Notre Dame Journal of Formal Logic
Volume 59, Number 4, 2018

Stable Forking and Imaginaries

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Abstract We prove that a theory T has stable forking if and only if T^{eq} has stable forking.

1 Introduction

We follow the standard conventions, where T is a complete theory of language L and $\mathfrak{C}$ is its monster model. A formula $\varphi(x, y)$ (where x, y are disjoint tuples of variables) is *stable* if there are not $(a_i \mid i < \omega)$ and $(b_i \mid i < \omega)$ such that $\models \varphi(a_i, b_j)$ if and only if $i < j$.

It is said that T has *stable forking* if whenever a type $p(x) \in S(B)$ forks over some subset $A \subseteq B$, there is some stable formula $\varphi(x, y) \in L$ and some tuple $b \in B$ such that $\varphi(x, b) \in p(x)$ and $\varphi(x, b)$ forks over A. The stable forking conjecture is the statement that every simple theory has stable forking.

A *complete φ-type over A* is a maximally consistent set of *φ-formulas over A*, that is, of formulas of the form $\varphi(x, a)$ and $\neg\varphi(x, a)$ with tuples $a \in A$. The set of all complete φ-types over A is $S_\varphi(A)$. We will also use generalized φ-types (see [1, Chapter 6]). The *generalized φ-type of a over A* is the set of all formulas in $\mathrm{tp}(a/A)$ which are equivalent to Boolean combinations of φ-formulas over the monster model. The main point is that nonforking is transitive for these types (if φ is stable) and over models they coincide with ordinary φ-types.

Remark 1.1

(1) If $\varphi(x, y)$ is a Boolean combination of stable formulas $\varphi_i(x_i, y_i)$ (where $x_i \subseteq x$, $y_i \subseteq y$, and $x_i \cap y_j = \emptyset$), then $\varphi(x, y)$ is stable.
(2) If $\varphi(x, y)$ is stable, then $\varphi^{-1}(y, x) = \varphi(x, y)$ (the same formula with the role of x, y interchanged) is stable.

Received September 29, 2015; accepted May 10, 2016
First published online July 17, 2018
2010 Mathematics Subject Classification: Primary 03C45
Keywords: imaginaries, stable forking
 10.1215/00294527-2018-0010

(3) If $\varphi(x,a) \equiv \psi(x,b)$ and $\psi(x,z)$ is stable, then for some $\mu(y) \in \mathrm{tp}(a)$, $\varphi(x,y) \wedge \mu(y)$ is stable.
(4) In order to check that T has stable forking, it is enough to consider complete types over models (forking over arbitrary subsets).
(5) If $\varphi(x,y)$ is stable and $p(x) \in S_\varphi(M)$, then $p(x)$ is definable by a Boolean combination of formulas $\varphi(m,y)$ for some tuples $m \in M$. The *canonical base* of $p(x)$ is an imaginary e, the canonical parameter of (any) definition of $p(x)$ over M. If $A \subseteq M$, then $p(x)$ divides over A if and only if $e \notin \mathrm{acl}^{\mathrm{eq}}(A)$. For any model $N \supseteq M$, $p(x)$ has a unique e-definable extension $p'(x) \in S_\varphi(N)$.
(6) If $\varphi(x,y)$ is stable, then for any model M, $\varphi(x,a)$ forks over M if and only if $\varphi(x,a)$ divides over M.

Proof For (1), (2), and (5), see [1, Chapters 6, 8] or [7, Chapter 1]. For (3) and (4), see [4]. For (6), see [1, Corollary 8.4]. □

We will need the following lemma on algebraic quantification of a stable formula, which seems to be folklore.

Remark 1.2 If the formula $\varphi(x,y) \in L$ is stable and $\theta(v,x) \vdash \exists^{=n} x\theta(v,x)$, then $\psi(v,y) = \exists x(\theta(v,x) \wedge \varphi(x,y))$ is stable.

Proof Assume that $\models \psi(a_i,b_j)$ if and only if $i < j$. For each $i < \omega$, $\models \exists x\theta(a_i,x)$ and hence there are distinct $c_i^1,\ldots,c_i^n$ such that $\models \theta(a_i,c_i^k)$ for all $k = 1,\ldots,n$. Whenever $i < j < \omega$ choose some k_{ij} such that $1 \leq k_{ij} \leq n$ and $\models \theta(a_i,c_i^{k_{ij}}) \wedge \varphi(c_i^{k_{ij}},b_j)$. By Ramsey's theorem, for some infinite $I \subseteq \omega$ there is some k such that $1 \leq k \leq n$ and $\models \theta(a_i,c_i^k) \wedge \varphi(c_i^k,b_j)$ for all $i,j \in I$ such that $i < j$. Then for $i,j \in I$, $\models \varphi(c_i^k,b_j)$ if and only if $i < j$, which shows that $\varphi(x,y)$ is unstable. □

Not much progress has been made on the stable forking conjecture. B. Kim [3] proved that simple one-based theories with elimination of hyperimaginaries have stable forking. A. Peretz [6] proved that types of SU-rank 2 elements in ω-categorical supersimple theories have stable forking. D. Palacín and F. O. Wagner [5] showed that supersimple CM-trivial ω-categorical theories have stable forking. Finally, let us mention that stable forking implies *weak elimination of hyperimaginaries*, which means that every hyperimaginary is interbounded with a sequence of imaginaries (see [4]).

One says that T *has stable (simple) dividing* if dividing is witnessed by a stable (simple) formula. If $\varphi(x,y)$ is stable and $p(x)$ is a φ-type over a model M, then p does not divide over its canonical base, an imaginary $e \in \mathrm{dcl}^{\mathrm{eq}}(M)$. Therefore, if T has stable dividing, then for every type $p(x) \in S(M)$ there is a subset $A \subseteq M$ such that $|A| \leq |T|$ and p does not divide over A. This means (see, e.g., [8, Proposition 29.5]) that if T has stable dividing, then T is simple. More generally, A. Chernikov has shown (see [2, Proposition 4.14]) that if T has simple dividing, then T is simple. The next proposition shows that theories with stable forking are also simple, which implies that stable forking and stable dividing coincide. Moreover, it includes a stronger version of item (4) of Remark 1.1, with a similar proof.

Proposition 1.3 *If whenever a type $p(x) \in S(N)$ forks over an elementary submodel $M \subseteq N$ there is an instance of a stable formula in $p(x)$ witnessing forking over M, then T is simple and has stable forking.*

Proof We show first that T is simple. Assume not. Then there is a dividing chain $(\varphi(x, a_i) \mid i < \kappa)$ with $\kappa = |T|^+$. This means that $\{\varphi(x, a_i) \mid i < \kappa\}$ is consistent and $\varphi(x, a_i)$ divides over $a_{<i}$ for all $i < \kappa$. One can construct an elementary chain of models $(M_i \mid i < \kappa)$ such that $a_{<i} \in M_i$ and $\varphi(x, a_i)$ divides over M_i for all $i < \kappa$. Let $M = \bigcup_{i<\kappa} M_i$, and let $p(x) \in S(M)$ contain $\{\varphi(x, a_i) \mid i < \kappa\}$. Then $p(x)$ divides (and forks) over each M_i. By the assumption, there is a family $(\psi_i(x, y_i) \mid i < \kappa)$ of stable formulas $\psi_i(x, y) \in L$ and a corresponding sequence of parameters $m_i \in M$ such that $\psi_i(x, m_i) \in p(x)$ and $\psi_i(x, m_i)$ forks over M_i for all $i < \kappa$. By item (6) of Remark 1.1, every $\psi_i(x, m_i)$ divides over M_i. By the choice of κ we may assume that $\psi_i(x, y_i) = \psi(x, y)$ for all i. We have obtained a stable formula with a dividing chain of uncountable length and hence with the tree property. But it is well known that stable formulas do not have the tree property (see, e.g., [1, Proposition 2.21]).

For the second part, assume that $A \subseteq B$ and $a \not\perp_A B$. Choose a model $M \supseteq A$ such that $M \perp_A Ba$ and note that $a \not\perp_M B$. Now choose a model $N \supseteq MB$ such that $N \perp_{MB} a$, and note that $a \not\perp_M N$. By the assumption, there is a stable formula $\varphi(x, y) \in L$ and some tuple $n \in N$ such that $\models \varphi(a, n)$ and $\varphi(x, n)$ forks over M. Let $p(x)$ be the φ-type of a over N, and let $q(x)$ be the generalized φ-type of a over B. Since $p(x)$ forks over A but does not fork over B, $q(x)$ forks over A. Hence there is some formula $\psi(x, z) \in L$ and some tuple $b \in B$ such that $\psi(x, b)$ is in $q(x)$ and forks over A. By Remark 1.1(3), there is some formula $\mu(z) \in \mathrm{tp}(b)$ such that $\psi'(x, z) = \psi(x, z) \wedge \mu(z)$ is stable. But $\models \psi'(a, b)$ and $\psi'(x, b)$ forks over A. □

2 Main Result

In this section we will assume that T is a simple theory, but since stable forking implies simplicity, in fact it is not necessary to add this assumption to the propositions below. In particular, the notions of forking and dividing coincide in T.

Proposition 2.1 *If T has stable forking, then T^{eq} has stable forking over real parameters.*

Proof Let $A \subseteq B \subseteq \mathfrak{C}$, let $e \in \mathfrak{C}^{\mathrm{eq}}$ be an imaginary of sort E, and assume that $e \not\perp_A B$. Choose an $|A|^+ + \omega$-saturated model $M \supseteq B$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(M)$ and a representative a of $e = a_E$ such that $a \perp_e M$. Note that $a \not\perp_A B$. By assumption, there is some stable formula $\delta(x, y) \in L$ and some tuple $b \in B$ such that $\models \delta(a, b)$ and $\delta(x, b)$ forks over A. Consider the type $p(x) = \mathrm{tp}_\delta(a/M)$. It has a definition $d_p x\delta(x, y)$ which is a Boolean combination of formulas of the form $\delta(m, y)$ for some tuples $m \in M$. The definition is, therefore, an instance of a stable formula. Note that $p(x)$ does not fork over e and hence its canonical basis $d_F \in M^{\mathrm{eq}}$ is in $\mathrm{acl}^{\mathrm{eq}}(e)$. For some $\chi(w, y) \in L^{\mathrm{eq}}$, $\chi(d_F, y)$ defines $p(x)$. Since $d_p x\delta(x, y) \equiv \chi(d_F, y)$, by Remark 1.1(3) for some $\mu(w) \in \mathrm{tp}(d_F)$, the formula $\chi(w, y) \wedge \mu(y)$ is stable. Without loss of generality, we assume that $\chi(w, y)$ is stable. Note that since $\delta(x, b) \in p(x)$, $\models \chi(d_F, b)$.

Claim 1 *If $q(w) = \mathrm{tp}(d_F)$, then $q(w) \cup \{\chi(w, b)\}$ forks over A.*

Proof Assume not. We will prove that $\delta(x,b)$ does not divide over A, which is a contradiction. Let $(b_i \mid i < \omega)$ be an A-indiscernible sequence of tuples $b_i \equiv_A b$, and let us check that $\{\delta(x,b_i) \mid i < \omega\}$ is consistent. By the saturation of M, we may assume that $b_i \in M$ for all $i < \omega$. By our assumption in the proof, $q(w) \cup \{\chi(w,b_i) \mid i < \omega\}$ is consistent and hence we can find some realization $d'_F \in M^{\mathrm{eq}}$ of this set of formulas. Since $d_F \equiv d'_F$, there is some sequence $(b'_i \mid i < \omega)$ in M^{eq} such that $d_F(b'_i \mid i < \omega) \equiv d'_F(b_i \mid i < \omega)$. Then $\models \chi(d_F, b'_i)$ for every $i < \omega$, which implies that $\delta(x,b'_i) \in p(x)$ and $\models \delta(a,b'_i)$ for all $i < \omega$. Since $\{\delta(x,b'_i) \mid i < \omega\}$ is consistent, $\{\delta(x,b_i) \mid i < \omega\}$ is consistent too. □

With Claim 1 we can now choose some $\mu(w) \in q(w)$ such that $\chi(w,b) \wedge \mu(w)$ forks over A. Note that $\chi'(w,y) = \chi(w,y) \wedge \mu(w)$ is stable. Since $\chi'(w,b)$ divides over A, this can be witnessed by an A-indiscernible sequence $(b_i \mid i < \omega)$ with $b_i \equiv_A b$ and some $k < \omega$ for which $\{\chi'(w,b_i) \mid i < \omega\}$ is k-inconsistent. Since $d_F \in \mathrm{acl}^{\mathrm{eq}}(e)$, there is some formula $\theta(v,w) \in L^{\mathrm{eq}}$ and some $n < \omega$ such that $\models \theta(e,d_F)$ and $\theta(v,w) \vdash \exists^{=n} w\theta(v,w)$. Let $\varphi(v,y) = \exists w(\theta(v,w) \wedge \chi'(w,y))$. By Remark 1.2, $\varphi(v,y)$ is stable. Since $\models \varphi(e,b)$, it only remains to check that $\varphi(v,b)$ forks over A. This is done in the next claim.

Claim 2 *We have that $\varphi(v,b)$ divides over A with respect to $l = n(k-1)+1$, witnessed by $(b_i \mid i < \omega)$.*

Proof Otherwise, $\{\varphi(v,b_i) \mid i < \omega\}$ is consistent and it is realized by some $e' \in \mathfrak{C}^{\mathrm{eq}}$. For each $i < l$, choose some d^i_F such that $\models \theta(e',d^i_F) \wedge \chi'(d^i_F,b_i)$. The number of d^i_F's is at most n, and therefore, by choice of l, the mapping $i \mapsto d^i_F$ has some fiber of cardinality at least k. This shows that $\{\chi'(w,b_i) \mid i < \omega\}$ is k-consistent, which is a contradiction with the choice of k. □

□

Proposition 2.2 *If T^{eq} has stable forking over real parameters, then T^{eq} has stable forking.*

Proof By item (4) of Remark 1.1, it is enough to consider types over models. Assume that $e \not\downarrow_A M^{\mathrm{eq}}$, where $M \subseteq \mathfrak{C}$ is a model, $M^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}(M)$ is the corresponding imaginary model, $A \subseteq M^{\mathrm{eq}}$, and $e \in \mathfrak{C}^{\mathrm{eq}}$. Choose a set A' of representatives of the elements of A such that $A' \downarrow_A Me$. Then $e \not\downarrow_{A'} M$, and, by the assumption, there is some stable formula $\delta(v,y) \in L^{\mathrm{eq}}$ and some tuple $a \in A'M$ such that $\models \delta(e,a)$ and $\delta(v,a)$ forks over A'. Let $p(v) = \mathrm{tp}_\delta(e/M)$, and let c be its canonical base. Since $e \downarrow_M A'$, the unique global δ-type $\mathfrak{p}(v) \supseteq p(v)$ which is definable over c extends $\mathrm{tp}_\delta(e/A'M)$. Since c is the canonical base of $\mathfrak{p}$ and $\mathfrak{p}$ forks over A', $c \notin \mathrm{acl}^{\mathrm{eq}}(A')$. It follows that $c \notin \mathrm{acl}^{\mathrm{eq}}(A)$. Hence $p(v)$ forks over A. Let $\varphi(v,b)$ be a finite conjunction of formulas of $p(v)$ which forks over A. Since $\varphi(v,b)$ is a conjunction of δ-formulas, it is an instance of a stable formula. Moreover, $\models \varphi(e,b)$. □

Corollary 2.3 *T has stable forking if and only if T^{eq} has stable forking.*

Proof One direction follows from Propositions 2.1 and 2.2. The rest is clear since L^{eq}-formulas with real free variables are equivalent to L-formulas. □

3 An Example and Some Open Problems

We describe a theory T. Its language contains two binary relation symbols E, F, both of which are being interpreted as equivalence relations on the universe with some specific cross-cutting. The equivalence relation E has infinitely many classes, all infinite. On the other hand, F has exactly one class of size n for every $n \geq 1$, say, consisting of the elements $a_1^n, \ldots, a_n^n$. For each $k \geq 1$, the elements $a_k^k, a_k^{k+1}, \ldots$ build an E-class. With these specifications, the set $\{a_k^n \mid 1 \leq k \leq n < \omega\}$ is the universe of a model M of T. Note that $\operatorname{dcl}(\emptyset) = M$. The formula $E(x, y)$ has the nonfinite cover property, and $F(x, y)$ is stable. But $\exists y(E(x, y) \wedge F(y, z))$ is unstable, as witnessed by the sequences $(a_i^i \mid i \geq 1)$ and $(a_1^j \mid j \geq 1)$. This answers a question of M. C. Laskowski: Lemma 1.2 cannot be generalized to the case where $\theta(v, x)$ is a nonfinite cover property formula. On the other hand, it shows that the proof of Proposition 2.1 cannot be carried out by trying to prove that the formula $\exists x(\pi_E(x) = v \wedge \delta(x, y))$ is stable (where π_E is the mapping sending each tuple to its E-equivalence class). Since T is interpretable in Presburger arithmetic, it is dp-minimal. But T has the strict order property; hence it is not simple. This can be checked by observing that the E-class of a_{11} is infinite and has a definable linear ordering. It would be interesting to find a similar example in a simple theory.

A. Chernikov has raised the question of whether Corollary 2.3 can be generalized to dependent forking. (See [2] for the relevant definitions.)

Let us finally mention a connected question asked by Ziegler. Assume that all 1-types in T have stable forking. Does it follow that T has stable forking? A positive answer would be very helpful.

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Acknowledgments

Casanovas and Potier's work was partially supported by Spanish government grants MTM 2011-26840 and MTM2014-59178-P and by Catalan government grant 2014SGR-437.

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