciously circular. Wittgenstein's *Tractatus Logico-Philosophicus* had burst upon the scene after the Great War, promising to show how classical logic and mathematics are necessarily delivered, without any impredicativity, by the primitive fact that the world is pictured through human language. Yet the highly stringent semantics of the *Tractatus* deemed nonsensical not only the truths of logic, but also its own logico-philosophical propositions, and ended in metaphysical solipsism and mystical silence. So vicious impredicativity turned out to be harder to eradicate than Wittgenstein had thought. Russell's logical atomism and Carnap's logical empiricism in the 1920s and early 1930s had also failed, due to equally stubborn vicious impredicativity problems about the verifiability criterion of meaningfulness (which deemed itself meaningless) and the conventionalist theory of logical truth (which presupposed nonconventional logic). Then in the course of the 1930s things had gone from bad to worse, and finally logicism—like God and laissez faire capitalism—was dead. It was Gödel's incompleteness theorems that killed it.

Informally put, Gödel's incompleteness theorems say

(i) that there are logically unprovable true sentences in any elementary or classical second-order logical system that also includes enough axioms of Peano arithmetic, and
(ii) that all such logical systems are consistent (i.e., noncontradictory) if and only if they are incomplete (i.e., not all the truths of the system are theorems of the system) and have their ground of truth outside the system itself.

The incompleteness theorems together show that formal logical proof is not sufficient for mathematical truth—and, in particular, that Hilbert-style formalism and finitism as applied to formal logical proof, do not yield an explanation of mathematical truth—and also that mathematical truth is not a Turing-computable function that could be realized on a machine, on the further assumptions that every effectively decidable procedure is a recursive or Turing-computable function (Church's thesis, a.k.a. the Church-Turing thesis). Church's thesis, in turn, restricts effectively decidable procedures to machine computation, which rules out Brouwerian or *mentalistic* computation as a source of mathematical truth by means of decidable formal proof. It is of course possible to claim that Church's thesis is false, and that Brouwerian or mentalistic computation outruns machine computation, and still formally logically proves mathematical truths. But Church had also shown that even valid formal logical proofs in classical first-order predicate logic are not effectively decidable procedures, so there is no guarantee that a mentalistic computation will ever yield a formal logical proof in humanly finite time. Furthermore, it is arguable that Brouwerian mentalistic computation is solipsistic and incommunicable, and therefore ultimately unintelligible. Logically private proofs (i.e., proofs that necessarily only the prover can understand) are as deeply problematic as logically private languages (i.e., languages that necessarily only the speaker can understand), which Wittgenstein later compellingly rejected in *Philosophical Investigations*. In any case, Gödel's incompleteness theorems show that any known formalizable conception of logical proof, whether intuitionistic or classical, decidable or undecidable, is insufficient for mathematical truth.

Therefore, mathematical truth requires semantic resources that exceed those of either elementary logic or classical second-order logic plus enough of the Peano axioms, and also exceed the semantic resources of machine-based computation theory and intuitionist mentalistic computation theory alike. Correspondingly, mathematical knowledge requires cognitive resources that exceed our capacities for logical analysis and logical inference as classically understood by Frege, early Russell, early Wittgenstein, and Carnap, as well as exceeding the computational powers of Turing machines and Brouwerian intuitionistic minds. More generally then, how can we come fully to terms with the incompleteness results? And how can we account for the manifest difference, made evident by iterative set theory, between benign impredicativity and vicious impredicativity? To be sure, Tarski's semantic conception of truth and his regimented hierarchy of languages formally encodes and philosophically finesses these deeply troubling facts about logic and mathematics by shifting over much of the burden of demonstrating mathematical truth from proof theory to model theory—but it neither *explains* these facts, nor explains them *away*. What *is* truth, really? Where do all the models come from? In view of iterative set theory, how big can the models be? How can we tell the difference between the intended or standard models and nonstandard models of the same sentences or theories? And how do we know the intended or standard models in any case?

Granting all those very hard questions, and also resisting the strong temptations to deflationism, radical empiricism, and skepticism about mathematical truth and knowledge which had already emerged in Quine's "Truth by Convention," then it would not have been at all unreasonable by the end of the 1930s to think that those extra semantic resources required for mathematical truth must include ontologically rich, nonlogical *nonempirical concepts* together with their corresponding abstract properties, relations, objects, and truth-making states of affairs; and it would also not have been at all unreasonable to think that the extra cognitive resources required for mathematical knowledge must include a nonempirical, noninferential, noncomputational, nonmechanical, nonsolipsistic cognitive capacity—*mathematical intuition in a non-Brouwerian sense*—for grasping the contents of those concepts and also for referring to the abstract properties, relations, objects, and states of affairs that fall under those concepts and correspond to true mathematical statements.

This is precisely the view of Husserl in his doctrine of *phenomenology*, at least insofar as it applies to logic and mathematics, and therefore it is not at all surprising that Gödel himself—who, despite being the discoverer of the incompleteness results, was a serious antideflationist, rationalist, and antiskeptic about mathematical truth and knowledge—later explicitly adopted Husserl's phenomenological doctrine, and extended it. According to the Husserl/Gödel phenomenology of logic and mathematics then, the extra semantic resources for explaining mathematical truth are "essences" (*Wesen*) and their corresponding "objective manifolds" (*objektive Mannigfaltigkeiten*); the extra cognitive resources for explaining mathematical knowledge are "intentionality" and the capacity for "intuiting essences" (*Wesensschau*). With Tieszen's help, I will explicate these phenomenological notions in Section 3. The crucial point for the moment is that if the Husserl/Gödel phenomenology of logic and mathematics had actually *worked*, then it would have resolved the foundational crisis in logic and mathematics which Gödel himself had triggered, and which Husserl himself explicitly and compellingly linked to a deeper foundational "crisis of European sciences." And that is a philosophically exciting thought.

All of this is known, at least nominally, to mainstream contemporary philosophers of logic and mathematics, via the Gödel connection. What is far less well known (although Parsons[3] and Wang[4] are notable exceptions) is precisely what the Husserl/Gödel phenomenological theory says and implies—and especially, precisely what "essences," "objective manifolds," "intentionality," and "intuiting essences" are supposed to be—and also whether the phenomenological theory is in fact either fully intelligible or adequately defensible. We can call this *the Phenomenology Gap* in mainstream contemporary philosophy of logic and mathematics. Given the central and indeed seminal importance of Gödel's formal work in the development of contemporary logic and mathematics, and given the philosophically exciting results that could possibly flow from closing the Phenomenology Gap, then the Gap needs to be faced up to, and, if possible, closed.

Tieszen valiantly attempts to close the Gap. In Section 3, I will spell out the main arguments, issues, and themes of PLPoM, and then follow that up with a critical section. As a sneak preview, however, what I am going to argue is

(1) that the Husserl/Gödel phenomenological theory on its own is *not* adequately defensible, even if it is fully intelligible,
(2) that the Phenomenology Gap *can* be closed, but only by way of significant metaphysical, semantic, and epistemological supplements from a broadly *Kantian* philosophy of logic and mathematics, and
(3) that when the Gap is closed in this Husserlian/Gödelian and Kantian way, then it arguably yields a fully intelligible and adequately defensible theory of mathematical truth and knowledge.

Or, in other and fewer words, I will argue *that Husserl/Gödel + Kant = The Truth*. If this argument is sound, then some irreducible facts about the conscious, intentional human mind are essential to the foundations of logic and mathematics in a way that also avoids the classical problems of idealism, psychologism, and intuitionism.

3 The Husserl/Gödel Phenomenology of Logic and Mathematics

Unfortunately, PLPoM is not a monograph with a single, unified line of argument: instead it is a collection of Tieszen's essays, most of them written for various conferences and edited collections, that share a common, criss-crossing, or overlapping set of philosophical concerns and topics. This means that the philosophical narrative which Tieszen wants to tell emerges only as a kind of patchwork from the several essays, and not in a smooth linear way.

The book is divided into three unequal parts, preceded by an Introduction which previews and summarizes the basic issues and themes. The first part, which contains three essays, describes Husserl's later phenomenology and relates it to the philosophy of mathematics. The second part, which comprises seven essays and provides the philosophical core of the book, spells out the Husserl-Gödel relationship and relates it to fundamental problems in the philosophy of logic and mathematics, and then adds two essays—one on Maddy's philosophy of mathematics and one on Penrose's Gödel-inspired view on minds and machines—somewhat as an afterthought. Finally the third part, which has five essays in it, develops the interesting (although, I think, ultimately problematic) idea that the Husserl/Gödel phenomenology of logic and mathematics should also be construed as a version of intuitionist constructivism, and then adds another essay which compares and contrasts Husserl's and Frege's philosophies of arithmetic, again somewhat as afterthought.

Leaving aside the patchwork way in which the main ideas of PLPoM are presented, Tieszen's work is nevertheless extremely valuable for its serious, systematic attempt to combine analytic philosophy in the core tradition that runs from Frege to Quine and beyond, with phenomenology in the core tradition that runs from Brentano to later Husserl and beyond.

The defining feature of analytic philosophy in its core tradition is a central concern with logic, meaning, and necessary truth, and in particular with the notions of *analyticity* and *analysis*. Analyticity is necessary truth in virtue of conceptual meanings alone, independently of facts in the world, and analysis is the reductive decomposition of propositions into their constituent, simple meanings (corresponding to simple objects, properties, and relations in the world), followed by their systematic logical reconstruction. This logical reconstruction phase of analysis is specifically intended to yield the a priori discovery of nontrivial or informative, cognitively unexpected necessary connections between conceptual meanings and thereby avoid the critical charge that analysis is either trivial (because it is based on mere identities of meanings) or impossible (because nontriviality or informativeness seems to imply the nonidentity of meanings). The leading special program within the core tradition of analytic philosophy is *logicism*, which is the thesis that mathematics (or in Frege's case, arithmetic) is explanatorily and ontologically reducible to logic.

Correspondingly but contrastively, the defining feature of phenomenology in its core tradition is a central concern with *consciousness and intentionality*, both of which are taken to be primitive, irreducible facts about the human mind. Consciousness is *subjective experience*, either in the form of first-order reflexive awareness (a.k.a. "internal time-consciousness") or higher-order reflexive awareness (a.k.a. "self-consciousness"). Intentionality, in turn, is the spontaneous *directedness* of a conscious human mind, via "intentional contents" (i.e., informative, intersubjectively accessible modes-of-presentation) or to existing or nonexisting objects and states of affairs, which in turn are systematically grouped under various categories or kinds into "objective manifolds" (i.e., domains of objects). This spontaneous conscious directedness can be either directly referential or propositional. Phenomenological method begins with careful, nonreductive descriptions of intentional activities such as sense perception, imagination, temporal awareness (including memory and anticipation), judgment, and logical or mathematical reasoning, and then advances to a priori insights into "essences" (i.e., necessary conceptual connections) that link the various facts picked out by phenomenological descriptions. The leading special program within the core tradition of phenomenology is *transcendental phenomenology*, which is the thesis that the objects of intentionality derive both their existence and their specific character from the intentional subject, or transcendental ego, which in turn is a unified, universal intersubjective set of agential, normative, rational structures realized in every individual act of human intentionality. Otherwise put, transcendental phenomenology is an updated version of Kant's transcendental idealism.

Now Kant's transcendental idealism, as presented in the *Critique of Pure Reason*,[5] says

(i) that all the proper objects of human cognition are appearances (phenomena) and not things-in-themselves (noumena), and that we should be consistently agnostic about the existence or nonexistence of things-in-themselves precisely because they are humanly unknowable,
(ii) that the universal, basic structures of the apparent world necessarily conform to the innate nonempirical basic structures of the human cognitive faculties, and
(iii) that all the objects of actual or possible human experience are token-identical to the well-formed contents of judgments of experience, understood as instances of intersubjectively accessible representation-types.[6]

Correspondingly, transcendental phenomenology says

(i∗) that all the objects of intentionality are ontologically dependent on the transcendental ego which is necessarily "in" all of us individual thinkers, and that we should "bracket" the question of the extra-intentional existence or nonexistence of those objects, a.k.a. "the phenomenological reduction,"
(ii∗) that the universal, basic structures of intentional objects must conform to the universal nonempirical basic structures of human intentionality, and
(iii∗) that intentional objects are token-identical to the intentional contents of the intentional acts directed toward them, understood as instances of intersubjectively accessible representation-types.

In fact, Husserl also worked out a realistic, *non*transcendental version of phenomenology in his early books, *Philosophy of Arithmetic* (1891) and *Logical Investigations* (1900–1901). But Tiezsen, like Gödel, focuses exclusively on Husserl's transcendental phenomenology, which he systematically developed in "Philosophy as Rigorous Science" (1911), *Ideas I* (1913), *Formal and Transcendental Logic* (1929), *Cartesian Meditations* (1931), "The Crisis of European Sciences and Transcendental Phenomenology" (1936), and *Experience and Judgment* (1939). As I mentioned above, Husserl's phenomenology of logic and mathematics has four basic elements: (1) essences, (2) objective manifolds, (3) intentionality, and (4) the cognitive capacity for intuiting essences. In a transcendental-phenomenological framework, these are all metaphysically grounded on the transcendental ego in the strong "upward determination" or "supervenience" sense that fixing the existence and specific character of the transcendental ego metaphysically necessitates a corresponding fixation of the existence and specific character of essences, objective manifolds, intentionality, and the capacity for intuiting essences, and there cannot be a change in the properties of the latter without a corresponding change in the properties of the former. Within that metaphysical framework, in essays 1–3, 11, and 13, Tieszen then focuses on what I take to be the four basic themes of Husserl's later work on the nature and status of logic and mathematics.

First, from his earliest writings, Husserl conceives of pure logic, in the tradition of Descartes, Leibniz, Kant, and Bolzano, as the *Ur*-science, the *mathesis universalis* or *Wissenschaftslehre*—the ultimate science which provides strictly universal (and in Kant's case, categorically normative) laws and truths for every other actual or possible science. The specific subject-matter of pure logic is the set of ideal analytic meaning-relations between higher-order concepts, and the specific sort of knowledge which is generated by logical insight is a priori. So Husserl's pure logic is an *intensional* logic and not reducible to extensional notions or relations, whether these are functions from objects to objects or functions from objects to truth-values (i.e., Frege's "concepts") or the collections or totalities of objects which are the inputs to functions that map from objects to the True (i.e., Frege's classes or sets). Husserl's theory of pure logic, like Frege's, is strongly antipsychologistic, hence, nonreducible to empirical psychology, whether introspective, behavioral, or otherwise naturalistic. Yet Husserl also holds, again like Frege, that pure logic is inherently directly accessible to individual human minds. The obvious Benacerraf-style problem which arises here is how individual human minds, which obviously stand in real spatiotemporal causal acquaintive relations with ordinary worldly objects in sense perceptual knowledge, also manage to stand in acquaintive relations with causally inert abstract logical objects in pure logical cognition. This deep problem is not in fact resolved until Husserl shifts from phenomenological realism to a specifically phenomenological version of transcendental idealism after 1911. According to this transcendental-phenomenological solution to the Benacerraf dilemma, abstract logical intentional objects are inherently accessible to human minds just because *all* intentional objects whatsoever, even the causally efficacious objects of ordinary sense perceptual knowledge, are transcendentally ideal and therefore inherently accessible to human minds. Or in Kant's terminology, transcendental idealism equally yields the real possibility of a priori logical knowledge and the real possibility of empirical (i.e., direct perceptual) realism.[7]

Second, pure logic for Husserl has three distinct levels and introduces distinctive universal laws governing linguistic meanings and in particular sentential meanings or propositional contents corresponding to each of the levels. Level 1 is "universal grammar," which provides both rules of syntax (i.e., well-formedness conditions) and also semantic categories (i.e., sortal correctness conditions) for fine-grained intensions or concepts. Violations of universal grammar yield syntactical or semantical "nonsense" (*Unsinn*), and it is possible to have syntactically well-formed terms that nevertheless violate sortal correctness conditions—to use Russell's example, "Quadruplicity drinks procrastination." Level 2 is "the logic of noncontradiction," which provides both rules of formal (i.e., truth-functional and deductive) consistency and also material (i.e., fine-grained intensional or conceptual) consistency. Violations of the consistency laws yield formal or material contradictions, or "countersense" (*Widersinn*). Finally, level 3 is the logic of truth or "truth logic" (*Wahrheitslogik*), which provides rules for fully interpreting meaningful, consistent sentences and evaluating their propositional contents as either true or false. This is also the level of "meaning-fulfillment," which is a characteristic feature of logical intentionality and all other types of intentionality as well.

The notion of meaning-fulfillment says that when a meaningful, consistent sentence is evaluated as true under the logical rules of interpretation, then the meaning of the sentence or its propositional content is isomorphically mapped directly onto its truth-making objects, properties, relations, and states of affairs, that is, directly onto the relevant inhabitants of objective manifolds. This is meaning-fulfillment at the level of *formal* semantics. Correspondingly, the judging intentional subject is *also* consciously aware of this isomorphic semantic mapping. This is meaning-fulfillment at the level of *cognitive* semantics and is also what Husserl calls "categorial intuition," because it is a direct conscious awareness which is immanently structured by precisely the same categories or higher-order concepts that also determine the basic object-structures of the corresponding objective manifolds. Under normal epistemic conditions, categorial intuition generates "self-evidence" (*Evidenz*), which is the subjective experience of sufficiently justified true belief. In this way, for Husserl, cognitive intentionality mirrors logical intensionality and extensionality under semantic categories and formal rules of grammar, consistency, and interpretation, and thus the epistemology of pure logic mirrors the syntax and semantics of pure logic. Again: the cognizing mind mirrors the truth-making world, which in turn mirrors pure logic, which in turn is mirrored in human language. Obviously, there are important parallels here between Husserl's theory of pure logic and Wittgenstein's theory of logic in the *Tractatus*.

Third, Husserl has an importantly distinctive position on the issue of platonism vs. antiplatonism (whether nominalism, formalism, conventionalism, Quinean pragmatism, fictionalism, or whatever). As we saw above, Husserl defends a phenomenological version of Kant's transcendental idealism. Specifically applied to mathematical objects, this transcendental phenomenology entails that mathematical objects are nothing but the intersubjectively accessible intentional contents of the intentional acts directed toward the corresponding truth-makers of mathematical truths. Otherwise put, according to Tieszen,

> On Husserl's view we are to take mathematical language at face value, and to take mathematical theorems as true.... On Husserl's view language is in fact meaningful only insofar as it expresses intentions.... Perhaps the single most important thing to say about the conception of objects of cognition in Husserl's transcendental phenomenology, whether the objects be mathematical or physical, is that they are to be understood in terms of the "invariants" or "identities" in our [intentional] experience.... [It is] the relatively harmless idea of mathematical objects as invariants that persist across [intentional] acts carried out by different mathematicians at different times and places. (PLPoM, pp. 53, 55, and 59)

In this way, for Husserl mathematical objects are nothing but the constituent *higher-order conceptual contents* of necessarily true mathematical statements insofar as they are grasped in mathematical intentionality, and, furthermore, these abstract objects are inherently directly accessible to rational human minds just because conceptual content is known by acquaintance in all and only those intentional linguistic acts that manifest *concept-possession*. So the platonic abstractness of mathematical objects is the same as their transcendentally ideal conceptual-linguistic character, which in turn is the same as their being "essences" in a broadly Platonic and broadly Aristotelian sense which, precisely because the essences are transcendentally ideal, remains metaphysically intermediate as between the classical *ante rem* and *in rebus* versions of essentialism:

> Our beliefs and other cognitive acts at any given time are... always about certain *categories* of objects. The mind always categorizes in this way. It cannot help doing so, for our experience would have to be very different otherwise. Thus many types or categories are always at work in our experience, even in everyday sense experience. I shall refer to these categories as "concepts." In order to introduce some specific texts of Husserl that clearly influenced Gödel, I now introduce a new term for what I have been calling "categories" or "concepts." I will call these categories "essences".... I only mean for it to be another way of speaking of these categories in our experience. There are several reasons why it makes sense to use this term.... First, we are concerned with *what* the experience is about at a given stage, and "whatness" has been associated with the notion of an essence at least since the time of Aristotle. Second, we can think of essence in this sense as a universal. For example, I can see different instances of the category "chair." I can believe that there are different instances of the category "natural number" or of the category "function" or "set." I cannot believe that just anything is an instance of the category "natural number." The essences are not arbitrary. There are constraints on them and they are not subject to being changed at will. They are not freely variable. (PLPoM, pp. 126–27)

Fourth, this identification of platonic abstractness, transcendentally ideal conceptual-linguistic character, and essence carries over directly into Husserl's theory of the formal ontology of mathematical objective manifolds. Objective manifolds are what results when intentional linguistic acts proceed from "empty" to "filled" intentions; hence, objective mathematical manifolds are nothing but the categorially-structured truth-makers (i.e., interpretation-theoretic models) that are consciously experienced in mathematical cognitive-linguistic acts of intentional fulfillment. Tieszen also interprets Husserl's theory of essences and objective manifolds as a nonsolipsistic, non-Dummettian version of *intuitionist constructivism*, according to which the intentional linguistic activities of generating mathematical proofs cognitively *create* the very objective manifolds that are the truth-makers of the theorems being proved:

> We start with the idea that mathematical cognition, like other forms of cognition, exhibits intentionality. The contents of our mathematical acts (i.e., mathematical intentions) are expressed in mathematical sentences. Mathematical intentions are not essentially private possessions locked in some Cartesian theater. They are or may be shared in the mathematical community. Scientific investigation is a matter of group intentionality. Many philosophers have argued that human beings are so constituted as to have at least some [mutually] isomorphic cognitive structure and that this characteristic is what makes learnability of language and communication possible. Learnability of language and communication should be possible in the case of both... intuitionistic mathematics and classical mathematics. Following Heyting and Becker, one can argue that intuitionistic constructions in particular may be thought of as fulfilled or fulfillable mathematical intentions. As such, they are in the first instance forms of consciousness or possible experience that are sharable but that are not completely exhausted in observable linguistic behavior. Not all mathematical intentions need to be fulfilled or to be fulfillable at a given stage in time. In the case in which mathematical intentions are fulfillable we have *constructive meaning*. Not all meaning or content, however, is to be identified with constructive meaning or content. In the case in which we do have constructive meaning or content, we can recognize different degrees or types of fulfullment. That is, we can recognize

> different degrees or types of mathematical evidence. Finally, the concept of intuition in intuitionism can be defined in terms of the fulfillment of a mathematical intention. In this sense, mathematical constructions as cognitive processes carried out in time by human subjects are just mathematical intuitions. (PLPoM, pp. 246–47)

Otherwise put, according to Tieszen, Husserl's transcendental phenomenology of logic and mathematics simply identifies the *intended models* of mathematical truths or theories with the *intentional fulfillments* of the corresponding intentional acts of mathematical proof and mathematical intuition. Constructively and intuitively proving that P makes it the case that P. I have some substantive worries about this doctrine that I will come back to in Section 4.

As I mentioned above, the philosophical core of PLPoM is the Husserl-Gödel relationship, which Tieszen spells out in Chapters 4–8 and which basically completes the Husserl/Gödel phenomenological theory of logic and mathematics.

Gödel began seriously studying Husserl's phenomenology in 1959 and focused exclusively on Husserl's transcendental phenomenology. Gödel was also importantly influenced by Plato, Leibniz, and Kant, but viewed them all through Husserlian transcendental-phenomenological lenses. Gödel applied transcendental phenomenology to the two fundamental problems of logic and mathematics as he understood them: (1) the implications of the incompleteness theorems and (2) the implications of impredicativity. More precisely, given these fundamental problems, and as a serious antideflationist, rationalist, and antiskeptic who *also* accepts transcendental phenomenology, Gödel proposes

(i) that mathematical truth can be explained only as a set of analytically necessary and a priori relations between higher-order concepts which transcend the proof procedures contained within formalized systems of mathematical logic,

(ii) that a nonempirical, noninferential, nonmechanical capacity for the mathematical intuition of these higher-order concepts is required in order to know mathematical truth, and

(iii) that not all impredicativity is viciously circular because there is, at the very least, a mathematically robust benign impredicativity in the transfinite hierarchy of the constructible universe of sets (a.k.a. "L"), which in turn is knowable via our mathematical intuition of the higher-order concepts contained in the axioms of Zermelo-Fraenkel set theory.

Or in Gödel's own words,

> P. Bernays has pointed out on several occasions that in view of the fact that the consistency of a formal system cannot be proved by any deduction procedures available in the system itself, it is necessary to go beyond the framework of finitary mathematics in Hilbert's sense in order to prove the consistency of classical mathematics or even of classical number theory. Since finitary mathematics is defined as the mathematics of *concrete intuition*, this seems to imply that *abstract concepts* are needed for the proof of consistency of number theory.... By abstract concepts, in this context, are meant concepts which are essentially of the second or higher level, i.e., which do not have as their content properties or relations of *concrete objects* (such as combinations of symbols), but rather of *thought structures* or *thought contents* (e.g., proofs, meaningful propositions, and so on), where in the proofs of propositions about these mental objects insights are needed which are not derived from reflection upon the combinatorial (space-time) properties of the symbols representing

> them, but rather from a reflection upon the *meanings* involved. (Gödel, as quoted in PLPoM, pp. 133–34)
>
> [We do have] something like a perception of the objects of [Zermelo-Fraenkel] set theory [which is] seen in the fact that the axioms force themselves upon us as being true.... I don't see any reason why we should have less confidence in this kind of perception, i.e., mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and moreover, to believe that a question not decidable now has meaning and may be decided in the future. (Gödel, as quoted in PLPoM, p. 140)
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> The generalized undecidability results do not establish any bounds for the powers of human reason, but rather for the potentialities of pure formalism in mathematics.... Turing's analysis of mechanically computable functions is independent of the question whether there exist finite nonmechanical procedures... such as involve the use of abstract terms on the basis of their meaning. (Gödel, as quoted in PLPoM, p. 153)
>
> If ... it is a question of objects that exist independently of our construction there is nothing in the least absurd in the existence of totalities containing members which can be described (i.e., uniquely characterized) only by reference to this totality. (Gödel, as quoted in PLPoM, p. 258)

Translated into Husserlian transcendental-phenomenological teminology, these Gödelian claims amount to claiming

(i∗) that the extra semantic resources which are required by the incompleteness results for explaining mathematical truth are higher-order concepts, or essences, and their corresponding objective manifolds,

(ii∗) that the extra cognitive resources which are required by the incompleteness results for explaining mathematical knowledge are intentionality and the capacity for possessing (i.e., understanding, deploying, and reusing) higher-order concepts, or intuiting essences, and

(iii∗) that the abstractness of mathematical objects is nothing but the transcendental-phenomenological ideality of higher-order concepts, or essences, and the constructible universe of sets or L is categorially intuitable with self-evidence just insofar as we are capable of possessing the higher-order concepts which are contained in the axioms of Zermelo-Fraenkel set theory.

Indeed, Gödel himself explicitly confirms this translation:

> [There] exists today the beginnings of a science which claims to possess a systematic method for such clarification of [abstract conceptual] meaning and that is the phenomenology of Husserl. Here clarification of meaning consists in concentrating more closely on the concepts in question by directing our attention in a certain way, namely, onto our own acts in the use of those concepts, onto our own powers in carrying out those acts, etc. In so doing, one must keep clearly in mind that this phenomenology is not a science in the same sense as the other sciences. Rather it is a procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other, hitherto unknown, basic concepts. (Gödel, as quoted in PLPoM, p. 154)
>
> Some reductionism is right: reduce to concepts and truths, but not to sense perceptions.... Platonic ideas [i.e., essences or abstract concepts] are what things are to be reduced to (Gödel, as recorded by Wang, as quoted in PLPoM, p. 155)

4 How to Close the Phenomenology Gap

It seems to me undeniable that Tieszen has shown that the Husserl/Gödel phenomenology of logic and mathematics is fully intelligible, whether or not one agrees with it. But is it adequately defensible? Here it also seems to me that Tieszen's account has three serious weaknesses: first, an inadequate modal semantics; second, an inadequate modal epistemology; and third, no adequate explanation for the difference between benign impredicativity and vicious impredicativity. I will unpack each of these worries in turn.

4.1 An inadequate modal semantics It seems to me that Tieszen's defense of the Husserl/Gödel account of mathematical truth is hampered by an inadequate modal semantics of necessity, particularly with respect to analyticity (pp. 185–90). Tieszen distinguishes between analyticity in the narrow sense (i.e., the truths of either elementary logic or the classical second-order logic favored by Frege and Russell) and analyticity in the wide sense (i.e., truths that are translatable into the truths of either elementary logic or classical second-order logic by replacing intensionally identical concepts by intensionally identical concepts) and then places the logically unprovable necessarily true statements of mathematics in the class of wide or conceptual analyticities, which would make them logically unprovable yet still conceptually true. But the wide or conceptual analyticity gambit has been thoroughly tried by Frege via his "logical definitions," by the Carnap of *Meaning and Necessity* via his "meaning postulates," and by the neo-Fregeans via their "Hume's Principle." The main problem here is that Frege's notion of a logical definition, Carnap's notion of a meaning postulate, and the neo-Fregeans' appeal to Hume's Principle, not to mention Husserlian essences plus their corresponding manifolds, all add special *ontological* commitments to logic in a way that semantically overburdens any appeal to the inherently *conceptual* or *fine-grained intensional* character of wide analyticity.

Here are three arguments for that claim. Since conceptual semantics is assumed to be legitimate by Frege, Carnap, the neo-Fregeans, Husserl-Gödel, and Tieszen alike, then we are rationally entitled, on grounds that they already accept, to hold that conceivability entails (broad) logical and (weak) metaphysical possibility. But, first, for every ontological commitment added by logical definitions, meaning postulates, Hume's Principle, or Husserlian essences plus their corresponding objective manifolds, we can consistently conceive of a metaphysically possible world in which the postulated properties, relations, objects, or states of affairs in those objective manifolds do *not* exist, and indeed in which *nothing whatsoever exists*. Such a null world would be, in Husserlian terms, an intentionally *unfulfillable* world. In null or unfulfillable worlds, at least some of the statements of simple arithmetic—say, "$7 + 5 = 12$"—that are held to be analytically true in a wide sense, are thereby shown to be *not true* by model-theoretic means. Therefore, the wide or conceptual notion of analyticity cannot explain even simple arithmetic truth, and a fortiori it cannot explain mathematical truth. I will call this *the Null World Argument*.

Moreover, second, for every ontological commitment added by logical definitions, meaning postulates, Hume's Principle, or Husserlian essences plus their corresponding objective manifolds, we can also consistently conceive of a logically and metaphysically possible world in which the postulated properties, relations, objects, or states of affairs are allowed to grow explosively by benign impredicative reasoning.

According to the Löwenheim-Skolem theorem, which says that every countable first-order theory having an infinite model has many distinct infinite models, this allows for many nonstandard transfinite models of every statement of simple arithmetic—say, again, "$7 + 5 = 12$"—that is held to be analytically true in a wide sense. Such explosive worlds would be, in Husserlian terms, intentionally *over-fulfillable* worlds. Given the metaphysical possibility of explosive or over-fulfillable worlds, the wide or conceptual notion of analyticity fails to uniquely determine the *ontology* of even simple arithmetic. So again the wide or conceptual notion of analyticity cannot explain even simple arithmetic truth, and again a fortiori it cannot explain mathematical truth. I will call this *the Explosive World Argument*.

Third, and finally, for every impredicatively explosive possible world, again according to the Löwenheim-Skolem theorem, there is a countable submodel for each true simple arithmetic statement that is held to be analytically true in a wide or conceptual sense such that this submodel does not uniquely determine the total function expressed by that statement. Consider, for example, the indeterminacy between the standard "plus" function in elementary arithmetic and Kripke's nonstandard "quus" function,[8] as applied to the statement "$7 + 5 = 12$." Such quus-like worlds would, in Husserlian terms, be intentionally *weirdly-fulfillable* worlds. Given the metaphysical possibility of quus-like or weirdly-fulfillable worlds, it follows that the wide or conceptual notion of analyticity fails to uniquely determine the *rules* of even simple arithmetic. So yet again the wide or conceptual notion of analyticity cannot explain even simple arithmetic truth, and yet again a fortiori it cannot explain mathematical truth. I will call this *the Quus-like World Argument*.

It follows directly from a natural generalization of the Null World Argument that all true mathematical statements are consistently deniable; it follows directly from a natural generalization of the Explosive World Argument that unless the ontology of mathematics is nonarbitrarily restricted, then mathematics has no determinate ontology; and it follows directly from a natural generalization of the Quus-like World Argument that unless the ontology of mathematics is nonarbitrarily restricted, then mathematical rule-following is impossible. Therefore, it seems very plausible to claim, along with Kant, that true statements of mathematics are *synthetic a priori*, not analytic, since the Kantian criteria of synthetic apriority are (i) consistent deniability, (ii) the nonarbitrary restriction of ontology by means of pure formal spatiotemporal intuition or *Anschauung*, (iii) necessity, and (iv) nonempiricality, or the underdetermination of semantic content by (i.e., the nonsupervenience of content upon) all specific actual or possible sensory information (CPR: B1-10, A6-10/B10-21, A19-49/B33-73, B160-161, A154-158/B193-197).[9] Or, in other words, for Kant a proposition P is synthetic a priori if and only if P is true in all and only the worlds of possible human experience (a.k.a. "phenomenal worlds")—that is, P is true in all the phenomenal worlds and never false otherwise, precisely because P is *truth-valueless* in all the logically possible worlds which are inaccessible to human cognition (a.k.a. "noumenal worlds"). Ironically enough, Husserl explicitly recognized the syntheticity and apriority of mathematical truth in *Logical Investigations*. In his later transcendental-phenomenological writings, however, this explicit recognition seems to have been mostly repressed if not explicitly denied.

In any case, in my opinion the Husserl/Gödel phenomenological theory of mathematical truth needs to be explicitly and robustly supplemented by a Kantian *modal* *dualist* semantics of necessity, according to which there are two essentially different types of necessity—

(1) logical (i.e., both narrow and wide or conceptual) or analytic, a.k.a. "weak" metaphysical necessity = truth in all logically possible worlds, by virtue of concepts or fine-grained intensions alone, and
(2) nonlogical (i.e., nonconceptual or intuitional) or synthetic, a.k.a. "strong" metaphysical necessity = truth in all and only the phenomenal worlds, by virtue of concepts together with nonconceptual pure formal intuitions.[10]

This modal dualist semantics, in turn, would adequately explain Gödel's incompleteness results in a specifically Kantian way, according to which the logically unprovable Gödel sentences are necessarily true although logically unprovable precisely because (a) they are *synthetic a priori*, not analytic, and (b) their ground of truth depends on benign impredicative reasoning together with model-theoretic reasoning according to the Löwenheim-Skolem theorems, which in turn jointly allow for explosive worlds and quus worlds, and these forms of reasoning are appropriately constrained by *pure formal intuition* to finite models, denumerably infinite models (i.e., models exemplifying time-structure as described in the Transcendental Aesthetic), or nondenumerable models that go only as far up the transfinite hierarchy as the real numbers (i.e., models exemplifying the structure of the extensive and intensive spacetime continuum as described in the Axioms of Intuition and the Anticipations of Perception).

The notion of logical provability that is shown to be too weak for mathematical truth by the incompleteness results is, of course, formal provability in either elementary logic or classical second-order logic, insofar as it is conservatively extended by logical definitions, meaning-postulates, Hume's principles, or Husserlian essences plus their corresponding objective manifolds. But this leads to another perhaps surprising Kantian point: The essentially polyadic part of elementary logic or classical second-order logic is itself *synthetic a priori too*, not analytic, precisely because its truths are *also* consistently deniable in possible worlds with empty domains, that is, the null or unfulfillable worlds. In this way, the nondecidability of elementary logic and classical second-order logic is also semantically explained in a specifically Kantian way.

Now if the logical truths of elementary logic and classical second-order logic are synthetic to the extent that they are essentially polyadic, hence logical truth in those systems is nondecidable even if formally logically provable, then what remaining part of logic, if any, is analytic? The perhaps surprising Kantian answer is *second-order intensional monadic logic*. Now monadic logic is classical sentential logic plus quantification into one-place predicates only. First-order logic is classical predicate logic with quantification over individuals only. And intensional logic is a conservative extension of classical predicate logic that is nonextensional in that it permits universal intersubstitution salva veritate only for intensionally equivalent terms, not extensionally equivalent terms, in which, therefore, extensionally equivalent terms can have different intensions, and in which intensions determine cross-possible-worlds extensions or (as C. I. Lewis dubbed them) *comprehensions*. Then second-order intensional monadic logic is second-order monadic logic plus quantification over fine-grained, decomposable intensions and lowest-order or roughgrained intensions (i.e., comprehensions), instead of quantification over purely extensional predicates. The very idea of the fundamental importance of second-order intensional logic is a *Kantian* idea precisely because second-order intensional monadic logic is essentially the same as what Kant calls *pure general logic* (CPR: A50-55/B74-79).[11]

In any case, both first-order and second-order *extensional* monadic logic are decidable, consistent, and complete,[12] so their truths all count as analyticities in the narrow sense. By contrast, second-order *intensional* monadic logic conservatively adds a theory of conceptual structure (i.e., the fine-grained, decomposable intensions) and cross-possible-worlds extensions (i.e., the lowest-order or roughgrained intensions, a.k.a. comprehensions) to first-order monadic logic; hence, all of its truths then count as analytic in the *minimally wide* sense so elegantly sketched by Quine in "Two Dogmas of Empiricism"—that is, the class of statements translatable into logical truths of *monadic logic* by replacing synonyms by synonyms—although, of course, ultimately even this minimally wide sense of analyticity was skeptically repudiated by Quine.

But *what about* that minimally wide class of analytic statements? It is arguable that all of Quine's basic arguments against the analytic-synthetic distinction in "Two Dogmas" are unsound.[13] Of course, this Awful Truth has long been muttered quietly under their breaths by generations of clever undergraduate philosophy students who had not yet been properly socialized into the philosophical profession, and who could not understand why Quine's "Aristotelian essence wedded to the word" argument against explaining analyticity by appeal to intensions seemed so cheap and rhetorical; who could not figure out why Quine's translating the minimally wide class of analytic statements into the narrow class of analytic statements in classical first-order monadic logic did not entail a perfectly acceptable analytic-synthetic distinction between the narrow class of logically true statements and all others; who wondered why Quine was allowed to get away with an argument-by-cases against explicating the notion of synonymy when he had not shown that he had exhausted the set of cases; and who got dizzy when they realized that the universal revisability of all statements entails that the *law of noncontradiction* is revisable, and also that the universal revisability principle is *itself* revisable, both of which are clearly incoherent, if not downright self-refuting—and indeed, in *Philosophy of Logic* Quine himself later explicitly asserted the incoherence of any attempt to revise the law of noncontradiction, his reasoning being that to *deny this doctrine* would be to *change the subject*. In other words, it seems to me (and to the clever undergraduates) that Kant was right about the analytic-synthetic distinction, and Quine was wrong.

So, bracketing out Quine's scientific naturalism, his radical empiricism, and his meaning-skepticism, and instead following the lead of, for example, the iconoclastic rationalistic analytic philosophers Pap and Katz,[14] it is very plausible to think that analyticity in the minimally wide sense can in fact be provided with some or another adequate formal semantic analysis and structural characterization, and thereby theoretically endorsed. But that semantically *exhausts* the total domain of analyticity and also leaves mathematical truth and necessity maximally wide open for the semantics of syntheticity, since the logic of mathematics is inherently polyadic, not monadic, and thus ontologically committed from the get-go—as is obvious in the striking fact that

$$(\exists x)x = x$$

(i.e., that some object exists) is a valid sentence of elementary logic, and derivable from the empty set of premises.

4.2 An inadequate modal epistemology Tieszen's defense of the Husserl/Gödel phenomenological theory of mathematical knowledge is also hampered by an inadequate modal epistemology, particularly with respect to the notion of mathematical intuition (pp. 185–90). I am all for endorsing mathematical intuition. But there are two serious problems with Tieszen's theory of it.

First, Tieszen's response to the Benacerraf Dilemma is just to deny that our best overall theory of knowledge involves direct spatiotemporal perceptual contact with the objects known. But that, in effect, is just to deny—without any independent argument to support the denial—that all knowledge, including mathematical knowledge, involves a sense perceptual component. But what is knowledge when it is detached from all possible human experience? Kant's faculty-based nonempirical innateness thesis to the effect that all our cognition *begins in* experience but is not all *derivable from* experience precisely because some of it is transcendental, a priori, and necessary (CPR: B1-2) seems to conform much more closely to the actual facts. Moreover, how can human minds ever grasp non-spatiotemporal abstract objects? Kant's account of the abstractness of mathematical objects and of logical laws, which says

(i) that those objects and those laws are transcendentally ideal and based respectively on our *pure formal intuitions of space and time* and also *our pure higher-order concepts of the understanding (Verstand)*,

and says also that

(ii) space and time themselves, insofar as they necessarily conform to our pure formal intuitions of them, are restricted nonlogical abstract structures in which the real material objects of direct perception are *inherently embedded*,

seems again to conform much more closely to the actual facts. We have direct contact with mathematical objects by having direct cognitive contact, via pure formal intuition, with the transcendentally ideal, a priori, and synthetically necessary *ante rem* mathematical structures in which numbers and other mathematical objects are nothing but positions or roles, and in which all the real directly perceivable material objects are inherently embedded. I call this doctrine *Kantian Structuralism*.[15]

Second, Tieszen's theory of mathematical intuition says that intuition is wholly conceptual and also based on "free imagination" (p. 197), but this begs the question in favor of the thesis of *conceptualism* about mental content against *nonconceptualism* and also against *imagism*. Conceptualism says that all mental content is either solely or wholly determined by our conceptual capacities.[16] Nonconceptualism, by contrast, says that at least some mental content is solely and wholly determined by our nonconceptual capacities.[17] And imagism says that the content of mental imagery is neither solely nor wholly determined by our propositional capacities.[18] A Kantian account of mathematical intuition, by sharp contrast with Tieszen's, is innatist, apriorist, nonconceptualist, and imagist,[19] and holds that our possession of mathematical concepts in the "a priori construction" of mathematically intuitable truths—for example, the simpler truths of elementary arithmetic such as "7+5 = 12" and the like—is ultimately grounded on our innate capacities to run (in effect) primitive recursive calculation operations over the natural numbers, to imaginatively manipulate spatial imagery, and to imaginatively manipulate linguistic imagery.[20] It is historically significant that Husserl's early theory of categorial intuition in the first edition of *Logical Investigations*, and also Wittgenstein's Tractarian account of self-evidence, are arguably both innatist, apriorist, nonconceptualist, and imagist in precisely these ways. I call this doctrine the *Husserl-Wittgenstein Phenomenological Theory of Logical and Mathematical Self-Evidence*, or the HW Theory for short.[21]

4.3 No adequate explanation for the difference between benign and vicious impredicativity Tieszen correctly and explicitly points out that "impredicative specifications in mathematics do not always yield contradictions" (PLPoM, p. 258) and ties this fact directly to the phenomenological notion of constructivity as fulfilled or fulfillable intentions (PLPoM, pp. 276–93). The main idea here is that some human constructions using mathematical intuition, despite being finite, are inherently nonmechanical and also capable of consciously grasping denumerably infinite and even transfinite structures generated by impredicative means. True enough. But this is merely to reformulate Gödel's deep point in Husserlian terms, not to *explain* the difference between benign impredicativity and vicious impredicativity.

By sharp contrast, it seems to me that Kantian cognitive semantics and metaphysics can provide at least the beginnings of an adequate explanation of the difference between the two kinds of impredicativity. For Kant, the cognitive fact of (what we now think of as) impredicative reasoning, as it is presented in the Transcendental Aesthetic, Analytic of Principles, and Transcendental Dialectic sections of the first *Critique*, has two distinct primitive sources: the pure formal intuitions of time and space, and pure theoretical reason. The first source, the pure formal intuitions of time and space, allows for impredicative constructions of finite, denumerably infinite, and nondenumerably infinite (up to the real numbers) totalities via the successive synthesis of homogeneous moments intuited in inner sense, together with the extensive continuum consisting of all the proper parts of space, and the intensive continuum of all possible degrees of quality experienceable in phenomenal spacetime, when presented in the unified dual framework of time and space, insofar as they are immediately and immanently "given" in human experience as infinite wholes. So this kind of impredicativity is specifically temporal, spatial, qualitative, and mereological. The second source, pure theoretical reason, allows for impredicative constructions of "actually infinite" nondenumerable totalities exceeding the transfinite numerosity of the real numbers, via the intellectual synthesis of higher-order pure concepts detached from any sensory application, a.k.a. "notions," "concepts of pure reason," or "transcendental ideas" (CPR: A310-333/B366-390). So this kind of impredicativity is specifically *higher-order intensional*, and purely *logical* in character rather than spatiotemporally *mereo*logical in character.

In turn, the Kantian idea about the distinction between benign and vicious impredicativity is taken from the Antinomies of Pure Reason and says that benign impredicativity *heeds* the fundamental distinction between (1) appearances or phenomena and (2) things-in-themselves or noumena, whereas vicious impredicativity does *not* heed this fundamental distinction. Kant's theory of the phenomenon vs. noumenon distinction is to be found in the chapter of the first *Critique* entitled "On the Ground of the Distinction of All Objects in General into **Phenomena and Noumena**" (CPR: A235-260/B294-315). Here we learn that noumena or things-in-themselves are nonsensory, nonspatiotemporal, mind-independent substances which are defined and constituted by their intrinsic nonrelational properties, for example, Platonic forms or Leibnizian monads. Noumena can have a "lonely" existence, which is to say that it is (weakly) metaphysically possible for them to exist even if nothing else exists.

In short, noumena are humanly unperceivable abstract objects that are cognitively accessible only by means of pure higher-order conceptual thinking, or ideas of pure theoretical reason, which is why Kant calls them mere "entities of the understanding" (*Verstandeswesen*). By contrast, phenomena or appearances are sensory, spatiotemporal, mind-dependent substances which are defined and constituted by intrinsic relational properties, for example, ordinary macroscopic material objects. Phenomena are metaphysically incapable of "lonely" existence because they require the existence of space, time, causation, and actual or really possible human perceivers in every possible world in which they exist. In short, phenomena are humanly directly perceivable concrete objects that are cognitively accessible only by means of empirical intuition and pure formal intuition.

More precisely, then, the Kantian doctrine about the distinction between benign and vicious impredicativity is that benign impredicativity allows for the impredicative construction of phenomenal totalities via pure formal intuition alone, and also for the impredicative construction of noumenal totalities via pure theoretical reason alone. Vicious impredicativity systematically fails to recognize the fundamental difference between phenomena and noumena, and the corresponding fundamental difference between pure formal intuition and pure theoretical reason. Hence vicious impredicativity involves either the attempt to construct phenomenal totalities using self-including noumenal totalities, or the attempt to construct noumenal totalities using self-including phenomenal totalities.

Take, for example, Russell's paradoxical set K, that is, the set of all non-self-membered sets, which is a member of itself if and only if it is not a member of itself. For Kant, a set is the same as the complete or partial "comprehension" (*Umfang*) of a concept, that is, the cross-possible-worlds extension of a concept. This Kantian notion of a set as a complete or partial comprehension can be usefully compared and contrasted with *Frege's* notion of a set as the extension of a concept in *his* sense, that is, the collection or totality of inputs to a given function that maps from objects to the True. In turn, this means that for Kant, a set is *the lowest-order intensional entity* or *a rough-grained intension*. Comprehensions necessarily belong to concepts in *Kant's* sense, since every concept or fine-grained intension (*Inhalt*) uniquely determines its own comprehension—so, to coin a Kantian semantic slogan, *Inhalt* uniquely determines *Umfang*—but a comprehension is never exactly the same as a Kantian concept, which is always more or less fine-grained, whereas comprehensions are rough-grained intensions. Because sets are intensional entities, they are nonsensory and also abstract in the minimal sense that they are inherently repeatable or multiply realizable (i.e., they are repeated or realized wherever and whenever their corresponding concepts are). Sets in the Kantian sense, then, are intensional collections or totalities of objects *under* concepts, where the notion of "objects" is the strictly neutral notion that Kant calls "objects in general."

Consider, first, what I will call *phenomenal sets*. Phenomenal sets are sets containing only phenomenal objects. Sets themselves, as we have just seen, are nonsensory objects; hence, no set is a member of a phenomenal set. So there cannot be a phenomenal set containing *any* sets as members, much less the set of *all* sets of any kind. Otherwise put, there can be all sorts of phenomenal sets, and all sorts of phenomenal collections or totalities of phenomenal objects, including spatiotemporally self-containing phenomenal collections or totalities up to the transfinite numerosity of the real numbers, but none of these phenomenal sets contain any sets as members. Therefore, there cannot be a phenomenal set of all non-self-membered sets.

Consider, second, what I will call *noumenal sets*. Noumenal sets are sets containing only noumenal objects and that are *themselves* noumenal objects. Since all noumena are robustly abstract entities, and noumenal sets contain only noumena, then obviously noumenal sets can contain sets. Indeed, all sorts of collections or totalities of noumenal objects, including sets, can be constructed, including all those of transfinite numerosity greater than that of the real numbers. Moreover and more importantly, however, because noumena are defined and constituted by their intrinsic *nonrelational* properties, then they are essentially self-containing—otherwise put, noumena are strictly *identical with* what Leibniz would have called their "complete individual concepts." Therefore, every noumenal object essentially contains itself. So since every noumenal set is itself a noumenal object, then every noumenal set essentially contains itself and thereby is a *member* of itself, since every set is uniquely determined by its membership, which in turn is uniquely determined by a concept in the Kantian sense. Consequently, there are no non-self-membered noumenal sets.

Hence there are no phenomenal sets that contain any sets as members, including any non-self-membered sets, and there are no noumenal sets that are not members of themselves. Or otherwise put, since all sets are either phenomenal sets or noumenal sets, and since there are no sets that are both phenomenal and noumenal, then there is no such thing as a set containing all non-self-membered sets. This gives the Kantian a nonarbitrary way of separating the vicious impredicativity of the Russell set (which fatally blurs the fundamental difference between phenomenal sets and noumenal sets) from the nonvicious impredicativity of benignly impredicative phenomenal sets (which contain self-containing non-self-membered phenomenal collections or totalities only) and also the nonvicious impredicativity of benignly impredicative noumenal sets (which contain self-containing and self-membered noumenal collections or totalities only).

Here, then, is a positive four-part proposal about the Husserl/Gödel phenomenological theory of mathematical truth and knowledge. First, we should accept the two basic Husserl-Gödel ideas to the effect that

(i) the extra semantic resources required for mathematical truth over and above what is provided by elementary logic and the Peano axioms alone include ontologically rich, nonlogical higher-order concepts together with their corresponding abstract properties, relations, objects, and truth-making states of affairs, and

(ii) the extra cognitive resources required for mathematical knowledge over and above our capacities for logical analysis and logical inference include a nonempirical, innate, noninferential capacity—mathematical intuition—for grasping the contents of those concepts and also for referring to the abstract properties, relations, objects, and states of affairs that fall under those concepts and correspond to true mathematical statements.

Second, we should reject the Husserl/Gödel modal *monist* approach to the semantics of necessity and replace it with a Kantian modal *dualist* theory of the analytic-synthetic distinction, especially including his theory of the synthetic a priori, as based on Kantian Structuralism. Third, we should reject the Husserl/Gödel approach to modal epistemology, and replace it with the HW Theory. Fourth, we should supplement and enrich the Husserl/Gödel approach to impredicativity with the Kantian explanation of the difference between benign impredicativity and vicious impredicativity.

The basic advantages of this Husserl/Gödel + Kant (H/G + K) theory of mathematical truth and knowledge over other standard theories are at least sixfold. First, the H/G + K theory is seriously antideflationist and antiskeptical, but at the same time it is not problematically inflationist. Mathematical ontology on this view is a *mentalistic* ontology, so nothing over and above irreducible mental facts and properties needs to be added to the foundations of logic and mathematics insofar as they are constituted for the purposes of Gödel's incompleteness theorems. This is not a *sparse ontology*, but it remains a comparatively *parsimonious* ontology. Second, the H/G + K theory is moderately rationalist and nonreductively explains the nature of our a priori knowledge. Third, the H/G + K theory smoothly accommodates the failure of logicism. Fourth, as I have argued elsewhere,[22] the H/G + K theory solves not only the Caesar or identification problem of how to individuate the numbers, but also solves the classical application problem for arithmetic, by explicitly adopting a specifically nonreductive transcendental idealist version of ante rem Structuralism. Fifth, as I also argued in the same place, the H/G + K theory provides a positive or nonskeptical solution to the Bencerraf Dilemma, thereby effectively getting between inflationist and strongly rationalist classical platonism on the one hand and deflationist empiricist skeptical accounts on the other. Sixth and finally, the H/G + K theory promises to explain the previously unexplicated difference between benign impredicativity and vicious impredicativity in terms of a fundamental distinction between two essentially and exhaustively different kinds of objects—phenomenal objects and noumenal objects—and the corresponding idea of a fundamental distinction between phenomenal sets and noumenal sets.

Therefore, the H/G + K theory of mathematical truth and knowledge not only closes the Phenomenological Gap and shows how logic and mathematics essentially require irreducible facts about the conscious, intentional human mind, but also opens up a Great Gate to an equally post-inflationist and post-deflationist, moderately rationalist, and seriously antiskeptical condition of Paradise Regained in the philosophy of mathematics. Tieszen's *Phenomenology, Logic, and the Philosophy of Mathematics* provides us with a working key to that Great Gate.

Notes

1. For simplicity's sake, in this paper, by "the human mind" I mean the same as "the mind of a rational human animal" and "the mind of a human person," and I am leaving aside the actual fact that some minded human animals are either not rational (e.g., infants and other prelinguistic children) or not persons (e.g., permanently insane humans, or humans in the final stages of Alzheimer's, that is, so-called post-persons).

2. See, for example, Hanna [8].

3. See Parsons [20] and [21].

4. See Wang [23], [24], and [25].

5. [11]. For convenience, I will follow the common practice of citing page numbers from the A (1781) and B (1787) German editions only; so internal references to the first Critique will include the abbreviation "CPR" and the corresponding A or B edition numbers.

6. See, for example, Hanna [4], Chapters 1–2.

7. See, for example, Hanna [7], Chapters 1–4.

8. See Kripke [17].

9. See Hanna [4], Chapters 4–5.

10. See also Hanna [8], Chapter 6.

11. See Boolos and Jeffrey [1] and Denyer [2], pp. 220–24.

12. See Boolos and Jeffrey [1] and Denyer [2], pp. 220–24.

13. See, for example, Pap [19]; Katz [14], pp. 689–719; and Katz [15], pp. 1–28. See also Katz [12], Chapters 4–8, 10, and 13; and Katz [13], Chapter 5.

14. See, for example, Pap [19]; Katz [14], pp. 689–719; and Katz [15], pp. 1–28. See also Katz [12], Chapters 4–8, 10, and 13; and Katz [13], Chapter 5.

15. This is only the sketchiest of sketches. The crucial point is just that a Kantian philosophy of arithmetic can be plausibly presented as a version of mathematical structuralism. For details, see Hanna [5] and Hanna [10].

16. See, for example, McDowell [18].

17. See, for example, Hanna [9].

18. See, for example, Kosslyn, Thompson, and Ganis [16].

19. See, for example, Hanna [6].

20. See Hanna [7], Chapter 6.

21. This is also only the sketchiest of sketches, and the crucial point is just that a Kantian theory of mathematical intuition can be plausibly presented as a *nonintuitionistic* theory. For details, see Hanna [10], Section V.

22. See Hanna [10], Section IV.

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