Stable Formulas in Intuitionistic Logic

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Abstract   In 1995 Visser, van Benthem, de Jongh, and Renardel de Lavalette introduced NNIL-formulas, showing that these are (up to provable equivalence) exactly the formulas preserved under taking submodels of Kripke models. In this article we show that NNIL-formulas are up to frame equivalence the formulas preserved under taking subframes of (descriptive and Kripke) frames, that NNIL-formulas are subframe formulas, and that subframe logics can be axiomatized by NNIL-formulas. We also define a new syntactic class of ONNILLI-formulas. We show that these are (up to frame equivalence) the formulas preserved in monotonic images of (descriptive and Kripke) frames and that ONNILLI-formulas are stable formulas as introduced by Bezhanishvili and Bezhanishvili in 2013. Thus, ONNILLI is a syntactically defined set of formulas axiomatizing all stable logics. This resolves a problem left open in 2013.

1   Introduction

Intermediate logics are logics situated between intuitionistic propositional calculus (IPC) and classical propositional calculus (CPC). One of the central topics in the study of intermediate logics is their axiomatization. Jankov [14], by means of Heyting algebras, and de Jongh [12], via Kripke frames, developed an axiomatization method for intermediate logics using the so-called *splitting formulas*. These formulas are also referred to as *Jankov–de Jongh formulas*. In algebraic terminology, for each finite subdirectly irreducible Heyting algebra A, its Jankov formula is refuted in an algebra B, if there is a one-one Heyting homomorphism from A into a homomorphic image of B. In other words, the Jankov formula of A axiomatizes the greatest variety of Heyting algebras that does not contain A. In terms of Kripke frames, for each finite rooted frame $\mathfrak{F}$, the Jankov–de Jongh formula of $\mathfrak{F}$ is refuted in a frame

$\mathfrak{G}$ if and only if $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. In fact, the Jankov–de Jongh formula of $\mathfrak{F}$ axiomatizes the least intermediate logic that does not have $\mathfrak{F}$ as its frame. Large classes of intermediate logics (splitting and join-splitting logics) are axiomatizable by Jankov–de Jongh formulas. However, not every intermediate logic is axiomatizable by such formulas (see, e.g., Chagrov and Zakharyaschev [11, Section 9.4]).

Zakharyaschev [18], [19] introduced new classes of formulas called *subframe* and *cofinal subframe formulas* that axiomatize large classes of intermediate logics not axiomatizable by Jankov–de Jongh formulas. For each finite rooted frame $\mathfrak{F}$ the (cofinal) subframe formula of $\mathfrak{F}$ is refuted in a frame $\mathfrak{G}$ if and only if $\mathfrak{F}$ is a p-morphic image of a (cofinal) subframe of $\mathfrak{G}$. Logics axiomatizable by subframe and cofinal subframe formulas are called *subframe* and *cofinal subframe logics*, respectively. There is a continuum of such logics and each of them enjoys the finite model property. Moreover, Zakharyaschev showed that subframe logics are exactly those logics whose frames are closed under taking subframes. He also showed that an intermediate logic L is a subframe logic if and only if it is axiomatizable by $(\wedge, \rightarrow)$-formulas, and L is a cofinal subframe logic if and only if it is axiomatizable by $(\wedge, \rightarrow, \bot)$-formulas. However, there exist intermediate logics that are not axiomatizable by subframe and cofinal subframe formulas (see, e.g., [11, Section 9.4]). Finally, Zakharyaschev [18] introduced *canonical formulas* that generalize these three types of formulas and showed that every intermediate logic is axiomatizable by these formulas.

Zakharyaschev's method was model-theoretic. G. Bezhanishvili and Ghilardi [6] developed an algebraic approach to subframe and cofinal subframe logics which was later extended by G. Bezhanishvili and N. Bezhanishvili [1] to a full algebraic treatment of canonical formulas. This approach is based on identifying locally finite reducts of Heyting algebras. Recall that a variety V of algebras is called *locally finite* if the finitely generated V-algebras are finite. In logical terminology, the corresponding notion is called *local tabularity*. A logic L is called *locally tabular* if there exist only finitely many non-L-equivalent formulas in finitely many variables. Note that $\vee$-free reducts of Heyting algebras are locally finite.

Based on the above observation, for a finite subdirectly irreducible Heyting algebra A, [1] defined a formula that encodes fully the structure of the $\vee$-free reduct of A, and only partially the behavior of $\vee$. This results in a formula that has properties similar to the Jankov formula of A, but captures the behavior of A not with respect to Heyting homomorphisms, but rather morphisms that preserve the $\vee$-free reduct of A. This formula is called the $(\wedge, \rightarrow)$-*canonical formula of A*, and such $(\wedge, \rightarrow)$-canonical formulas axiomatize all intermediate logics. In [1], it was shown, via the Esakia duality for Heyting algebras, that $(\wedge, \rightarrow)$-canonical formulas are frame-equivalent to Zakharyaschev's canonical formulas, and that so defined subframe and cofinal subframe formulas are frame-equivalent to Zakharyaschev's subframe and cofinal subframe formulas.

However, Heyting algebras also have other locally finite reducts, namely, $\rightarrow$-free reducts. Recently, G. Bezhanishvili and N. Bezhanishvili [4] developed a theory of canonical formulas for intermediate logics based on these reducts of Heyting algebras. For a finite subdirectly irreducible Heyting algebra A, [4] defined the $(\wedge, \vee)$-*canonical formula of A* that encodes fully the structure of the $\rightarrow$-free reduct of A, and only partially the behavior of $\rightarrow$. One of the main results of [4] is that each

intermediate logic is axiomatizable by $(\wedge, \vee)$-canonical formulas. This parallels the result on the axiomatization of intermediate logics via $(\wedge, \rightarrow)$-canonical formulas.

The $(\wedge, \vee)$-canonical formulas produce a new class of formulas called *stable formulas*. It was shown in [4], via the Esakia duality, that for each finite rooted frame $\mathfrak{F}$ the stable formula of $\mathfrak{F}$ is refuted in a frame $\mathfrak{G}$ if and only if $\mathfrak{F}$ is an monotonic image of $\mathfrak{G}$. *Stable logics* are intermediate logics axiomatizable by stable formulas. There is a continuum of stable logics and all stable logics have the finite model property. Also, an intermediate logic is stable if and only if the class of its rooted frames is preserved under monotonic images (see [4]).

Thus, stable formulas play the same role for $(\wedge, \vee)$-canonical formulas that subframe formulas play for $(\wedge, \rightarrow)$-canonical formulas. Also, the role that subframes play for subframe formulas are played by monotonic images for stable formulas. A syntactic characterization of stable formulas was left in [4] as an open problem. The goal of this article is to resolve this problem. This is done via the NNIL-formulas of [16].

NNIL-formulas were introduced in Visser, van Benthem, de Jongh, and Renardel de Lafayette [16]. NNIL stands for *no nested implication to the left*. It was shown in [16] that these formulas are exactly the formulas that are closed under taking submodels of Kripke models. This implies that these formulas are also preserved under taking subframes. Moreover, for each finite rooted frame $\mathfrak{F}$, N. Bezhanishvili [7] constructs its subframe formula as an NNIL-formula. In Section 3 of this article we recall this characterization and use it to show that the class of NNIL-formulas is (up to frame equivalence) the same as the class of subframe formulas. Hence, an intermediate logic is a subframe logic if and only if it is axiomatized by NNIL-formulas. This also implies that each NNIL-formula is frame-equivalent to a $(\wedge, \rightarrow)$-formula. We refer to Yang [17] for more details on this.

In this article we define a new class of ONNILLI-formulas. ONNILLI stands for *only NNIL to the left of implications*. We show that each ONNILLI-formula is closed under monotonic images of rooted frames. For each finite rooted frame $\mathfrak{F}$, we also construct an ONNILLI-formula as its stable formula. This shows that the class of stable formulas (up to frame equivalence) is the same as the class of ONNILLI-formulas. We deduce from this that an intermediate logic is stable if and only if it is axiomatizable by ONNILLI-formulas.

The article is organized as follows. In Section 2 we recall Kripke and descriptive frames and models of intuitionistic logic and basic operations on them. In Section 3 we discuss in detail the connection between NNIL-formulas and subframe logics. In Section 4 we introduce ONNILLI-formulas and prove that they axiomatize stable logics.

2 Preliminaries

For the definition and basic facts about intuitionistic propositional calculus IPC, we refer to [11], van Dalen [15], or [7]. Here we briefly recall the Kripke semantics of intuitionistic logic.

Let $\mathcal{L}$ denote a *propositional language* consisting of

- infinitely many propositional variables (letters) $p_0, p_1, \ldots$,
- propositional connectives $\wedge, \vee, \rightarrow$,
- a propositional constant $\perp$.

We denote by Prop the set of all propositional variables. Formulas in $\mathcal{L}$ are defined as usual. Denote by Form($\mathcal{L}$) (or simply by Form) the set of all well-formed formulas in the language $\mathcal{L}$. We assume that $p, q, r, \ldots$ range over propositional variables and $\varphi, \psi, \chi, \ldots$ range over arbitrary formulas. For every formula φ and ψ, we let $\neg\varphi$ abbreviate $\varphi \to \bot$ and $\varphi \leftrightarrow \psi$ abbreviate $(\varphi \to \psi) \wedge (\psi \to \varphi)$. We also let $\top$ abbreviate $\neg\bot$.

We now quickly recall the Kripke semantics for intuitionistic logic. Let R be a binary relation on a set W. For every $w, v \in W$, we write wRv if $(w, v) \in R$, and we write $\neg(wRv)$ if $(w, v) \notin R$.

Definition 2.1

(1) An *intuitionistic Kripke frame* is a pair $\mathfrak{F} = (W, R)$, where $W \neq \emptyset$ and R is a partial order, that is, a reflexive, transitive, and antisymmetric relation on W.

(2) An *intuitionistic Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{F}$ is an intuitionistic Kripke frame and V is an *intuitionistic valuation*, that is, a map V from Prop to the powerset $\mathcal{P}(W)$ of W satisfying the condition:

$$w \in V(p) \text{ and } wRv \text{ implies } v \in V(p).$$

The definition of the satisfaction relation $\mathfrak{M}, w \models \varphi$, where $\mathfrak{M} = (W, R, V)$ is an intuitionistic Kripke model, $w \in W$, and $\varphi \in$ Form, is given in the usual manner (see, e.g., [11]). We will write $V(\varphi)$ for $\{w \in W \mid w \models \varphi\}$. The notions $\mathfrak{M} \models \varphi$ and $\mathfrak{F} \models \varphi$ (where $\mathfrak{F}$ is a Kripke frame) are also introduced as usual.

Let $\mathfrak{F} = (W, R)$ be a Kripke frame. $\mathfrak{F}$ is called *rooted* if there exists $w \in W$ such that for every $v \in W$, we have wRv. It is well known that IPC is complete with respect to finite rooted frames (see, e.g., [11, Theorem 5.12]).

Theorem 2.2   *For every formula φ, we have*

IPC $\vdash \varphi$ *if and only if φ is valid in every finite rooted Kripke frame.*

Next we recall the main operations on Kripke frames and models. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. For each $w \in W$ and $U \subseteq W$, let $R(w) = \{v \in W : wRv\}$, $R^{-1}(w) = \{v \in W : vRw\}$, $R^{-1}(w) = \{v \in W : vRw\}$, $R(U) = \bigcup_{w \in U} R(w)$, and $R^{-1}(U) = \bigcup_{w \in U} R^{-1}(w)$.

A subset $U \subseteq W$ is called an *upset of $\mathfrak{F}$* if for every $w, v \in W$, we have that $w \in U$ and wRv imply $v \in U$. A frame $\mathfrak{F}' = (U, R')$ is called a *generated subframe of $\mathfrak{F}$* if $U \subseteq W$, U is an upset of $\mathfrak{F}$, and R' is the restriction of R to U, that is, $R' = R \cap U^2$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a Kripke model. A model $\mathfrak{M}' = (\mathfrak{F}', V')$ is called a *generated submodel of $\mathfrak{M}$* if $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$ and V' is the restriction of V to U, that is, $V'(p) = V(p) \cap U$. We write $\mathfrak{M}_w$ for the *submodel of $\mathfrak{M}$ generated by w*, that is, with the domain $R(w)$.

Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be Kripke frames. A map $f : W \to W'$ is called a *p-morphism*[1] *between $\mathfrak{F}$ and $\mathfrak{F}'$* if for every $w, v \in W$ and $w' \in W'$:

(1) wRv implies $f(w)R'f(v)$,

(2) $f(w)R'w'$ implies that there exists $u \in W$ such that wRu and $f(u) = w'$.

We call the conditions (1) and (2) the "forth" and "back" conditions, respectively. We say that f is *monotonic* if it satisfies the forth condition. If f is a surjective p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$, then $\mathfrak{F}'$ is called a *p-morphic image of $\mathfrak{F}$*. Let $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{M}' = (\mathfrak{F}', V')$ be Kripke models. A map $f : W \to W'$ is called

a *p-morphism between* $\mathfrak{M}$ *and* $\mathfrak{M}'$ if f is a p-morphism between $\mathfrak{F}$ and $\mathfrak{F}'$, and for every $w \in W$ and $p \in \text{Prop}$:

$$\mathfrak{M}, w \models p \text{ iff } \mathfrak{M}', f(w) \models p.$$

If a map between models satisfies the above condition, then we call it *valuation-preserving*. If f is surjective, then $\mathfrak{M}$ is called a *p-morphic image of* $\mathfrak{M}'$; surjective p-morphisms are also called *reductions* (see, e.g., [11]). A *monotonic map* between two models is a monotonic map between the underlying frames which, in addition, is valuation-preserving.

Next we recall the definition of general frames (see, e.g., [11, Sections 8.1 and 8.4]).

Definition 2.3   An *intuitionistic general frame* or simply a *general frame* is a triple $\mathfrak{F} = (W, R, \mathcal{P})$, where (W, R) is an intuitionistic Kripke frame and $\mathcal{P}$ is a set of upsets such that $\emptyset$ and W belong to $\mathcal{P}$, and $\mathcal{P}$ is closed under $\cup$, $\cap$, and $\Rightarrow$ defined by

$$U_1 \Rightarrow U_2 := \{w \in W : \forall v(wRv \wedge v \in U_1 \to v \in U_2)\} = W \setminus R^{-1}(U_1 \setminus U_2).$$

Note that every Kripke frame can be seen as a general frame where $\mathcal{P}$ is the set of all upsets of $\mathfrak{F} = (W, R, \mathcal{P})$. A *valuation* on a general frame is a map $V : \text{Prop} \to \mathcal{P}$. The pair $(\mathfrak{F}, V)$ is called a *general model*. The validity of formulas in general models is defined exactly the same way as for Kripke models.

Definition 2.4   Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a general frame.

(1) We call $\mathfrak{F}$ *refined* if for every $w, v \in W$: $\neg(wRv)$ implies that there is $U \in \mathcal{P}$ such that $w \in U$ and $v \notin U$.

(2) We call $\mathfrak{F}$ *compact* if for every $\mathcal{X} \subseteq \mathcal{P} \cup \{W \setminus U : U \in \mathcal{P}\}$, if $\mathcal{X}$ has the *finite intersection property* (i.e., every intersection of finitely many elements of $\mathcal{X}$ is nonempty), then $\bigcap \mathcal{X} \neq \emptyset$.

(3) We call $\mathfrak{F}$ *descriptive* if it is refined and compact.

We call the elements of $\mathcal{P}$ *admissible sets*.

Definition 2.5   Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. A *descriptive valuation* is a map $V : \text{Prop} \to \mathcal{P}$. A pair $(\mathfrak{F}, V)$ where V is a descriptive valuation is called a *descriptive model*.

Validity of formulas in a descriptive frame (model) is defined similarly to the Kripke case except that it ranges over all descriptive valuations. It is well known that every intermediate logic L is complete with respect to a class of descriptive frames (see, e.g., [11, Theorem 8.36]).

Next we recall the definitions of generated subframes and p-morphisms of descriptive frames.

Definition 2.6

(1) A descriptive frame $\mathfrak{F}' = (W', R', \mathcal{P}')$ is called a *generated subframe* of a descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ if (W', R') is a generated subframe of (W, R) and $\mathcal{P}' = \{U \cap W' : U \in \mathcal{P}\}$.

(2) A map $f : W \to W'$ is called a *p-morphism* between $\mathfrak{F} = (W, R, \mathcal{P})$ and $\mathfrak{F}' = (W', R', \mathcal{P}')$ if f is a p-morphism between (W, R) and (W', R') and for every $U' \in \mathcal{P}'$ we have $f^{-1}(U') \in \mathcal{P}$. If a map between descriptive models satisfies the latter condition, it is called *admissible*.

Generated submodels and p-morphisms between descriptive models are defined as in the case of Kripke semantics. For convenience, we will sometimes denote a descriptive frame, just as a pair (W, R), dropping the set $\mathcal{P}$ of admissible sets from the signature.

3 Subframe Logics and NNIL-Formulas

Subframe formulas for modal logic were first introduced by Fine [13]. Subframe formulas for intuitionistic logic were defined by Zakharyaschev [18]. For an overview of these results, see [11, Section 9.4]. For an algebraic approach to subframe formulas, we refer to [6] and [1]. We will define subframe formulas differently and connect them to the NNIL-formulas of [16]. Most of the results in this section have appeared in the PhD dissertation [7].

We first recall from [16] and [17] some facts about NNIL-formulas. NNIL-formulas are known to have the following normal form.

Definition 3.1   NNIL-formulas (*no nested implication to the left*) in *normal form* are defined by

$$\varphi := \bot \,|\, p \,|\, \varphi \wedge \varphi \,|\, \varphi \vee \varphi \,|\, p \rightarrow \varphi.$$

Definition 3.2

(1) Let $\mathfrak{F} = (W, R)$ be a Kripke frame. A frame $\mathfrak{F}' = (W', R')$ is called a *subframe* of $\mathfrak{F}$ if $W' \subseteq W$ and R' is the restriction of R to W'.

(2) Let $\mathfrak{F} = (W, R, \mathcal{P})$ be a descriptive frame. A descriptive frame $\mathfrak{F}' = (W', R', \mathcal{P}')$ is called a subframe of $\mathfrak{F}$ if (W', R') is a subframe of (W, R), $\mathcal{P}' = \{U \cap W' : U \in \mathcal{P}\}$ and the following condition, which we call the *topo-subframe condition*, is satisfied:

For every $U \subseteq W'$ such that $W' \setminus U \in \mathcal{P}'$ we have $W \setminus R^{-1}(U) \in \mathcal{P}$.

For a detailed discussion about the topological motivation behind the notion of subframes and its connection to nuclei of Heyting algebras, we refer to [6] (see also [7]). Here we just note how we are going to use this condition.

Remark 3.3   The reason for adding the topo-subframe condition to the definition of subframes of descriptive frames is explained by the next proposition. The topo-subframe condition allows us to extend a descriptive valuation V' defined on a subframe $\mathfrak{F}'$ of a descriptive frame $\mathfrak{F}$ to a descriptive valuation V of $\mathfrak{F}$ such that the restriction of V to $\mathfrak{F}'$ is equal to V'.

Proposition 3.4   *Let $\mathfrak{F} = (W, R, \mathcal{P})$ and $\mathfrak{F}' = (W', R', \mathcal{P}')$ be descriptive frames. If $\mathfrak{F}'$ is a subframe of $\mathfrak{F}$, then for every descriptive valuation V' on $\mathfrak{F}'$ there exists a descriptive valuation V on $\mathfrak{F}$ such that the restriction of V to W' is V'.*

Proof   For every $p \in \text{Prop}$, let $V(p) = W \setminus R^{-1}(W' \setminus V'(p))$. By the topo-subframe condition, $V(p) \in \mathcal{P}$. Now suppose $x \in W'$. Then $x \notin V(p)$ if and only if $x \in R^{-1}(W' \setminus V'(p))$ if and only if (there is $y \in W'$ such that $y \notin V'(p)$ and xRy) if and only if $x \notin V'(p)$, since $V'(p)$ is an upset of $\mathfrak{F}'$. Therefore, $V(p) \cap W' = V'(p)$.   □

We say that a formula φ is *preserved under submodels*, if for all models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{N} = (W', R', V')$, if w is in the domain of $\mathfrak{N}$, $\mathfrak{N}$ is a submodel of $\mathfrak{M}$, and $\mathfrak{M}, w \models \varphi$, then $\mathfrak{N}, w \models \varphi$. We say that a formula φ is *preserved under subframes*,

if for all (descriptive or Kripke) frames $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (W', R')$, if $\mathfrak{G}$ is a subframe of $\mathfrak{F}$ and $\mathfrak{F} \models \varphi$, then $\mathfrak{G} \models \varphi$.

We have the following characterization theorem showing that NNIL-formulas are exactly the ones that are preserved under submodels (see [16]).

Theorem 3.5

(1) *Every $\varphi \in$ NNIL is preserved under submodels.*
(2) *If φ is preserved under submodels, then there exists $\psi \in$ NNIL such that IPC $\vdash \psi \leftrightarrow \varphi$.*

Corollary 3.6 NNIL-*formulas are preserved under subframes.*

Proof Assume that an NNIL-formula is not preserved under subframes. Then there exists an NNIL-formula φ, (descriptive or Kripke) frames $\mathfrak{G}$ and $\mathfrak{F}$ such that $\mathfrak{F}$ is a subframe of $\mathfrak{G}$, $\mathfrak{G} \models \varphi$, and $\mathfrak{F} \not\models \varphi$. So there exists a valuation V on $\mathfrak{F}$ such that $(\mathfrak{F}, V) \not\models \varphi$. Let V' be a valuation on $\mathfrak{G}$ such that $(\mathfrak{F}, V)$ is a submodel of $(\mathfrak{G}, V')$. If $\mathfrak{F}$ and $\mathfrak{G}$ are descriptive frames, then such a V' exists by Proposition 3.4. If $\mathfrak{F}$ and $\mathfrak{G}$ are Kripke frames, then we again use the valuation V' defined in Proposition 3.4. Thus, we obtain that φ is not preserved under submodels, which contradicts Theorem 3.5. $\qquad\qquad\square$

A formula is called a *subframe formula* if it is preserved under subframes. An intermediate logic is called a *subframe logic* if it is axiomatizable by subframe formulas. Chagrov and Zakharyaschev proved (see, e.g., [11, Theorem 11.25]) that an intermediate logic L is a subframe logic if and only if L is axiomatizable by $(\wedge, \rightarrow)$-formulas if and only if descriptive frames of L are closed under subframes. Also, every subframe logic has the finite model property (see [11, Theorem 11.20]).

Definition 3.7 Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a descriptive model. We fix n propositional variables $p_1, \ldots, p_n$. With every point w of $\mathfrak{M}$, we associate a sequence $i_1 \ldots i_n$ such that for $k = 1, \ldots, n$:

$$i_k = \begin{cases} 1, & \text{if } w \models p_k, \\ 0, & \text{if } w \not\models p_k. \end{cases}$$

We call the sequence $i_1 \ldots i_n$ associated with w the *color* of w (or, more specifically, the *n-color* of w) and denote it by $\mathrm{col}(w)$.

A finite model $\mathfrak{M} = (W, R, V)$ is *colorful* if the number of propositional variables is $|W|$ and, for each $w \in W$, there is a propositional variable p_w such that $v \models p_w$ if and only if wRv.

Definition 3.8 Let $i_1 \ldots i_n$ and $j_1 \ldots j_n$ be two colors. We write

$$i_1 \ldots i_n \leq j_1 \ldots j_n \text{ if and only if } i_k \leq j_k \text{ for each } k = 1, \ldots, n.$$

We also write $i_1 \ldots i_n < j_1 \ldots j_n$ if $i_1 \ldots i_n \leq j_1 \ldots j_n$ and $i_1 \ldots i_n \neq j_1 \ldots j_n$.

Let $\mathfrak{F}$ be a finite rooted frame. For every point w of $\mathfrak{F}$, we introduce a propositional letter p_w and let V be such that $V(p_w) = R(w)$. We denote the model $(\mathfrak{F}, V)$ by $\mathfrak{M}$. Then $\mathfrak{M}$ is colorful. Thus, with any finite frame $\mathfrak{F}$ we can associate a colorful model $(\mathfrak{F}, V)$. We call $(\mathfrak{F}, V)$ the *colorful model corresponding to* $\mathfrak{F}$.

Lemma 3.9 *Let $(\mathfrak{F}, V)$ be a colorful model. Then, for every $w, v \in W$, we have*

(1) $w \neq v$ and wRv if and only if $\mathrm{col}(w) < \mathrm{col}(v)$,
(2) $w = v$ if and only if $\mathrm{col}(w) = \mathrm{col}(v)$.

Proof   The proof is just spelling out the definitions.   □

Let $\mathfrak{F} = (W, R)$ be a finite rooted frame, and let $\mathfrak{M} = (\mathfrak{F}, V)$ be the corresponding colorful model. Next, we inductively define the subframe formula $\beta(\mathfrak{F})$ in the NNIL form. For every $v \in W$, let

$$\text{prop}(v) := \{p_k : v \models p_k, k \leq n\}, \qquad \text{notprop}(v) := \{p_k : v \not\models p_k, k \leq n\}.$$

Definition 3.10   We define $\beta(\mathfrak{F})$ by induction. If v is a maximal point of $\mathfrak{M}$, then let

$$\beta(v) := \bigwedge \text{prop}(v) \to \bigvee \text{notprop}(v).$$

Let w be a point in $\mathfrak{M}$, and let $w_1, \ldots, w_m$ be all the immediate successors of w. We assume that $\beta(w_i)$ is already defined, for every w_i. We define $\beta(w)$ by

$$\beta(w) := \bigwedge \text{prop}(w) \to \bigvee \text{notprop}(w) \vee \bigvee_{i=1}^{m} \beta(w_i).$$

Let r be the root of $\mathfrak{F}$. We define $\beta(\mathfrak{F})$ by

$$\beta(\mathfrak{F}) := \beta(r).$$

We call $\beta(\mathfrak{F})$ the *subframe formula* of $\mathfrak{F}$.

Note that $\beta(\mathfrak{F})$ is an NNIL-formula. We will need the next lemma for establishing the crucial property of subframe formulas. We first recall the definition of depth of a frame and of a point.

Definition 3.11   Let $\mathfrak{F}$ be a (descriptive or Kripke) frame.

(1) We say that $\mathfrak{F}$ is of *depth $n < \omega$*, denoted $d(\mathfrak{F}) = n$, if there is a chain of n points in $\mathfrak{F}$ and no other chain in $\mathfrak{F}$ contains more than n points. The frame $\mathfrak{F}$ is of finite depth if $d(\mathfrak{F}) < \omega$.

(2) We say that $\mathfrak{F}$ is of an *infinite depth*, denoted $d(\mathfrak{F}) = \omega$, if for every $n \in \omega$, $\mathfrak{F}$ contains a chain consisting of n points.

(3) The *depth* of a point $w \in W$ is the depth of $\mathfrak{F}_w$, that is, the depth of the subframe of $\mathfrak{F}$ generated by w. We denote the depth of w by $d(w)$.

Lemma 3.12   *Let $\mathfrak{F} = (W, R)$ be a finite rooted frame, and let V be defined as above. Let $\mathfrak{M}' = (W', R', V')$ be an arbitrary (descriptive or Kripke) model. For every $w, v \in W$ and $x \in W'$, if wRv, then*

$$\mathfrak{M}', x \not\models \beta(w) \text{ implies } \mathfrak{M}', x \not\models \beta(v).$$

Proof   The proof is a simple induction on the depth of v. If $d(v) = d(w) - 1$ and wRv, then v is an immediate successor of w. Then $\mathfrak{M}', x \not\models \beta(w)$ implies $\mathfrak{M}', x \not\models \beta(v)$, by the definition of $\beta(w)$. Now suppose that $d(v) = d(w) - (k+1)$ and the lemma is true for every u such that wRu and $d(u) = d(w) - k$, for every k. Let u' be an immediate predecessor of v such that wRu'. Such a point clearly exists since we have wRv. Then $d(u') = d(w) - k$ and by the induction hypothesis $\mathfrak{M}, x \not\models \beta(u')$. This, by definition of $\beta(u')$, means that $\mathfrak{M}', x \not\models \beta(v)$.   □

The next theorem states the crucial property of subframe formulas (see also [7, Theorem 3.3.16]).

Theorem 3.13 *Let $\mathfrak{G} = (W', R', \mathcal{P}')$ be a descriptive frame, and let $\mathfrak{F} = (W, R)$ be a finite rooted frame. Then*

$$\mathfrak{G} \not\models \beta(\mathfrak{F}) \text{ if and only if } \mathfrak{F} \text{ is a p-morphic image of a subframe of } \mathfrak{G}.$$

Proof Suppose that $\mathfrak{G} \not\models \beta(\mathfrak{F})$. Then there exists a valuation V' on $\mathfrak{G}$ such that $(\mathfrak{G}, V') \not\models \beta(\mathfrak{F})$. For every $w \in W$, let $\{w_1, \ldots, w_m\}$ denote the set of all immediate successors of w. Let $p_1, \ldots, p_n$ be the propositional variables occurring in $\beta(\mathfrak{F})$ (in fact, $n = |W|$). Then, V' defines a coloring of $\mathfrak{G}$. Let

$$P_w := \left\{ x \in W' : \mathrm{col}(x) = \mathrm{col}(w) \text{ and } x \not\models \bigvee_{i=1}^{m} \beta(w_i) \right\}.$$

Take $Y := \bigcup_{w \in W} P_w$ and $\mathfrak{H} := (Y, S, \mathcal{Q})$, where S is the restriction of R' to Y, and $\mathcal{Q} = \{U' \cap Y : U' \in \mathcal{P}'\}$. We show that $\mathfrak{H}$ is a subframe of $\mathfrak{G}$ and $\mathfrak{F}$ is a p-morphic image of $\mathfrak{H}$.

For the proof that $\mathfrak{H}$ is a subframe of $\mathfrak{G}$, we just check the topo-subframe condition. The other conditions are clear from the definition of $\mathfrak{H}$. So, assume that $Y \setminus U' \in \mathcal{Q}$. We have to show that $W' \setminus R'^{-1}(U') \in \mathcal{P}'$.

Note that

$$
\begin{aligned}
x \in W' \setminus R'^{-1}(U') \quad &\text{iff} \quad x \notin R'^{-1}(U') \\
&\text{iff} \quad \neg \exists y (xRy \wedge y \in U') \\
&\text{iff} \quad \forall y (xRy \rightarrow y \notin U') \\
&\text{iff} \quad \forall y (xRy \rightarrow y \notin Y \vee y \in Y \setminus U') \\
&\text{iff} \quad \forall y (xRy \wedge y \in Y \rightarrow y \in Y \setminus U') \\
&\text{iff} \quad \text{for } U'' \in \mathcal{P}' \text{ such that } Y \setminus U' = U'' \cap Y, \\
&\qquad \forall y (xRy \wedge y \in Y \rightarrow y \in U'').
\end{aligned}
$$

Since $Y = \bigcup_{w \in W} P_w$, the latter is equivalent to the conjunction of all the $\forall y (xRy \wedge y \in P_w \rightarrow y \in U'')$ for $w \in W$. Then

$$
\begin{aligned}
\forall y (xRy \wedge y \in P_w \rightarrow y \in U'') \quad &\text{iff} \quad \forall y (xRy \wedge \mathrm{col}(y) = \mathrm{col}(w) \wedge \\
&\qquad y \not\models \textstyle\bigvee_{i=1}^{m} \beta(w_i) \rightarrow y \in U'') \\
&\text{iff} \quad \forall y (xRy \wedge y \models \bigwedge \mathrm{prop}(w) \rightarrow \\
&\qquad y \models \bigvee \mathrm{notprop}(w) \vee \\
&\qquad y \models \textstyle\bigvee_{i=1}^{m} \beta(w_i) \vee y \in U'').
\end{aligned}
$$

Now the sets $\{x \mid \forall y (xRy \wedge y \models \bigwedge \mathrm{prop}(w) \rightarrow y \models \bigvee \mathrm{notprop}(w) \vee y \models \bigvee_{i=1}^{m} \beta(w_i) \vee y \in U'')\}$ are equal to the sets $V'(\bigwedge \mathrm{prop}(w)) \Rightarrow (V'(\bigvee \mathrm{notprop}(w)) \cup V'(\bigvee_{i=1}^{m} \beta(w_i)) \cup U'')$ and therefore are in $\mathcal{P}'$, as $\mathcal{P}'$ is closed under $\Rightarrow$ and union. So their intersection (the conjunction of the corresponding formulas) is also in $\mathcal{P}'$.

Define a map $f : Y \rightarrow W$ by

$$f(x) = w \text{ if } x \in P_w.$$

We show that f is a well defined onto p-morphism. By Lemma 3.9, distinct points of W have distinct colors. Therefore, $P_w \cap P_{w'} = \emptyset$ if $w \neq w'$. This means that f is well defined.

To prove that f is onto, by the definition of f, it is sufficient to show that $P_w \neq \emptyset$ for every $w \in W$. If r is the root of $\mathfrak{F}$, then since $(\mathfrak{G}, V') \not\models \beta(\mathfrak{F})$, there exists a point $x \in W'$ such that $x \models \bigwedge \mathrm{prop}(r)$ and $x \not\models \bigvee \mathrm{notprop}(r)$ and $x \not\models \bigvee_{i=1}^{m} \beta(r_i)$.

This means that $x \in P_r$. If w is not the root of $\mathfrak{F}$, then we have rRw. Therefore, by Lemma 3.12, we have $x \not\models \beta(w)$. This means that there is a successor y of x such that $y \models \bigwedge \text{prop}(w)$, $y \not\models \bigvee \text{notprop}(w)$, and $y \not\models \beta(w_i)$, for every immediate successor w_i of w. Therefore, $y \in P_w$ and f is surjective.

To show that f is admissible, we first note that to show an onto p-morphism to a finite frame to be admissible, it is sufficient to show that for every upset U of W we have $f^{-1}(U) \in \mathcal{Q}$. It is clear that $U = R(u_1) \cup \cdots \cup R(u_k)$ for some $u_1, \ldots, u_k \in W$. Since $(\mathfrak{F}, V)$ is a colorful model, we have $R(u_i) = V(p_{u_i})$ for each $i = 1, \ldots, k$. Now observe that by the definition of f, for each $x \in Y$ we have $\text{col}(f(x)) = \text{col}(x)$. So $f(x) \in V(p_{u_i})$ if and only if $x \in V'(p_{u_i})$. This means that $f^{-1}(U) = (V'(p_{u_1}) \cup \cdots \cup V'(p_{u_k})) \cap Y$, which clearly is in $\mathcal{Q}$.

Next assume that $x, y \in Y$ and xSy. Note that by the definition of f, for every $t \in Y$ we have

$$\text{col}(t) = \text{col}(f(t)).$$

Obviously, xSy implies $\text{col}(x) \leq \text{col}(y)$. Therefore, $\text{col}(f(x)) = \text{col}(x) \leq \text{col}(y) = \text{col}(f(y))$. By Lemma 3.9, this yields $f(x)Rf(y)$. Now suppose $f(x)Rf(y)$. Then by the definition of f, we have that $x \not\models \beta(f(x))$, and by Lemma 3.12, $x \not\models \beta(f(y))$. This means that there is $z \in W'$ such that $xR'z$, $\text{col}(z) = \text{col}(f(y))$, and $z \not\models \beta(u)$, for every immediate successor u of $f(y)$. Thus, $z \in P_{f(y)}$ and $f(z) = f(y)$. Therefore, $\mathfrak{F}$ is a p-morphic image of $\mathfrak{S}$.

Conversely, suppose that $\mathfrak{S}$ is a subframe of a descriptive frame $\mathfrak{G}$ and that $f : \mathfrak{S} \to \mathfrak{F}$ is a p-morphism. Clearly, $\mathfrak{F} \not\models \beta(\mathfrak{F})$ and since f is a p-morphism, we have that $\mathfrak{S} \not\models \beta(\mathfrak{F})$. This means that there is a valuation V' on $\mathfrak{S}$ such that $(\mathfrak{S}, V') \not\models \beta(\mathfrak{F})$. By Proposition 3.4, V' can be extended to a valuation V on $\mathfrak{G}$ such that the restriction of V to $\mathfrak{G}'$ is equal to V'. Finally, recall that $\mathfrak{G}$ is an NNIL-formula. This, by Theorem 3.5(1), implies that $\mathfrak{G} \not\models \beta(\mathfrak{F})$. $\square$

Zakharyaschev [18] showed that every subframe logic is axiomatizable by the formulas satisfying the condition of Theorem 3.13. We will now put this result in the context of the first-named author's frame-based formulas of [7] and [8]. We will use the same argument in the next section for stable logics and ONNILI-formulas.

For each intermediate logic L, let $\mathbb{DF}(L)$ be the class of rooted descriptive frames of L. Note that [7] and [8] work with finitely generated descriptive frames. But for our purposes this restriction is not essential.

Definition 3.14   Call a reflexive and transitive relation $\trianglelefteq$ on $\mathbb{DF}(\mathsf{IPC})$ a *frame order* if the following two conditions are satisfied.

(1) For every $\mathfrak{F}, \mathfrak{G} \in \mathbb{DF}(\mathsf{IPC})$, $\mathfrak{G}$ is finite and $\mathfrak{F} \lhd \mathfrak{G}$ imply that $|\mathfrak{F}| < |\mathfrak{G}|$.
(2) For every finite rooted frame $\mathfrak{F}$ there exists a formula $\alpha(\mathfrak{F})$ such that, for every $\mathfrak{G} \in \mathbb{DF}(\mathsf{IPC})$,

$$\mathfrak{G} \not\models \alpha(\mathfrak{F}) \text{ if and only if } \mathfrak{F} \trianglelefteq \mathfrak{G}.$$

The formula $\alpha(\mathfrak{F})$ is called the *frame-based formula for* $\trianglelefteq$.

Definition 3.15   Let L be an intermediate logic. We let

$$\mathbf{M}(L, \trianglelefteq) := \min_{\trianglelefteq}\big(\mathbb{DF}(\mathsf{IPC}) \setminus \mathbb{DF}(L)\big).$$

Theorem 3.16 ([7, Theorem 3.4.12], [8, Theorem 3.9])   *Let L be an intermediate logic, and let $\trianglelefteq$ be a frame order on $\mathbb{DF}(\mathsf{IPC})$. Then L is axiomatized by frame-based formulas for $\trianglelefteq$ if and only if the following two conditions are satisfied.*

(1) $\mathbb{DF}(L)$ *is a $\trianglelefteq$-downset. That is, for every $\mathfrak{F}, \mathfrak{G} \in \mathbb{DF}(\mathsf{IPC})$, if $\mathfrak{G} \in \mathbb{DF}(L)$ and $\mathfrak{F} \trianglelefteq \mathfrak{G}$, then $\mathfrak{F} \in \mathbb{DF}(L)$.*

(2) *For every $\mathfrak{G} \in \mathbb{DF}(\mathsf{IPC}) \setminus \mathbb{DF}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)$ such that $\mathfrak{F} \trianglelefteq \mathfrak{G}$.*

The formula $\beta(\mathfrak{F})$ is a particular case of a frame-based formula for a relation $\preccurlyeq$, where $\mathfrak{F} \preccurlyeq \mathfrak{G}$ if $\mathfrak{F}$ is a p-morphic image of a subframe of $\mathfrak{G}$. Theorem 3.16(2) is always satisfied by $\preccurlyeq$ (see [11, Theorem 9.36]); for an algebraic proof of this fact, see [6] and [1]. So an intermediate logic L is a subframe logic if and only if L is axiomatizable by these formulas if and only if $\mathbb{DF}(L)$ is a $\preccurlyeq$-downset. As p-morphic images preserve the validity of formulas, we obtain that $\mathbb{DF}(L)$ is a $\preccurlyeq$-downset if and only if $\mathbb{DF}(L)$ is closed under subframes. Thus, L is a subframe logic if and only if L is axiomatizable by these formulas if and only if $\mathbb{DF}(L)$ is closed under subframes.

We say that formulas φ and ψ are *frame-equivalent* if for any (descriptive) frame $\mathfrak{F}$, we have $\mathfrak{F} \models \varphi$ if and only if $\mathfrak{F} \models \psi$.

Corollary 3.17

(1) *An intermediate logic L is a subframe logic if and only if L is axiomatizable by* NNIL-*formulas.*

(2) *The class of* NNIL-*formulas (up to frame equivalence) coincides with the class of subframe formulas.*

(3) *Each* NNIL-*formula is frame-equivalent to a $(\wedge, \rightarrow)$-formula.*

Proof (1) As we showed above, L is a subframe logic if and only if it is axiomatizable by the formulas of type $\beta(\mathfrak{F})$. As each $\beta(\mathfrak{F})$ is NNIL, subframe logics are axiomatizable by NNIL-formulas. Conversely, by Corollary 3.6, every NNIL-formula is preserved under subframes. Therefore, if L is axiomatizable by NNIL-formulas, $\mathbb{DF}(L)$ is closed under subframes. Thus, L is a subframe logic.

(2) By Corollary 3.6, every NNIL-formula is preserved under subframes. So every NNIL-formula is a subframe formula. Now suppose that φ is preserved under subframes. Then $\mathsf{IPC} + \varphi$ (where $\mathsf{IPC} + \varphi$ is the least intermediate logic containing formula φ) is a subframe logic. By (1) subframe logics are axiomatizable by the formulas $\beta(\mathfrak{F})$. Then there exists $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ such that $\mathsf{IPC} + \varphi = \mathsf{IPC} + \bigwedge_{i=1}^{n} \beta(\mathfrak{F}_i)$. Note that $n \in \omega$, otherwise $\mathsf{IPC} + \varphi$ is infinitely axiomatizable, which is a contradiction. Each $\beta(\mathfrak{F}_i)$ is an NNIL-formula, so $\bigwedge_{i=1}^{n} \beta(\mathfrak{F}_i)$ is also an NNIL-formula. Thus, φ is frame-equivalent to an NNIL-formula and NNIL is (up to frame equivalence) the class of formulas preserved under subframes.

(3) This also follows from (1) and the fact that subframe formulas are frame-equivalent to $(\wedge, \rightarrow)$-formulas (see [11, Theorem 11.25]). A direct syntactic proof that each NNIL-formula is frame-equivalent to some $(\wedge, \rightarrow)$-formula can be found in [17]. □

We do not treat cofinal subframe logics here as they are not axiomatized by NNIL-formulas. We refer to [11, Section 9.4] for a detailed treatment of these logics, to [6] and [1] for their algebraic analysis, and to [7, Section 3.3.3] for the details on how to obtain cofinal subframe formulas from the subframe formulas introduced in this article.

4 Stable Logics and ONNILLI-Formulas

In this section we construct a new class of formulas, ONNILLI, that turns out to be
the class of formulas preserved by onto monotonic maps. This class is defined using
the class of NNIL-formulas.

Proposition 4.1   *Let* $\mathfrak{M} = (X, R, V)$ *and* $\mathfrak{N} = (Y, R', V')$ *be two intuitionistic
(Kripke or descriptive) models, and let* $f : X \to Y$ *be a monotonic map on these
models. Then, for each* $x \in X$ *and each* $\varphi \in$ NNIL*, we have*

$$f(x) \models \varphi \Rightarrow x \models \varphi.$$

Proof   This is by induction on the normal form of φ as in Definition 3.1. The basic
steps are trivial. Assume that the induction hypothesis holds for φ and ψ and that
$f(x) \models \varphi \wedge \psi$. Then, $f(x) \models \varphi$ and $f(x) \models \psi$. By the induction hypothesis,
$x \models \varphi$ and $x \models \psi$. So, $x \models \varphi \wedge \psi$. The case for $\vee$ is similar.

So, finally assume that the induction hypothesis holds for φ and that $f(x) \models
p \to \varphi$. Now let xRy and $y \models p$. Then $f(x)Rf(y)$ and, as f is valuation-
preserving, $f(y) \models p$. So, $f(y) \models \varphi$. By the induction hypothesis, $y \models \varphi$. So
$x \models p \to \varphi$.   □

Corollary 4.2   *For each formula* ψ *there exists an* NNIL-*formula* φ *such that*
IPC $\vdash \varphi \leftrightarrow \psi$ *if and only if for any pair of intuitionistic (Kripke or descriptive)
models* $\mathfrak{M} = (X, R, V)$ *and* $\mathfrak{N} = (Y, R', V')$ *with a monotonic map* $f : X \to Y$
and $x \in X$, *we have*

$$f(x) \models \psi \Rightarrow x \models \psi. \tag{1}$$

Proof   The left-to-right direction follows from Proposition 4.1. Conversely, note
that the identity function from a submodel into the larger model is always a mono-
tonic map. Thus, if ψ satisfies (1), then ψ is preserved in submodels and, by Theo-
rem 3.5, is equivalent to some NNIL-formula φ.   □

Definition 4.3

  (1) BASIC is the closure of the set of the propositional variables plus $\top$ and $\bot$
      under conjunctions and disjunctions.
  (2) Formulas of the form $\varphi \to \psi$ with $\varphi \in$ NNIL and $\psi \in$ BASIC are called
      simple ONNILLI-*formulas*.
  (3) The class ONNILLI (*only* NNIL *to the left of implications*) is defined as the
      closure of the set of simple ONNILLI-formulas under conjunctions and dis-
      junctions.

Note that there are no iterations of implications in ONNILLI-formulas except inside
the NNIL-part. Note also that, if $\psi \in$ BASIC and f is valuation-preserving, then by
a simple induction argument we obtain that

$$y \models \psi \Leftrightarrow f(y) \models \psi. \tag{2}$$

Example 4.4 $\neg p \vee \neg\neg p$ is ONNILLI. To see this, write it as $(p \to \bot) \vee (\neg p \to \bot)$,
and note that $\neg p$ is in NNIL. It is well known that $\neg p \vee \neg\neg p$ is not preserved under
taking subframes. (Note, however, that $\neg p \vee \neg\neg p$ is preserved under taking cofinal
subframes; see, e.g., [11, Section 9.4].) So, by Corollary 3.17 it cannot be equivalent
to an NNIL-formula. Thus the class NNIL does not contain ONNILLI. We will see later
that ONNILLI also does not contain NNIL.

Proposition 4.5

   (1) *Let $\mathfrak{M} = (X, R, V)$ and $\mathfrak{N} = (Y, R', V')$ be two rooted intuitionistic (Kripke or descriptive) models, let $f : X \to Y$ be a surjective monotonic map, and let $\varphi \in$ ONNILLI such that $\mathfrak{M} \models \varphi$. Then $\mathfrak{N} \models \varphi$.*

   (2) *Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, R')$ be two rooted intuitionistic (Kripke or descriptive) frames, and let $f : X \to Y$ a monotonic map from $\mathfrak{F}$ onto $\mathfrak{G}$. Then, for each $\varphi \in$ ONNILLI, if $\mathfrak{F} \models \varphi$, then $\mathfrak{G} \models \varphi$.*

Proof   (1) This is proved by induction on the form of φ. For the base case, let us consider a simple ONNILLI-formula $\varphi = \psi \to \chi$ with $\psi \in$ NNIL and $\chi \in$ BASIC, and let $\mathfrak{M} \models \psi \to \chi$, that is, $x \models \psi \to \chi$ for all $x \in X$. Now assume that $yR'y'$ and $y' \models \psi$. Note that because f is surjective, all elements of Y are of the form $f(x)$ for some $x \in X$. So, assume that $y' = f(x)$. Then $f(x) \models \psi$. By Proposition 4.1, we know that $x \models \psi$. But then, since $x \models \psi \to \chi$, we have $x \models \chi$. As $\chi \in$ BASIC, we can, as we noted above (see (2)), conclude that $y' \models \chi$. Hence, $y \models \psi \to \chi$. Thus, $\mathfrak{N} \models \psi \to \chi$.

   The induction step for conjunctions is straightforward. For the induction step for disjunctions it is necessary to require that the models are rooted. Indeed, let $\varphi = \psi \vee \chi$, and let the induction step hold for ψ and χ. Furthermore, let r and r' be the roots of $\mathfrak{M}$ and $\mathfrak{N}$, respectively. Then $\mathfrak{M} \models \psi \vee \chi$ implies that $r \models \psi$ or $r \models \chi$. Then, since $f(r) = r'$, by the above, we have $r' \models \psi$ or $r' \models \chi$. So $r' \models \psi \vee \chi$ and therefore $\mathfrak{N} \models \psi \vee \chi$.

   (2) The proof is similar to the proof of Corollary 3.6 and follows immediately from part (1).   □

In general, this proposition holds definitely only for rooted models, and not for truth in a node. Also surjectivity is an essential feature. Let us give an example to show that rootedness is needed for the result.

Example 4.6   Consider a nonrooted model $\mathfrak{M}$ which is the disjoint union of two linear 2-point models $\mathfrak{M}_1$ and $\mathfrak{M}_2$, $\mathfrak{M}_1$ consisting of two nodes with colors 00 and 10, and $\mathfrak{M}_2$ consisting of two nodes with colors 00 and 01. Furthermore, consider the 3-point model $\mathfrak{N}$ consisting of a root of color 00 and successors with colors 01 and 10 (see Figure 1). We have $\mathfrak{M} \models (p \to q) \vee (q \to p)$ because $\mathfrak{M}_1 \models q \to p$ and $\mathfrak{M}_2 \models p \to q$. But although there is an obvious surjective monotonic map from $\mathfrak{M}$ to $\mathfrak{N}$, clearly $\mathfrak{N} \not\models (p \to q) \vee (q \to p)$ in spite of the fact that $(p \to q) \vee (q \to p)$ is even in the intersection of NNIL and ONNILLI.
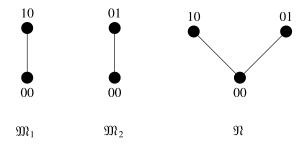

Figure 1

Definition 4.7

(1) If c is an n-color, we write ψ_c for $p_1 \wedge \cdots \wedge p_k \to q_1 \vee \cdots \vee q_m$ if $p_1 \ldots p_k$ are the propositional variables that are associated with 1 in c and $q_1 \ldots q_m$ are the ones that are associated with 0 in c.

(2) If $\mathfrak{M}$ is colorful and $w \in W$, we write $\text{Colors}(\mathfrak{M}_w)$ for the formula $\text{prop}(w) \wedge \bigwedge \{\psi_c \mid c \text{ is not the color of any point in } \mathfrak{M}_w\}$.

(3) If $\mathfrak{M} = (W, R, V)$ is finite and colorful, we let

$$\gamma(\mathfrak{M}) = \bigvee \{\text{Colors}(\mathfrak{M}_w) \to p_{w_1} \vee \cdots \vee p_{w_m} \mid w \in W, w_1, \ldots, w_m \text{ all}$$

immediate successors of $w\}$.

(4) Let $\mathfrak{F}$ be a finite rooted frame, and let $\mathfrak{M} = (\mathfrak{F}, V)$ be the colorful model corresponding to $\mathfrak{F}$. We define

$$\gamma(\mathfrak{F}) := \gamma(\mathfrak{M}).$$

We call $\gamma(\mathfrak{F})$ the *stable formula* of $\mathfrak{F}$.

Note that $\gamma(\mathfrak{F})$ is an ONNILLI-formula.

Lemma 4.8   *Let $\mathfrak{M} = (W, R, V)$ be a colorful model, and let $\mathfrak{N} = (W', R', V')$ be a model.*

(1) *The formula ψ_c expresses that the color c does not occur; more precisely, for $u' \in W'$ we have that $u' \models \psi_c$ if and only if, for all v' with $u' R' v'$, v' does not have color c.*

(2) *The formula $\text{Colors}(\mathfrak{M}_w)$ expresses that only the colors in $\mathfrak{M}_w$ occur; that is, if $w \in W$ and $u' \in W'$, then $u' \models \text{Colors}(\mathfrak{M}_w)$ if and only if, for all v' with $u' R' v'$, v' has the color of some node in $\mathfrak{M}_w$.*

(3) *Let $w, w' \in W$ such that wRw', and let $u' \in W'$. Then $u' \models \text{Colors}(\mathfrak{M}_w)$ implies that for any $v' \in W'$, $u' R' v'$ and $v' \models \text{prop}(w')$ if and only if $v' \models \text{Colors}(\mathfrak{M}_{w'})$.*

(4) *Let $w \in W$, $u' \in W'$, and $u' \not\models \text{Colors}(\mathfrak{M}_w) \to p_{w_1} \vee \cdots \vee p_{w_m}$, where $w_1, \ldots, w_m$ are the immediate successors of w in $\mathfrak{M}$. Then there is $v' \in W'$ such that $u' R' v'$, $v' \models \text{Colors}(\mathfrak{M}_w)$ and $\text{col}(v') = \text{col}(w)$.*

Proof   Assertion (1) is obvious. For (2) just note that the presence of $\text{prop}(w)$ means that only colors $\geq \text{col}(w)$ occur, and then apply (1).

For (3) apply (2), and note that then $v' \models \text{Colors}(\mathfrak{M}_{w'})$ if and only if $v' \models \text{Colors}(\mathfrak{M}_w)$ and $\text{col}(v') \geq \text{col}(w')$ (using for the $\Leftarrow$-direction the fact that $\mathfrak{M}$ is colorful) if and only if $v' \models \text{Colors}(\mathfrak{M}_w)$ and $v' \models \text{prop}(w')$.

(4) Let $u' \not\models \text{Colors}(\mathfrak{M}_w) \to p_{w_1} \vee \cdots \vee p_{w_m}$. Then there is $v' \in W'$ with $u' R' v'$ such that $v' \models \text{Colors}(\mathfrak{M}_w)$ and $v' \not\models p_{w_1} \vee \cdots \vee p_{w_m}$. By (2), $\text{col}(v')$ should occur in $\mathfrak{M}_w$. As $v' \not\models p_{w_1}, \ldots, p_{w_m}$, this color must be the color of w. $\qquad\square$

It may be good to stress the fact that truth of $\text{Colors}(\mathfrak{M}_w)$ in a node u does not imply that all colors in $\mathfrak{M}_w$ occur above u, but only that different colors do not occur above u.

Lemma 4.9   *Let $\mathfrak{F}$ be a finite rooted frame. Then $\mathfrak{F} \not\models \gamma(\mathfrak{F})$.*

Proof   It is easy to see that if $\mathfrak{M}$ is a finite rooted colorful model with a root r, then $r \not\models \text{Colors}(\mathfrak{M}_w) \to p_{w_1} \vee \cdots \vee p_{w_m}$ for each $w \in W$ and $w_1, \ldots, w_m$ all immediate successors of w. The result follows from Lemma 4.8(4). $\qquad\square$

Theorem 4.10   *Let $\mathfrak{F} = (W, R)$ be a finite rooted frame, and let $\mathfrak{G} = (W', R')$ be a rooted (Kripke or descriptive) frame. Then*

$\mathfrak{G} \not\models \gamma(\mathfrak{F})$ *if and only if there is a surjective monotonic map from $\mathfrak{G}$ onto $\mathfrak{F}$.*

Proof $\Leftarrow$: Let $\mathfrak{F}$ and $\mathfrak{G}$ be as in the above assumptions of the theorem, and assume that there is a surjective monotonic map from $\mathfrak{G}$ onto $\mathfrak{F}$. Furthermore, let $\mathfrak{M}$ be a colorful model on $\mathfrak{F}$. By Definition 4.7, $\gamma(\mathfrak{F}) = \gamma(\mathfrak{M})$. By Lemma 4.9, $\mathfrak{F} \not\models \gamma(\mathfrak{F})$. Since $\gamma(\mathfrak{F})$ is an ONNILLI-formula, by Proposition 4.5(2), it is preserved under monotonic images of rooted frames. Thus, $\mathfrak{G} \not\models \gamma(\mathfrak{F})$.

$\Rightarrow$: Let $\mathfrak{N} = (W', R', V')$ be a model on $\mathfrak{G}$ such that $\mathfrak{N}, u \not\models \gamma(\mathfrak{F})$ for some $u \in W'$. Then u has, for each element $w \in W$, a successor u' that makes Colors($\mathfrak{M}_w$) true and $p_{w_1}, \ldots, p_{w_m}$ false if $p_{w_1}, \ldots, p_{w_m}$ are the immediate successors of w. This means, by Lemma 4.8(4) and 4.8(2), that u' has the color of w and its successors have colors of successors of w. Let U be the set of all such u's, that is, $U = \{u' \mid \exists w \in W(u' \models \text{Colors}(\mathfrak{M}_w) \text{ and } \text{col}(u') = \text{col}(w))\}$. By Lemma 4.8(2) and Lemma 4.8(3), U is an upset of W'.

Let r be the root of $\mathfrak{F}$. Define a map $f : W' \to W$ by

$$f(u) = \begin{cases} w, & \text{if } u \in U, u \models \text{Colors}(\mathfrak{M}_w) \ \& \ \text{col}(u) = \text{col}(w), \\ r, & \text{otherwise.} \end{cases}$$

Because each point of W has a distinct color, f is well defined.

If $u', v' \in U$ are such that $u'Rv'$, then there are $u, v \in W$ such that $\text{col}(u') = \text{col}(u)$ and $\text{col}(v') = \text{col}(v)$. By Lemma 3.9, we have uRv. So $f(u')Rf(v')$ and f is monotonic on U. Mapping the other nodes to the root of $\mathfrak{F}$ preserves this monotonicity.

We already saw that, for each $w \in W$, there exists $u \in U$ such that $u \models \text{Colors}(\mathfrak{M}_w)$ and $\text{col}(u) = \text{col}(w)$. Thus, $f(u) = w$ and f is also surjective. So, f is monotonic and surjective.

If $\mathfrak{N}$ is a descriptive model, it remains to prove that f is admissible. For that it is sufficient to prove that, for each $w \in W$, $f^{-1}(R(w))$ is definable, that is, $V'(\varphi)$ for some φ. But that is straightforward. If $f(r') = w$ for the root r' of $\mathfrak{G}$, it is trivial: $f^{-1}(R(w)) = W'$. Otherwise, $f^{-1}(R(w)) = V'(\text{Colors}(\mathfrak{M}_w))$. Namely, if $f(u) = w$, then $u \models \text{Colors}(\mathfrak{M}_w)$, and, if $f(u) = w'$ for some w' with wRw', then $u \models \text{Colors}(\mathfrak{M}_{w'})$, so $u \models \text{Colors}(\mathfrak{M}_w)$ as well. On the other hand, if $u \models \text{Colors}(\mathfrak{M}_w)$, then, by Lemma 4.8(2) and Lemma 4.8(3), for some w' with $wRw', u \models \text{Colors}(\mathfrak{M}_{w'})$ and $\text{col}(u) = \text{col}(w')$, so that $f(u) = w'$. $\square$

If we define an order $\leq$ on (Kripke or descriptive) frames by putting $\mathfrak{F} \leq \mathfrak{G}$ if $\mathfrak{F}$ is a monotonic image of $\mathfrak{G}$, then the formula $\gamma(\mathfrak{F})$ becomes a frame-based formula for $\leq$. Note that, similarly to subframe formulas, Theorem 3.16(2) is always satisfied by $\leq$ (see [4]). Thus, an intermediate logic L is axiomatizable by these formulas if and only if $\mathbb{DF}(L)$ is a $\leq$-downset. Intermediate logics axiomatizable by these formulas are called *stable logics*. Therefore, a logic L is stable if and only if $\mathbb{DF}(L)$ is closed under monotonic images. Formulas closed under monotonic images are called *stable formulas*. There are continuum many stable logics and all of them enjoy the finite model property (see [4]). Now we are ready to prove our main theorem resolving an open problem of [4] on syntactically characterizing formulas that axiomatize stable logics.

Theorem 4.11

(1) *An intermediate logic L is stable if and only if L is axiomatized by* ONNILLI-*formulas.*

(2) *The class of* ONNILLI-*formulas is (up to frame equivalence) the class of all stable formulas.*

Proof   (1) As each $\gamma(\mathfrak{F})$ is ONNILLI, all stable logics are axiomatized by ONNILLI-formulas. By Proposition 4.5(2), every ONNILLI-formula is preserved under monotonic images. Therefore, if L is axiomatized by ONNILLI-formulas, $\mathbb{DF}(L)$ is closed under monotonic images. So L is stable.

(2) By Proposition 4.5(2), every ONNILLI-formula is preserved under monotonic images. So ONNILLI-formulas are stable. Now suppose that φ is preserved under monotonic images. Then IPC $+ \varphi$ is a stable logic. Stable logics are axiomatized by the formulas $\gamma(\mathfrak{F})$. So there exist $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ such that IPC $+ \varphi =$ IPC $+ \bigwedge_{i=1}^{n} \gamma(\mathfrak{F}_i)$. Note that $n \in \omega$, otherwise IPC $+ \varphi$ is infinitely axiomatizable, which is a contradiction. Each $\gamma(\mathfrak{F}_i)$ is ONNILLI, so $\bigwedge_{i=1}^{n} \gamma(\mathfrak{F}_i)$ is also ONNILLI. Thus, φ is frame-equivalent to an ONNILLI-formula and ONNILLI is (up to frame equivalence) the class of formulas closed under monotonic images. □

Example 4.12   It is now easy to construct NNIL-formulas that are not equivalent to an ONNILLI-formula. Note that the logic BD_n of all frames of depth n for each $n \in \omega$ is closed under taking subframes. Thus, it is a subframe logic and hence by Corollary 3.17 is axiomatizable by NNIL-formulas. On the other hand, it is easy to see that there are frames of depth n having frames of depth $m > n$ as monotonic images. So BD_n is not a stable logic. Therefore, it cannot be axiomatized by ONNILLI-formulas. Thus, the class of ONNILLI-formulas does not contain the class of NNIL-formulas (up to frame equivalence).

Example 4.13   We list some more examples of stable logics. Let LC_n be the logic of all linear rooted frames of depth $\leq n$, let BW_n be the logic of all rooted frames of width $\leq n$, and let BTW_n be the logic of all rooted descriptive frames of cofinal width $\leq n$. (For the definition of width and cofinal width, we refer to [11].) Then, for each $n \in \omega$, the logics LC_n, BW_n, and BTW_n are stable. (For the proofs, we refer to [4].)

It remains an open problem whether ONNILLI-formulas are exactly the ones that are preserved under monotonic maps of models in the sense of Proposition 4.5(1).

We finish here by mentioning the connection to modal logic. Modal analogues of subframe formulas were defined by Fine [13]. Analogues of $(\wedge, \rightarrow)$-canonical formulas for transitive modal logics were investigated by Zakharyaschev (see [11, Chapter 9] for an overview). An algebraic approach to these formulas was developed in G. Bezhanishvili and N. Bezhanishvili [2] and generalized to weak transitive logics by these authors in [3]. Modal analogues of $(\wedge, \vee)$-canonical formulas are studied in [5], where modal analogues of stable logics are also defined. N. Bezhanishvili and Ghilardi [9] show that modal stable logics have nice proof-theoretic properties. In particular, they have the bounded proof property.

It still remains open how to define modal analogues of NNIL- and ONNILLI-formulas for transitive modal logics and whether these formulas axiomatize all subframe and stable transitive modal logics, respectively. We note that a syntactic

characterization of formulas axiomatizing subframe transitive modal logics is a long-standing open problem (see [11]).

Note

1. Some authors call such maps *bounded morphisms* (see, e.g., Blackburn, de Rijke, and Venema [10]).

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