# EFFECTIVENESS FOR THE DUAL RAMSEY THEOREM 

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#### Abstract

We analyze the Dual Ramsey Theorem for $k$ partitions and $\ell$ colors $\left(\mathrm{DRT}_{\ell}^{k}\right)$ in the context of reverse math, effective analysis, and strong reductions. Over $\mathrm{RCA}_{0}$, the Dual Ramsey Theorem stated for Baire colorings Baire- $\mathrm{DRT}_{\ell}^{k}$ is equivalent to the statement for clopen colorings $\mathrm{ODRT}_{\ell}^{k}$ and to a purely combinatorial theorem $\mathrm{CDRT}_{\ell}^{k}$.

When the theorem is stated for Borel colorings and $k \geq 3$, the resulting principles are essentially relativizations of $\mathrm{CDRT}_{\ell}^{k}$. For each $\alpha$, there is a computable Borel code for a $\Delta_{\alpha}^{0}$ coloring such that any partition homogeneous for it computes $\emptyset^{(\alpha)}$ or $\emptyset^{(\alpha-1)}$ depending on whether $\alpha$ is infinite or finite.

For $k=2$, we present partial results giving bounds on the effective content of the principle. A weaker version for $\Delta_{n}^{0}$ reduced colorings is equivalent to $\mathrm{D}_{2}^{n}$ over $\mathrm{RCA}_{0}+\mathrm{I} \Sigma_{n-1}^{0}$ and in the sense of strong Weihrauch reductions.


## 1. Introduction

This paper concerns the reverse mathematical and computational strength of variations of the Dual Ramsey Theorem. For $k \leq \omega$, let $(\omega)^{k}$ denote the set of all partitions of $\omega$ into exactly $k$ pieces. Such a partition can be represented as a surjective function from $\omega$ to $k$. Thus $(\omega)^{k}$ inherits a natural topology by considering it as a subset of $k^{\omega}$.
Dual Ramsey Theorem ([4], [14]). For any $k, \ell<\omega$, suppose we have a coloring $(\omega)^{k}=\cup_{i<\ell} C_{i}$. If for each $i<\ell, C_{i}$ has the property of Baire, then there is a partition $p \in(\omega)^{\omega}$ such that any coarsening of $p$ down to exactly $k$ pieces has the same color.

The reason that this theorem is dual to the original Ramsey's Theorem concerns what objects are being colored. In the original Ramsey's theorem, we color the $k$-element subsets of $\omega$, which correspond to injective functions from $k$ to $\omega$. In the Dual Ramsey Theorem, we color surjective functions from $\omega$ to $k$.

A straightforward choice argument shows that the Dual Ramsey Theorem fails if no regularity conditions on the $C_{i}$ are assumed. The theorem was first proved for Borel colorings by Carlson and Simpson [4, and extended to colorings with the Baire property by Prömel and Voigt [14. From the perspective of reverse mathematics or computational mathematics, the variation in hypothesis gives us two theorems to consider. We call them the Borel Dual Ramsey Theorem and the Baire Dual Ramsey Theorem respectively.

Carlson and Simpson asked for a recursion-theoretic analysis of the Borel Dual Ramsey Theorem. In order to answer this, it is necessary to choose a method for

[^0]encoding the coloring, and one must consider the potential effects of a topologically intricate coloring. Previous work side-stepped these issues by restricting attention to open colorings only [12] or by focusing attention only on the main combinatorial lemma which Carlson and Simpson used in their proof, and on its variable word variants [12, 7, 11.

From the work of [12], we know that over $\mathrm{RCA}_{0}, \mathrm{ODRT}_{\ell}^{k}$ implies $\mathrm{RT}_{\ell}^{k-1}$, where $\mathrm{ODRT}_{\ell}^{k}$ is the restriction of the Borel Dual Ramsey Theorem to open colorings only, and RT is the usual Ramsey's Theorem. This provides a lower bound on the strength of the Borel Dual Ramsey Theorem. Conversely, in unpublished work Slaman has shown that the Borel Dual Ramsey Theorem follows from $\Pi_{1}^{1}-\mathrm{CA}_{0}$ [18. No direct implication is known between the Dual Ramsey Theorems and the variable word theorems, because the Dual Ramsey Theorem does not require the "words" in its solution to be finite (and by Proposition 3.15, it cannot require this), while the proof of the Dual Ramsey Theorem from the variable word theorems uses infinitely many sequential applications of the latter (Theorem 3.18). Overall, this leaves a rather large gap, and we do not close it. However, we do provide significant clarification of the key difficulties. In particular, for the first time we directly tackle the topological aspect of the Borel version of the theorem.
1.1. Combinatorial core of the Borel Dual Ramsey Theorem. Since the Borel version follows from the Baire version plus the additional principle "Every Borel set has the property of Baire", our first step is to understand the Baire version.

To be clear, an instance of the Baire Dual Ramsey Theorem is a sequence of pairwise disjoint open sets $O_{0}, \ldots, O_{\ell-1}$ whose union is dense in $(\omega)^{k}$, and a sequence of dense open sets $\left\{D_{n}\right\}_{n \in \omega}$. Such an instance simultaneously represents all colorings $(\omega)^{k}=\cup_{i<\ell} C_{i}$ for which the symmetric difference $C_{i} \Delta O_{i}$ is disjoint from $\cap_{n} D_{n}$. There may be uncountably many such colorings, because no condition is placed on how $2^{\omega} \backslash \cap_{n} D_{n}$ is colored. Any solution $p \in(\omega)^{\omega}$ to the Baire version must have $(p)^{k} \subseteq \cap_{n} D_{n}$.

In Section 3.1 we define a purely combinatorial principle $\mathrm{CDRT}_{\ell}^{k}$, which precisely captures the strength of the Baire version. In the following, if $p \in(\omega)^{\omega}$ and $k \leq \omega$, let $(p)^{k}$ denote the set of coarsenings of $p$ into exactly $k$ pieces. Recalling that we consider $p$ as a surjective function $p: \omega \rightarrow k$, let

$$
p^{*}:=p \upharpoonright \min p^{-1}(k-1)
$$

In other words, $p^{*}$ is a string on alphabet $k-1$, it tells us by its length what is the smallest element of $p$ 's last block, and it tells us how $p$ partitions the finitely many smaller elements into its first $k-1$ blocks. Let $(<\omega)^{k-1}=\left\{p^{*}: p \in(\omega)^{k}\right\}$.
Theorem 1.1. Let $k, \ell<\omega$. Over $\mathrm{RCA}_{0}$, the following are equivalent.
(1) The Baire Dual Ramsey Theorem for $k$ partitions and $\ell$ colors.
(2) $\mathrm{ODRT}_{\ell}^{k}$
(3) $\mathrm{CDRT}_{\ell}^{k}$, which states: for every $c:(<\omega)^{k-1} \rightarrow \ell$, there is a $p \in(\omega)^{\omega}$ and a color $i<\ell$ such that for every $x \in(p)^{k}, c\left(x^{*}\right)=i$.
Thus we have reduced the Baire version of the theorem to a purely combinatorial statement. The proof of the equivalence is essentially an effectivization of [14].

Aside from the results in [12], the strengths of the $\mathrm{CDRT}_{\ell}^{k}$ statements are wide open. We include one more result, which was known to Simpson (see [4, page 268])

Figure 1. Implications over $\mathrm{RCA}_{0}$ between variants of the Dual Ramsey Theorem considered in this paper and some related principles. The parameter $k \geq 4$ is arbitrary.

and subsequently rediscovered by Patey [13: a proof of one case of the CarlsonSimpson Lemma from Hindman's Theorem. With minor modifications, we adapt this proof in Section 3.2 to show that Hindman's Theorem for $\ell$ colorings implies the stronger $\mathrm{CDRT}_{\ell}^{3}$. See Figure 1.1 for a summary of what is known about the combinatorial core of the Dual Ramsey Theorem.

We close Section 3 with a self-contained proof of $\mathrm{CDRT}_{\ell}^{k}$ from the CarlsonSimpson Lemma (Theorem 3.18). In our proof, the only non-constructive steps are $\omega \cdot(k-2)$ nested applications of the Carlson-Simpson Lemma.

The earliest claim we are aware of for a proof of $\mathrm{CDRT}_{\ell}^{k}$ is in [14, where a generalization of $\mathrm{CDRT}_{\ell}^{k}$ called Theorem $A$ is attributed to a preprint of Voigt titled "Parameter words, trees and vector spaces". However, as far as we can tell, this paper never appeared. Another proof of $\mathrm{CDRT}_{\ell}^{k}$ can be found in [19, but as a corollary of a larger theory.
1.2. Computational strength of the Borel Dual Ramsey Theorem. In Sections 4 and 5, we consider the Borel Dual Ramsey Theorem, or Borel-DRT, from the perspective of effective combinatorics. The behavior is different depending on the number of pieces $k$ in the partition, with the $k \geq 3$ case being addressed in Section 4 and the $k=2$ case in Section 5 .

When $k \geq 3$, given a fast-growing function $f$ one can design an open, $f$ computable coloring such that all of its homogeneous partitions compute a function which dominates $f$ (this was already essentially done in [12]). But if $f$ is hyperarithmetic, that same coloring has an effective Borel code as a $\Delta_{\alpha}^{0}$ set. Thus by sneaking the computation of $f$ into an effective Borel code, we obtain a computable
instance of Borel- $\mathrm{DRT}_{2}^{3}$. As a result, Borel- $\mathrm{DRT}_{2}^{3}$ can be informally considered as some kind of hyperjump of $\mathrm{ODRT}_{2}^{3}$. Formally, we have the following in Theorem 4.7.

Theorem 1.2. For every computable ordinal $\alpha>0$ and every $k \geq 3$, there is a computable Borel code for a $\Delta_{\alpha}^{0}$ coloring $c:(\omega)^{k} \rightarrow 2$ such that every infinite partition homogeneous for c computes $\emptyset^{(\alpha)}$ if $\alpha$ is infinite, or $\emptyset^{(\alpha-1)}$ if $\alpha$ is finite.

The preceding theorem gives a coding lower bound on the complexity of solutions for $k \geq 3$. In contrast, we remark that the best known basis theorem for the $k \geq 3$ case is still the following result of Slaman [18]: Every hyperarithmetic instance of the Borel Dual Ramsey Theorem has a hyperarithmetically low solution. This result can also be extracted from our analysis as follows. Given a Borel coloring, there is a hyperarithmetic witness that it has the property of Baire. Use Theorem 1.1 to computably reduce this instance of the Baire Dual Ramsey Theorem to an instance of CDRT. It is arithmetic to check whether a given partition $p \in(\omega)^{\omega}$ is a solution to a given instance $c$ of CDRT. Therefore the collection of solutions is non-empty $\Sigma_{1}^{1}$. Applying the Gandy Basis Theorem gives the desired solution.

When $k=2$, it is likewise possible to create effectively Borel instances which correspond to hyperarithmetically computable open colorings. However, there are two important differences with the $k=2$ case. First, $\mathrm{ODRT}_{\ell}^{2}$ is computably true. As a consequence, when $k=2$ the Borel variant has a sharper basis theorem.
Theorem 1.3. Every $\Delta_{n}^{0}$ instance of Borel-DRT ${ }_{\ell}^{2}$ has a $\Delta_{n}^{0}$ solution.
This result follows from the more general Theorem 5.4. Note that the $\Delta_{n}^{0}$ instance is a subset of $(\omega)^{k}$ which could be topologically intricate, while the solution is a single $\Delta_{n}^{0}$ partition $p \in(\omega)^{\omega}$.

The second difference in the $k=2$ case is that $\operatorname{CDRT}_{\ell}^{2}$ is Weihrauch equivalent to the infinite pigeonhole principle $\mathrm{RT}_{\ell}^{1}$. (Observe that an instance of $\mathrm{CDRT}_{\ell}^{2}$ is essentially a coloring of $\omega$.) This immediately offers lower bounds: for each $n$, $D_{\ell}^{n} \leq_{\mathrm{sW}}$ Borel- $\mathrm{DRT}_{\ell}^{2}$, where $D_{\ell}^{n}$ is the problem whose instances are $\Delta_{n}^{0}$ colorings $c: \omega \rightarrow \ell$ and whose solutions are the infinite sets monochromatic for $c$. The question is whether these could possibly be equivalences when Borel- $\mathrm{DRT}_{\ell}^{2}$ is likewise restricted to $\Delta_{n}^{0}$ instances. We are only able to show a partial result in this direction (Theorem 5.7).

Theorem 1.4. Let $\Delta_{n}^{0}-\mathrm{rDRT}{ }_{2}^{2}$ be the restriction of Borel- $\mathrm{DRT}_{2}^{2}$ to instances $c$ which are given by $\Delta_{n}^{0}$ formulas and for which $c$ is reduced, meaning that $c(p)$ depends only on $p^{*}$ for all $p \in(\omega)^{2}$. Then
(1) $\Delta_{n}^{0}-\mathrm{rDRT}_{2}^{2} \equiv_{\mathrm{sW}} \mathrm{D}_{2}^{n}$.
(2) Over $\mathrm{RCA}_{0}+\mathrm{I} \Sigma_{n-1}^{0}, \Delta_{n}^{0}-\mathrm{rDRT}{ }_{2}^{2}$ is equivalent to $\mathrm{D}_{2}^{n}$.
1.3. Reverse mathematics and Borel sets. In Section 6, we consider problems motivated by the reverse mathematics of the Borel Dual Ramsey Theorem. We observe that the Borel Dual Ramsey Theorem can be obtained by composing "Every Borel set has the property of Baire" (let us call it BP) with the Baire Dual Ramsey Theorem. So a natural next step is to understand the strength of BP. We show the following as a part of Theorem 6.9.
Theorem 1.5. Over $\mathrm{RCA}_{0}, \mathrm{ATR}_{0}$ is equivalent to the following statement. For every Borel code $B$, there is some point $x$ such that $x \in B$ or $x \notin B$.

This result mainly shows that the usual definition of Borel sets, which is given in [17] using ATR $_{0}$ as a base theory, really does not make sense in the absence of ATR ${ }_{0}$. This provides an obstacle to a satisfactory analysis of BP. While BP follows from $\mathrm{ATR}_{0}$, (Proposition6.5), in the reversal BP formally implies ATR ${ }_{0}$ only due to the technical reason above. We leave a deeper analysis of BP and the Borel Dual Ramsey Theorem to future work [2].

The proof of Theorem 1.5 uses a method of effective transfinite recursion, ETR, which is available in $A C A_{0}$ (and possibly in weaker systems). Greenberg and Montalbán [8] use ETR to establish equivalences of ATR $_{0}$ and claim that ETR is provable in $R C A_{0}$. However, their proof of ETR overlooks an application of $\Sigma_{1}^{0}$ transfinite induction, and in general, transfinite induction for $\Sigma_{1}^{0}$ formulas does not hold in $\mathrm{RCA}_{0}$. While the main results in [8] continue to hold because Greenberg and Montalbán show the classified theorems imply ACA $_{0}$ without reference to ETR (and hence can use ETR in $A C A_{0}$ to complete the equivalence with $A T R_{0}$ ), we have included a proof of ETR in Section 6 to make explicit the use of transfinite induction.

In the final Section 7 we list a number of open questions.

## 2. Notation

We use $\omega$ to denote the natural numbers, which in subsystems of $Z_{2}$ is the set $\{x: x=x\}$, often denoted by $\mathbb{N}$ in the literature. Despite this notation, we do not restrict ourselves to $\omega$-models. Second, when we refer to the parameters $k$ and $\ell$ in versions of the Dual Ramsey Theorem, we assume $k$ and $\ell$ are arbitrary standard numbers with $k, \ell \geq 2$. By a statement such as " $\mathrm{RCA}_{0}$ proves Borel-DRT ${ }_{\ell}^{k}$ implies Baire-DRT $T_{\ell}^{k}$, we mean, for all $k, \ell \geq 2, \mathrm{RCA}_{0} \vdash$ Borel-DRT $_{\ell}^{k} \rightarrow \mathrm{Baire}^{\mathrm{DR}} \mathrm{T}_{\ell}^{k}$. For many results, the quantification over $k$ and $\ell$ can be pulled inside the formal system. However, in some cases, issues of induction arise and we wish to set those aside in this work.

For $k \leq \omega$, let $k^{<\omega}$ denote the set of finite strings over $k$ and let $k^{\omega}$ denote the set of functions $f: \omega \rightarrow k$. As noted above, unless explicitly stated otherwise, we will always assume that $k \geq 2$. For $\sigma \in k^{<\omega},|\sigma|$ denotes the length of $\sigma$, and if $|\sigma|>0$, $\sigma(0), \ldots, \sigma(|\sigma|-1)$ denote the entries of $\sigma$ in order. For $p \in k^{\omega}$ and $\sigma \in k^{<\omega}$, we write $\sigma \prec p$ if $\sigma$ is an initial segment of $p$. Similarly, if $\sigma, \tau \in k^{<\omega}$, we write $\sigma \preceq \tau$ if $\sigma$ is an initial segment of $\tau$ and $\sigma \prec \tau$ if $\sigma$ is a proper initial segment of $\tau$. We write $p \upharpoonright n$ to denote the string obtained by restricting the domain of $p$ to $n$. The standard (product) topology on $k^{\omega}$ is generated by basic clopen sets of the form

$$
[\sigma]=\left\{p \in k^{\omega}: \sigma \prec p\right\}
$$

for $\sigma \in k^{<\omega}$.
We use the following notational conventions for partitions. For $k \leq s \leq \omega$, we use $(s)^{k}$ to denote the set of all partitions of $s$ into exactly $k$ pieces. The pieces are also called blocks. Each such partition can be viewed as a surjective function $p: s \rightarrow k$, where the blocks are the sets $p^{-1}(i)$ for $i<k$. More than one surjective function can describe the same partition, so we pick a canonical one. We say that $p: s \rightarrow k$ is ordered if for each $i<j<k, \min p^{-1}(i)<\min p^{-1}(j)$. We then more formally define the $k$-partitions of $s$ as

$$
(s)^{k}=\left\{p \in k^{s}: p \text { is surjective and ordered }\right\}
$$

We also let $(<\omega)^{k}$ denote $\cup_{r \in \omega}(r)^{k}$.

If $k \leq s \leq t \leq \omega$ and $p \in(t)^{s}$, then we define $(p)^{k}=\left\{x \circ p: x \in(s)^{k}\right\}$. In English, if $p$ is a partition of $t$ into exactly $s$ pieces, $(p)^{k}$ is the set of ways to further coarsen $t$ down to exactly $k$ pieces, so we call $(p)^{k}$ the set of $k$-coarsenings of $p$.

If $(\omega)^{k}=\cup_{i<\ell} C_{i}$ and $p \in(\omega)^{\omega}$ with $(p)^{k} \subseteq C_{i}$, then we say that $p$ is homogeneous for the color $C_{i}$.

The set $(\omega)^{k}$ inherits the subspace topology from $k^{\omega}$ with basic open sets of the form $[\sigma] \cap(\omega)^{k}$ for $\sigma \in k^{<\omega}$. This topology is also natural from the partition perspective. For example, if we considered a partition instead as an equivalence relation $R \subseteq \omega \times \omega$, the same topology is also generated by declaring $\{R:(n, m) \in$ $R\}$ to be clopen for each pair $(n, m) \in \omega \times \omega$.

The space $(\omega)^{k}$ is not compact since, for example, the collection of open sets [ $0^{n} 1$ ] for $n \geq 1$ cover $(\omega)^{2}$ but this collection has no finite subcover. However, if $\sigma \in(<\omega)^{k}$, then $[\sigma] \subseteq(\omega)^{k}$ and $[\sigma]$ is a compact clopen subset of $(\omega)^{k}$. To generate the topology on $(\omega)^{k}$, it suffices to restrict to the basic clopen sets of the form $[\sigma]$ with $\sigma \in(<\omega)^{k}$. Although the notation $[\sigma]$ is ambiguous about whether the ambient space is $k^{\omega}$ or $(\omega)^{k}$ (or $\ell^{\omega}$ or $(\omega)^{\ell}$ for some $\ell>k$ ), the meaning will be clear from context.

We denote the $i$ th block of the partition $p$ by $p^{-1}(i)$ (we start counting the blocks at 0 , so the last block of a $k$-partition is indexed by $i=k-1$ ). We denote the least element of $p^{-1}(i)$ by $\mu^{p}(i)$. If $p \in(\omega)^{k}$, we will often have use for the string $p^{*}=p \upharpoonright \mu^{p}(k-1)$. We can also apply this notation if $p \in(s)^{k}$ for any $s \geq k$.

Sometimes it is convenient to consider colorings of $(p)^{k}$ for some $p \in(\omega)^{\omega}$, and then ask for a homogeneous partition $q \in(p)^{\omega}$. This is not really more general than the case we have been considering, because a coloring $(p)^{k}=\cup_{i<\ell} C_{i}$ corresponds canonically to the coloring of $(\omega)^{k}$ defined by

$$
\begin{equation*}
x \in \widehat{C}_{i} \Longleftrightarrow x \circ p \in C_{i} \tag{1}
\end{equation*}
$$

In this case any $y \in(\omega)^{\omega}$ is homogeneous for $\left\{\widehat{C}_{i}\right\}_{i<\ell}$ if and only if $y \circ p$ is homogeneous for $\left\{C_{i}\right\}_{i<\ell}$.

## 3. The Baire Dual Ramsey Theorem

3.1. Three versions of the Baire Dual Ramsey Theorem. We formulate three versions of the Baire Dual Ramsey Theorem in second order arithmetic and show they are equivalent over $\mathrm{RCA}_{0}$.

Coding colorings or sets with the Baire property in second order arithmetic is complicated by the fact that there are $2^{\mathfrak{c}}$ (where $\mathfrak{c}=2^{\aleph_{0}}$ ) many subsets of $(\omega)^{k}$ or $k^{\omega}$ with the Baire property. However, if we identify colorings which are the same after discarding a meager set, then there are only continuum many with the Baire property. Specifying only an equivalence class of colorings is consistent with how theorems which hypothesize the Baire property usually work. They start by fixing a comeager approximation to the set in question and then proceed by working exclusively with this approximation. This classical observation motivates our definition of a code for a Baire coloring.
Definition $3.1\left(\mathrm{RCA}_{0}\right)$. A code for an open set in $(\omega)^{k}$ is a set $O \subseteq \omega \times(<\omega)^{k}$. We say that a partition $p \in(\omega)^{k}$ is in the open set coded by $O$ (or just in $O$ and write $p \in O)$ if there is a pair $\langle n, \sigma\rangle \in O$ such that $p \in[\sigma]$.

A code for an closed set in $(\omega)^{k}$ is also a set $V \subseteq \omega \times(<\omega)^{k}$. In this case, we say $p \in(\omega)^{k}$ is in $V$ (and write $p \in V$ ) if for all pairs $\langle n, \sigma\rangle \in V, p \notin[\sigma]$.

Definition $3.2\left(\mathrm{RCA}_{0}\right)$. An open set $O \subseteq(\omega)^{k}$ is dense if for all $\tau \in(<\omega)^{k}$, $[\tau] \cap O \neq \emptyset$. That is, for all $\tau$, there is a pair $\langle n, \sigma\rangle \in O$ such that $\sigma$ and $\tau$ are comparable as strings.
Definition 3.3 $\left(\mathrm{RCA}_{0}\right)$. A code for a Baire $\ell$-coloring of $(\omega)^{k}$ is a sequence of dense open sets $\left\{D_{n}\right\}_{n<\omega}$ together with a sequence of pairwise disjoint open sets $\left\{O_{i}\right\}_{i<\ell}$ such that $\bigcup_{i<\ell} O_{i}$ is dense in $(\omega)^{k}$.

Recall that $\mathrm{RCA}_{0}$ suffices to prove the Baire Category Theorem: if $\left\{D_{n}\right\}_{n<\omega}$ is a sequence of dense open sets, then $\cap_{n<\omega} D_{n}$ is dense. Classically, if a coloring $\cup_{i<\ell} C_{i}=(\omega)^{k}$ has the Baire property, then it has a comeager approximation given by sequences of open sets $\left\{O_{i}\right\}_{i<\ell}$ and $\left\{D_{n}\right\}_{n<\omega}$ such that each $D_{n}$ is dense and for each $p \in \cap_{n<\omega} D_{n}, p \in C_{i}$ if and only if $p \in O_{i}$.

We abuse terminology and refer to the Baire code as a Baire $\ell$-coloring of $(\omega)^{k}$. Similarly, an open $\ell$-coloring is a coloring $(\omega)^{k}=\cup_{i<\ell} O_{i}$ in which the $O_{i}$ are open and pairwise disjoint.
Definition 3.4. For each (standard) $k, \ell \geq 2$, we define Baire- $\mathrm{DRT}_{\ell}^{k}, \mathrm{ODRT}_{\ell}^{k}$ and $\mathrm{CDRT}_{\ell}^{k}$ in $\mathrm{RCA}_{0}$ as follows.
(1) Baire-DRT ${ }_{\ell}^{k}$ : For every Baire $\ell$-coloring $\left\{O_{i}\right\}_{i<\ell}$ and $\left\{D_{n}\right\}_{n<\omega}$ of $(\omega)^{k}$, there is a partition $p \in(\omega)^{\omega}$ and a color $i<\ell$ such that for all $x \in(p)^{k}$, $x \in O_{i} \cap \bigcap_{n} D_{n}$.
(2) $\mathrm{ODRT}_{\ell}^{k}$ : For every open $\ell$-coloring $(\omega)^{k}=\cup_{i<\ell} O_{i}$, there is a partition $p \in(\omega)^{\omega}$ and a color $i<\ell$ such that for all $x \in(p)^{k}, x \in O_{i}$.
(3) $\mathrm{CDRT}_{\ell}^{k}$ : For every coloring $c:(<\omega)^{k-1} \rightarrow \ell$, there is a partition $p \in(\omega)^{\omega}$ and a color $i<\ell$ such that for all $x \in(p)^{k}, c\left(x^{*}\right)=i$.
Our first goal is to show that the instances of $\mathrm{CDRT}_{\ell}^{k}$ are in one-to-one canonical correspondence with those instances of $\mathrm{ODRT}_{l}^{k}$ for which the coloring of $(\omega)^{k}$ is reduced. We define a reduced coloring without considering the coding method and note that any reduced coloring is open.
Definition 3.5. Let $y \in(\omega)^{\omega}$ and $m<k$. A coloring of $(y)^{k}$ is m-reduced if whenever $p, q \in(y)^{k}$ and $p \upharpoonright \mu^{p}(m)=q \upharpoonright \mu^{q}(m), p$ and $q$ have the same color. A coloring of $(y)^{k}$ is reduced if it is $(k-1)$-reduced.

Note that a coloring is reduced means that the color of each partition $p \in(y)^{k}$ depends only on $p^{*}$.

Proposition $3.6\left(\mathrm{RCA}_{0}\right)$. The following are equivalent.
(1) $\mathrm{CDRT}_{\ell}^{k}$.
(2) For every open reduced coloring $(\omega)^{k}=\cup_{i<\ell} O_{i}$, there are $p \in(\omega)^{\omega}$ and $i<\ell$ such that $(p)^{k} \subseteq O_{i}$.
(3) For every $y \in(\omega)^{\omega}$ and open reduced coloring $(y)^{k}=\cup_{i<\ell} O_{i}$, there are $p \in(y)^{\omega}$ and $i<\ell$ such that $(p)^{k} \subseteq O_{i}$.

Proof. Clearly (3) implies (2). To see that (2) implies (3), fix $y \in(\omega)^{\omega}$ and a reduced open coloring $(y)^{k}=\cup_{i<\ell} O_{i}$. Define

$$
\widehat{O}_{i}=\left\{\langle n, \tau\rangle: \tau \in(<\omega)^{k} \text { and } \tau \circ y \in O_{i}\right\}
$$

It is straightforward to check that the coloring $(\omega)^{k}=\cup_{i<\ell} \widehat{O}_{i}$ is also reduced, and that whenever $x$ is homogeneous for $\cup_{i<\ell} \widehat{O}_{i}$ then $x \circ y$ is homogeneous for $\cup_{i<\ell} O_{i}$.

To see (2) implies (1), fix $c:(<\omega)^{k-1} \rightarrow \ell$. For each $i<\ell$, let

$$
O_{i}=\left\{\left\langle 0, \sigma^{\frown}(k-1)\right\rangle: \sigma \in(<\omega)^{k-1} \text { and } c(\sigma)=i\right\}
$$

Then $(\omega)^{k}=\cup_{i<\ell} O_{i}$ is an open reduced coloring of $(\omega)^{k}$, and any infinite partition which is homogeneous for it is also homogeneous for $c$.

For the implication from (1) to (2), assume $\mathrm{CDRT}_{\ell}^{k}$, and suppose we are given a coloring $\cup_{i<\ell} O_{i}$. Now, for each $\sigma \in(<\omega)^{k-1}$, we define $c(\sigma)$ as follows. Note that for some $i<\ell$, some $\tau \succeq \sigma^{\wedge}(k-1)$, and some $n$, we have $\langle n, \tau\rangle \in O_{i}$. Letting $\langle n, \tau, i\rangle$ be the least triple with this property, we define $c(\sigma)=i$.

Let $i<\ell$ and $p \in(\omega)^{\omega}$ be the result of applying $\mathrm{CDRT}_{\ell}^{k}$ to $c$. Given $x \in(p)^{k}$, we know that $c\left(x^{*}\right)=i$. Let $n, \tau$ be the witnesses used in the definition of $c\left(x^{*}\right)$. Let $q \in(\omega)^{k}$ with $q \succ \tau$. Then $q \in O_{i}$. Since $O_{i}$ is reduced and $q^{*}=\tau^{*}=x^{*}$, $x \in O_{i}$. Therefore, $p$ is homogeneous for the coloring $\cup_{i<\ell} O_{j}$, as required.

It is now routine to show that the number of colors does not matter.
Proposition $3.7\left(\mathrm{RCA}_{0}\right) . \mathrm{CDRT}_{\ell}^{k}$ and $\mathrm{CDRT}_{2}^{k}$ are equivalent.
Proof. Collapse colors and iterate $\mathrm{CDRT}_{2}^{k}$ finitely many times, using Proposition 3.6 .

The next proof is essentially an effective version of an argument in [14].
Theorem $3.8\left(\mathrm{RCA}_{0}\right)$. Baire- $\mathrm{DRT}_{\ell}^{k}, \mathrm{ODRT}_{\ell}^{k}$ and $\mathrm{CDRT}_{\ell}^{k}$ are equivalent.
Proof. By setting $D_{n}=(\omega)^{k}$ in Baire- $\mathrm{DRT}_{\ell}^{k}, \mathrm{ODRT}_{\ell}^{k}$ is a special case of Baire- $\mathrm{DRT}_{\ell}^{k}$, and by Proposition 3.6. $\mathrm{CDRT}_{\ell}^{k}$ is a special case of $\mathrm{ODRT}_{\ell}^{k}$. It remains to prove in $\mathrm{RCA}_{0}$ that $\mathrm{CDRT}_{\ell}^{k}$ implies Baire- $\mathrm{DRT}_{\ell}^{k}$.

Let $\left\{O_{i}\right\}_{i<\ell},\left\{D_{n}\right\}_{n<\omega}$ be a Baire $\ell$-coloring of $(\omega)^{k}$ for which the open sets $O_{i}$ are pairwise disjoint. We construct a partition $y \in(\omega)^{\omega}$ such that $(y)^{k} \subseteq \cap_{n} D_{n}$ and $\cup_{i} O_{i}$ restricted to $(y)^{k}$ is reduced. By Proposition 3.6 and $\mathrm{CDRT}_{\ell}^{k}$, there is a homogeneous $z \in(y)^{\omega}$ for this open reduced coloring. Since $(z)^{k} \subseteq(y)^{k} \subseteq \cap_{n} D_{n}$, this partition $z$ is homogeneous for the original Baire coloring.

First we describe the construction in a classical way, and then remark on how it can be carried out in $\mathrm{RCA}_{0}$.

Build $y$ by initial segments in stages, $y=\lim _{s} y_{s}$, starting with $y_{0}$ being the empty string, and then continuing with stage $s=1$ as follows. Assume that at the start of stage $s, y_{s-1}$ is an $(s-1)$-partition. In stage $s$ begin by letting $y_{s}^{0}=$ $y_{s-1} \cap(s-1)$, so that $y_{s}^{0}$ is an $s$-partition. Let $x_{0}, \ldots, x_{r}$ be a list of the elements of $(s)^{k}$. For each $i=0, \ldots r$, let $q=x_{i} \circ y_{s}^{i}$. Let $\tau \in(<\omega)^{k}$ be such that $q \preceq \tau$ and $\tau$ meets $\cap_{n \leq s} D_{n}$ and $\cup_{i<\ell} O_{i}$. Then extend $y_{s}^{i}$ to $y_{s}^{i+1}$ in such a way that $x_{i} \circ y_{s}^{i+1}=\tau$. In general there is more than one way to do this, but which way does not matter. For concreteness, for each $n \geq\left|y_{s}^{i}\right|$ we could set $y_{s}^{i+1}(n)$ to be the least $m$ such that $x_{i}(m)=\tau(n)$. At the conclusion of these substages we are left with $y_{s}^{r+1}$. Let $y_{s}=y_{s}^{r+1}$. This completes the construction of $y$.

We need to justify why this construction can be carried out in $\mathrm{RCA}_{0}$. To that end, we make the following claims in $\mathrm{RCA}_{0}$ :
(1) For any $q \in(<\omega)^{k}$ and $s$, there is an extension $\tau \succeq q$ which meets $\cup_{i<\ell} O_{i}$ and $\cap_{n \leq s} D_{n}$. To see that for all $s$, such a $\tau$ exists, apply $\Sigma_{1}^{0}$ induction.
(2) There is a function $f:(<\omega)^{k} \times \omega \rightarrow(<\omega)^{k}$ with the properties above. This follows because in $\mathrm{RCA}_{0}$, we can select the $\tau$ with least witness.
(3) There is a function which outputs the sequence

$$
y_{1}^{0}, \ldots, y_{1}^{r_{1}}, y_{2}^{0}, \ldots, y_{2}^{r_{2}}, y_{3}^{0}, \ldots
$$

This can be obtained by primitive recursion using the function $f$.
Therefore, $y$ exists in $\mathrm{RCA}_{0}$. Next we show that $(y)^{k} \subseteq \cap_{n} D_{n}$. Let $w \in(y)^{k}$ and fix $n$. Let $x \in(\omega)^{k}$ with $x \circ y=w$. Let $s \geq n$ be large enough that $x \upharpoonright s \in(s)^{k}$. Then $x \upharpoonright s$ was one of the ways to coarsen considered during stage $s$ of the construction. By construction, $(x \upharpoonright s) \circ y_{s}$ meets $D_{n}$. So $x \circ y \in D_{n}$.

Finally, we claim that the restriction of $\cup_{i<\ell} O_{i}$ to $(y)^{k}$ is a reduced coloring. Given if $w_{1}, w_{2} \in(y)^{k}$ with $w_{1}^{*}=w_{2}^{*}$, let $x_{1}$ and $x_{2}$ be such that $x_{1} \circ y=w_{1}$ and $x_{2} \circ y=w_{2}$. Then $x_{1}^{*}=x_{2}^{*}$. Let $x=\left(x_{1}^{*}\right)^{\wedge}(k-1)$ and let $s=|x|$. Then $x \in(s)^{k}$ and $x$ was considered at stage $s$ of the construction. By construction, $x \circ y_{s}$ meets $O_{i}$ for some $i$. Since $x \circ y_{s}$ is an initial segment of both $w_{1}$ and $w_{2}$, it follows that $w_{1}$ and $w_{2}$ are both in $O_{i}$. Finally, as $w_{1}, w_{2} \in \cap_{n} D_{n}$, we have $w_{1}, w_{2} \in C_{i}$, as needed.

Since $\mathrm{ODRT}_{\ell}^{k+1}$ implies $\mathrm{RT}_{\ell}^{k}$ over $\mathrm{RCA}_{0}$ [12], we have the following corollary.
Corollary $3.9\left(\mathrm{RCA}_{0}\right)$. $\mathrm{CDRT}_{\ell}^{k+1}$ implies $\mathrm{RT}_{\ell}^{k}$.
Proposition 3.10. For any $\ell \geq 2, \mathrm{RCA}_{0}$ proves $\mathrm{CDRT}_{\ell}^{2}$ and hence also $\mathrm{ODRT}_{\ell}^{2}$.
Proof. Let $c:(<\omega)^{1} \rightarrow \ell$. Since $(<\omega)^{1}=\left\{0^{n}: n \in \omega\right\}, c$ can be viewed as an $\ell$-coloring of $\omega$. By $\mathrm{RT}_{\ell}^{1}$, there is a color $i$ and an infinite set $X$ such that for every $n \in X, c\left(0^{n}\right)=i$. Let $z$ be the partition which has a block of the form $\{n\}$ for each $n \in X$ and puts all the other numbers in $z^{-1}(0)$. Then $z$ is homogeneous for $c$.
3.2. Connections to Hindman's theorem. In this section, we show that Hindman's Theorem for $\ell$-colorings implies $\mathrm{CDRT}_{\ell}^{3}$. In [4], Simpson remarks that one case of the Carlson-Simpson Lemma follows from Hindman's Theorem. Ludovic Patey showed us a proof, and the same argument gives a strong form of $\mathrm{CDRT}_{\ell}^{3}$. We include Patey's proof here.

Definition $3.11\left(\mathrm{RCA}_{0}\right)$. Let $\mathcal{P}_{\text {fin }}(\omega)$ denote the set of (codes for) all non-empty finite subsets of $\omega$. $X \subseteq \mathcal{P}_{\text {fin }}(\omega)$ is an $I P$ set if $X$ is closed under finite unions and contains an infinite sequence of pairwise disjoint sets.

Theorem 3.12 (Hindman's theorem for $\ell$-colorings). For every $c: \mathcal{P}_{\text {fin }}(\omega) \rightarrow \ell$ there is an IP set $X$ and a color $i<\ell$ such that $c(F)=i$ for all $F \in X$.

Theorem 3.13 (essentially Patey [13, see also [4, page 268]). Over $\mathrm{RCA}_{0}$, Hindman's theorem for $\ell$-colorings implies $\mathrm{CDRT}_{\ell}^{3}$. In particular, $\mathrm{CDRT}_{\ell}^{3}$ is provable in $\mathrm{ACA}_{0}^{+}$.

Proof. Hindman's Theorem follows from $\mathrm{ACA}_{0}^{+}$by [3], so it suffices to prove the first statement. Fix $\ell \geq 2$ and assume Hindman's Theorem for $\ell$-colorings. Since Hindman's Theorem for 2-colorings implies $\mathrm{ACA}_{0}$, we reason in $\mathrm{ACA}_{0}$. By Proposition [3.6, it suffices to fix an open reduced coloring $(\omega)^{3}=\cup_{i<\ell} O_{i}$ and produce $p \in(\omega)^{\omega}$ and $i<\ell$ such that for all $x \in(p)^{3}, x \in O_{i}$. We write the coloring as $c:(\omega)^{3} \rightarrow \ell$ with the understanding that $c(x)=i$ is shorthand for $x \in O_{i}$.

For a nonempty finite set $F \subseteq \omega$ with $0 \notin F$ and a number $n>\max F$, we let $x_{F, n} \in(\omega)^{3}$ be the following partition.

$$
x_{F, n}(k)= \begin{cases}0 & \text { if } k \notin F \text { and } k \neq n \\ 1 & \text { if } k \in F \\ 2 & \text { if } k=n\end{cases}
$$

Thus, the blocks are $\omega-(F \cup\{n\}), F$ and $\{n\}$. Note that we can determine the color $c\left(x_{F, n}\right)$ as a function of $F$ and $n$ and that since $c$ is reduced, if $x \in(\omega)^{3}$ and $x \upharpoonright \mu^{x}(2)=x_{F, n} \upharpoonright n$, then $c(x)=c\left(x_{F, n}\right)$.

The remainder of the proof is most naturally presented as a forcing construction. After giving a classical description of this construction, we indicate how to carry out the construction in $\mathrm{ACA}_{0}$. The forcing conditions are pairs $(F, I)$ such that

- $F$ is a non-empty finite set such that $0 \notin F$,
- $I$ is an infinite set such that $\max F<\min I$, and
- for every nonempty subset $U$ of $F$ there is an $i<\ell$ such that $c\left(x_{U, n}\right)=i$ for all $n \in F \cup I$ with $\max U<n$.
Extension of conditions is defined as for Mathias forcing: $(\widehat{F}, \widehat{I}) \leq(F, I)$ if $F \subseteq$ $\widehat{F} \subseteq F \cup I$ and $\widehat{I} \subseteq I$.

By the pigeonhole principle, there is an $i<l$ such that $c\left(x_{\{1\}, n}\right)=i$ for infinitely many $n>1$. For any such $i$, the pair $\left(\{1\},\left\{n \in \omega: n>1\right.\right.$ and $\left.\left.c\left(x_{\{1\}, n}\right)=i\right\}\right)$ is a condition. More generally, given a condition $(F, I)$ there is an infinite set $\widehat{I} \subseteq I$ such that $(F \cup\{\min I\}, \widehat{I})$ is also a condition. To see this, let $U_{0}, \ldots, U_{s-1}$ be the nonempty subsets of $F \cup\{\min I\}$ containing $\min I$. By arithmetic induction, for each positive $k \leq s$, there exist colors $i_{0}, \ldots, i_{k-1}<\ell$ such that there are infinitely many $n \in I$ with $c\left(x_{U_{j}, n}\right)=i_{j}$ for all $j<k$. (If not, fix the least $k$ for which the fact fails, and apply the pigeonhole principle to obtain a contradiction.) Let $i_{0}, \ldots, i_{s-1}$ be the colors corresponding to $k=s$ and let $\widehat{I}$ be the infinite set $\left\{n \in I: \forall j<s\left(c\left(x_{U_{j}, n}\right)=i_{j}\right)\right\}$.

Fix a sequence of conditions $\left(F_{1}, I_{1}\right)>\left(F_{2}, I_{2}\right)>\cdots$ with $\left|F_{k}\right|=k$ and let $G=\bigcup_{k} F_{k}$. To complete the proof, we use $G$ to define a coloring $d: \mathcal{P}_{\text {fin }}(\omega) \rightarrow \ell$ to which we can apply Hindman's Theorem. However, first we indicate why we can form $G$ in $\mathrm{ACA}_{0}$.

The conditions $(F, I)$ used to form $G$ can be specified by the finite set $F$, the number $m=\min I$ and the finite sequence $\delta \in \ell^{M}$ where $M=2^{|F|}-1$ such that if $F_{0}, \ldots, F_{M-1}$ is a canonical listing of the nonempty subsets of $F$, then $I=\{n \geq$ $\left.m: \forall j<M\left(c\left(x_{F_{j}, n}\right)=\delta(j)\right)\right\}$. The extension procedure above can be captured by an arithmetically definable function $f(F, m, \delta)=\left\langle F \cup\{m\}, m^{\prime}, \delta^{\prime}\right\rangle$ where $F \cup\{m\}$, $m^{\prime}$ and $\delta^{\prime}$ describe the extension $(F \cup\{m\}, \widehat{I})$. Because the properties of this extension where verified using arithmetic induction and the pigeonhole principle, both of which are available in $\mathrm{ACA}_{0}$, we can define $f$ in $\mathrm{ACA}_{0}$ and form a sequence of conditions $\left(F_{1}, m_{1}, \delta_{1}\right)>\left(F_{2}, m_{2}, \delta_{2}\right)>\cdots$ giving $G=\bigcup_{k} F_{k}$.

It remains to use $G=\left\{g_{0}<g_{1}<\cdots\right\}$ to complete the proof. By construction, for each non-empty finite subset $U$ of $G$, there is color $i_{U}<\ell$ such that $c\left(x_{U, n}\right)=i_{U}$ for all $n \in G$ with $n>\max U$. Define $d: \mathcal{P}_{\text {fin }}(\omega) \rightarrow \ell$ by $d(F)=i_{\left\{g_{m}: m \in F\right\}}$. We apply Hindman's theorem to $d$ to obtain an IP set $X$ and a color $i<\ell$. Since $X$ contains an infinite sequence of pairwise disjoint members, we can find a sequence $E_{1}, E_{2}, \ldots$ of members of $X$ such that $\max E_{k}<\min E_{k+1}$. Define $p \in(\omega)^{\omega}$ to be
the partition whose blocks are $p^{-1}(0)=\omega-\bigcup_{k}\left\{g_{m}: m \in E_{k}\right\}$ and, for each $k \geq 1$, $p^{-1}(k)=\left\{g_{m}: m \in E_{k}\right\}$. Note that for all $k \geq 1$,

$$
\max p^{-1}(k)=\max \left\{g_{m}: m \in E_{k}\right\}<\min p^{-1}(k+1)=\min \left\{g_{m}: m \in E_{k+1}\right\}
$$

It remains to verify that $p$ and $i$ have the desired properties. Consider any $x \in(p)^{3}$; we must show that $c(x)=i$. Let $U=x^{-1}(1) \cap \mu^{x}(2)$ and let $n=\mu^{x}(2)$. Then $n=\mu^{x_{U, n}}(2)$ and $x \upharpoonright n=x_{U, n} \upharpoonright n$, so since $c$ is reduced, $c(x)=c\left(x_{U, n}\right)$. Therefore, it suffices to show $c\left(x_{U, n}\right)=i$.

We claim $U$ is a finite union of $p$-blocks. Because $x$ is a coarsening of $p, x^{-1}(1)$ is a (possibly infinite) union of $p$-blocks $p^{-1}\left(j_{1}\right) \cup p^{-1}\left(j_{2}\right) \cup \cdots$ with $0<j_{1}<j_{2}<\cdots$ and $n=\mu^{x}(2)=\min x^{-1}(2)=\min p^{-1}(b)$ for some $b \geq 2$. Let $j_{a}$ be the largest index such that $j_{a}<b$. Since the $p$-blocks are finite and increasing, $U=x^{-1}(1) \cap$ $\mu^{x}(2)=p^{-1}\left(j_{1}\right) \cup \cdots \cup p^{-1}\left(j_{a}\right)$. Note that $n \in G$ (because $\left.p^{-1}(b) \neq p^{-1}(0)\right)$ and $\max U<n$.

It follows that $U=\left\{g_{m} \mid m \in F\right\}$ where $F=E_{j_{1}} \cup \cdots \cup E_{j_{a}}$. Since our fixed IP set $X$ is closed under finite unions, $F \in X$ and therefore $d(F)=i$. By the definition of $d, d(F)=i_{\left\{g_{m} \mid m \in F\right\}}=i_{U}$, so $i=i_{U}$. Finally, $U$ is a finite subset of $G, n \in G$ and $\max U<n$, so $c\left(x_{U, n}\right)=i_{U}=i$ as required.

Observe that this proof of $\mathrm{CDRT}_{\ell}^{3}$ from HT produces a homogeneous $p$ with a special property: $\max p^{-1}(i)<\min p^{-1}(i+1)$ for all $i>0$. We show that this strengthened "ordered finite block" version of $\mathrm{CDRT}_{\ell}^{3}$ is equivalent to HT. However, there is no finite block version of $\mathrm{CDRT}_{\ell}^{k}$ for $k>3$.

Proposition $3.14\left(\mathrm{RCA}_{0}\right)$. If for every $\ell$-coloring of $(<\omega)^{2}$ there is an infinite homogeneous partition $p$ with $\max p^{-1}(i)<\min p^{-1}(i+1)$ for all $i>0$, then Hindman's Theorem for $\ell$-colorings holds.

Proof. Given $c: \mathcal{P}_{\text {fin }}(\omega) \rightarrow \ell$, define $\widehat{c}:(<\omega)^{2} \rightarrow \ell$ by $\widehat{c}(\sigma)=c(\{i<|\sigma|: \sigma(i)=$ $1\})$. Let $p$ be a homogeneous partition for $\widehat{c}$ with $\max p^{-1}(i)<\min p^{-1}(i+1)$ for all $i>0$. The set of all finite unions of the blocks $p^{-1}(i)$ for $i>0$ satisfies the conclusion of Hindman's Theorem.

Proposition 3.15. There is a 2-coloring of $(<\omega)^{3}$ such that any infinite homogeneous partition $p$ has $p^{-1}(i)$ infinite for all $i>0$.

Proof. For $\sigma \in(<\omega)^{3}$, set $c(\sigma)=1$ if $\sigma$ contains more 1's than 2's and set $c(\sigma)=0$ otherwise. Let $p$ be homogeneous for this coloring. Suppose for contradiction that $i>0$ is such that $p^{-1}(i)$ is finite. Let $N=i+2+\left|p^{-1}(i)\right|$ and let $x=w \circ p$ where

$$
w(n)= \begin{cases}1 & \text { if } n=i \\ 2 & \text { if } i<n \leq N \\ 3 & \text { if } n=N+1 \\ 0 & \text { otherwise }\end{cases}
$$

Since $x^{*}$ has more 2 's than 1 's, $c\left(x^{*}\right)=0$. Now coarsen in a different way: let $h \in[i+1, N]$ be chosen so that the size of $p^{-1}(h) \cap\left[0, \mu^{x}(3)\right]$ is minimized. Let
$y=z \circ p$ where

$$
z(n)= \begin{cases}1 & \text { if } i \leq n \leq N \text { and } n \neq h \\ 2 & \text { if } n=h \\ 3 & \text { if } n=N+1 \\ 0 & \text { otherwise }\end{cases}
$$

Since at least one $p$-block has moved from $x^{-1}(2)$ to $y^{-1}(1)$ and since $y^{-1}(2)$ contains only the smallest $p$-block from $x^{-1}(2), c\left(y^{*}\right)=1$. So $p$ was not homogeneous.
3.3. CDRT and the Carlson-Simpson Lemma. The Carlson-Simpson Lemma is the main technical tool in the original proof of the Borel version of the Dual Ramsey Theorem. The principle is usually stated in the framework of variable words, but it can also be understood as a special case of the Combinatorial Dual Ramsey Theorem.

Carlson-Simpson Lemma $(\operatorname{CSL}(m, \ell))$. For every coloring $c:(<\omega)^{m} \rightarrow \ell$, there is a partition $p \in(\omega)^{\omega}$ and a color $i$ such that for all $x \in(p)^{m+1}$, if $p^{-1}(j) \subseteq x^{-1}(j)$ for each $j<m$, then $c\left(x^{*}\right)=i$.

The condition $p^{-1}(j) \subseteq x^{-1}(j)$ for $j<m$ captures those $x \in(p)^{m+1}$ which keep the first $m$ many blocks of $p$ distinct in $x$. Therefore, $\operatorname{CSL}(m, \ell)$ is a special case of $\mathrm{CDRT}_{\ell}^{m+1}$. Two related principles, $\operatorname{OVW}(m, \ell)$ and $\mathrm{VW}(m, \ell)$ have also been studied (see [12, 7, 11). We do not deal with these principles, but it may be useful to note that $\operatorname{VW}(m, \ell)$ is the strengthening of $\operatorname{CSL}(m, \ell)$ which requires each nonzero block $p^{-1}(j)$ to be finite, and $\operatorname{OVW}(m, \ell)$ is the further strengthening which requires $\max p^{-1}(j)<\min p^{-1}(j+1)$ for all $j>0$.

In Proposition 3.16, we give three equivalent versions of the Carlson-Simpson Lemma. The version in Proposition 3.16(2) is (up to minor notational changes which are easily translated in $\mathrm{RCA}_{0}$ ) the statement from Lemma 2.4 of Carlson and Simpson 4].

Proposition $3.16\left(\mathrm{RCA}_{0}\right)$. The following are equivalent.
(1) $\operatorname{CSL}(m, \ell)$.
(2) For each coloring $c:(<\omega)^{m} \rightarrow \ell$, there is a partition $p \in(\omega)^{\omega}$ and a color $i$ such that for all $j<m, p(j)=j$ and for all $x \in(p)^{m+1}$, if $p^{-1}(j) \subseteq x^{-1}(j)$ for each $j<m$, then $c\left(x^{*}\right)=i$.
(3) For each $y \in(\omega)^{\omega}$ and open reduced coloring $(y)^{m+1}=\cup_{i<\ell} O_{i}$, there is a partition $p \in(y)^{\omega}$ and a color $i$ such that for all $j<m, y^{-1}(j) \subseteq p^{-1}(j)$ and for all $x \in(p)^{m+1}$, if $p^{-1}(j) \subseteq x^{-1}(j)$ for each $j<m$, then $x \in O_{i}$.

Proof. (2) implies (1) because $\operatorname{CSL}(m, \ell)$ is a special case of (2). The extra condition in (2) that $p(j)=j$ for $j<m$ says that the partition $p$ does not collapse any of the first $m$-many blocks of the trivial partition defined by the identity function. The equivalence between (2) and (3) is proved in a similar way to Proposition 3.6,

It remains to prove (1) implies (2). Fix an $\ell$-coloring $c:(<\omega)^{m} \rightarrow \ell$. Define $\tilde{c}:(<\omega)^{m} \rightarrow \ell$ by $\tilde{c}(\sigma)=c\left(0^{\wedge} 1^{\wedge} \ldots \wedge(m-1)^{\wedge} \sigma\right)$. Apply $\operatorname{CSL}(m, \ell)$ to $\tilde{c}$ to get $\tilde{p} \in(\omega)^{\omega}$ and $i<\ell$ such that for all $\tilde{x} \in(\tilde{p})^{m+1}$, if $\tilde{p}^{-1}(j) \subseteq \tilde{x}^{-1}(j)$ for all $j<m$, then $\tilde{c}\left(\tilde{x}^{*}\right)=i$.

Let $p \in(\omega)^{\omega}$ be the partition defined by

$$
p(j)= \begin{cases}j & \text { if } j<m \\ \tilde{p}(j-m) & \text { if } j \geq m\end{cases}
$$

We claim that $p$ satisfies the conditions in (2) for the coloring $c$ with the fixed color $i$. Fix $x \in(p)^{m+1}$ such that $p^{-1}(j) \subseteq x^{-1}(j)$ for all $j<m$. We need to show that $c\left(x^{*}\right)=i$. Since $x$ does not collapse any of the first $m$-many $p$-blocks, $x(j)=j$ for all $j<m$. Define $\tilde{x} \in(\tilde{p})^{m+1}$ by $\tilde{x}(j)=x(j+m)$. Then $\tilde{p}^{-1}(j) \subseteq \tilde{x}^{-1}(j)$ for all $j<m$. Therefore, $\tilde{c}\left(\tilde{x}^{*}\right)=i$. Now, $x^{*}=0^{\wedge} 1^{\wedge} \ldots \wedge(m-1)^{\wedge} \tilde{x}^{*}$. Therefore, $c\left(x^{*}\right)=\tilde{c}\left(\tilde{x}^{*}\right)=i$, as required to complete the proof that (1) implies (2).

Let $y \in(\omega)^{\omega}$ and $(y)^{k}=\cup_{i<\ell} C_{i}$ be an $m$-reduced coloring for some $1<m<k$. We define the induced coloring $(y)^{m+1}=\cup_{i<\ell} \widehat{C}_{i}$ as follows. For $\widehat{q} \in(y)^{m+1}$, $\widehat{q} \in \widehat{C}_{i}$ if and only if $q \in C_{i}$ for some (or equivalently all) $q \in(y)^{k}$ such that $\widehat{q} \upharpoonright \mu^{\widehat{q}}(m)=q \upharpoonright \mu^{q}(m)$. This induced coloring is a reduced coloring of $(y)^{m+1}$ and therefore we can apply $\operatorname{CSL}(m, \ell)$ to it.

Our proof of $\mathrm{CDRT}_{\ell}^{k}$ from the Carlson-Simpson Lemma will use repeated applications of the following lemma, which is proved using $\omega$ many nested applications of $\operatorname{CSL}(m, \ell)$.
Lemma 3.17. Fix $1<m<k$ and $y \in(\omega)^{\omega}$. Let $(y)^{k}=\cup_{i<\ell} C_{i}$ be an $m$-reduced coloring. There is an $x \in(y)^{\omega}$ such that the coloring restricted to $(x)^{k}$ is $(m-1)$ reduced.

Proof. Fix an $m$-reduced coloring $(y)^{k}=\cup_{i<\ell} C_{i}$. We define a sequence of infinite partitions $x_{m}, x_{m+1}, \cdots$ starting with index $m$ such that $x_{m}=y$ and $x_{s+1}$ is a coarsening of $x_{s}$ for which $x_{s}{ }^{-1}(j) \subseteq x_{s+1}{ }^{-1}(j)$ for all $j<s$. That is, we do not collapse any of the first $s$-many blocks of the partition $x_{s}$ when we coarsen it to $x_{s+1}$. This property guarantees that the sequence has a well-defined limit $x \in(\omega)^{\omega}$. We show this limiting partition $x$ satisfies the conclusion of the lemma.

Assume $x_{s}$ has been defined for a fixed $s \geq m$ and we construct $x_{s+1}$. Set $x_{s}^{0}=x_{s}$. Let $\sigma_{0}, \ldots, \sigma_{r}$ be a list of the elements of $(s)^{m}$. We define a sequence of coarsenings $x_{s}^{1}, \ldots, x_{s}^{r}$ and set $x_{s+1}=x_{s}^{r}$.

Assume that $x_{s}^{j}$ has been defined. Define

$$
\sigma_{j}^{+}(n)= \begin{cases}\sigma_{j}(n) & \text { if } n<s \\ m+(n-s) & \text { if } n \geq s\end{cases}
$$

and let $w_{s}^{j}=\sigma_{j}^{+} \circ x_{s}^{j}$. That is $w_{s}^{j}$ collapses the first $s$-many blocks of $x_{s}^{j}$ into $m$-many blocks in the $j$-th possible way and leaves the remaining blocks of $x_{s}^{j}$ unchanged. Since $w_{s}^{j}$ is a coarsening of $y$, the coloring $\cup_{i<\ell} C_{i}$ is also an $m$-reduced coloring of $\left(w_{s}^{j}\right)^{k}$. Let $\left(w_{s}^{j}\right)^{m+1}=\cup_{i<\ell} \widehat{C}_{i}$ be the induced coloring. This coloring is reduced, so let $i_{s}^{j}<\ell$ and $z_{s}^{j} \in\left(w_{s}^{j}\right)^{\omega}$ be the result of applying $\operatorname{CSL}(m, l)$ as stated in Proposition 3.16(3). Then $z_{s}^{j}$ leaves the first $m$ blocks of $w_{s}^{j}$ separate, and any coarsening of $z_{s}^{j}$ into at least $m+1$ pieces receives color $i_{s}^{j}$, provided the first $m$ blocks are left separate.

To define $x_{s}^{j+1}$, we want to "uncollapse" the first $m$-many blocks of $z_{s}^{j}$ to reverse the action of $\sigma_{j}^{+}$in defining $w_{s}^{j}$. Since $w_{s}^{j}$ collapsed the first $s$-blocks of $x_{s}^{j}$ to $m$ many blocks and since $z_{s}^{j}$ is a coarsening of $w_{s}^{j}$, if $x_{s}^{j}(u)<s$, then $z_{s}^{j}(u)<m$. We define $x_{s}^{j+1}$ by cases as follows.
(1) If $x_{s}^{j}(u)<s$, then $x_{s}^{j+1}(u)=x_{s}^{j}(u)$.
(2) If $x_{s}^{j}(u) \geq s$ and $z_{s}^{j}(u)=a<m$, then $x_{s}^{j+1}(u)=x_{s}^{j}\left(\mu^{z_{s}^{j}}(a)\right)$.
(3) If $z_{s}^{j}(u) \geq m$, then $x_{s}^{j+1}(u)=z_{s}^{j}(u)+(s-m)$.

Below we verify that $x_{s}^{j+1}$ is an infinite partition coarsening $x_{s}^{j}$ which does not collapse any of the first $s$-many blocks of $x_{s}^{j}$. This completes the construction of $x_{s}^{j+1}$ and hence of $x_{s+1}$ and $x$.

We verify the required properties of $x_{s}^{j+1}$. By (1), $x_{s}^{j^{-1}}(a) \subseteq x_{s}^{j+1^{-1}}(a)$ for all $a<s$, so we do not collapse any of the first $s$-many blocks of $x_{s}^{j}$ in $x_{s}^{j+1}$. There is no conflict between (1) and (3) because $x_{s}^{j}(u)<s$ implies $z_{s}^{j}(u)<m$. Furthermore, (3) renumbers the $z_{s}^{j}$-blocks starting with index $m$ to $x_{s}^{j+1}$-blocks starting with index $s$ without changing any of these blocks. Therefore, $x_{s}^{j+1}$ is an infinite partition.

In (2), we handle the case when the $x_{s}^{j}$-block containing $u$ is not changed by $w_{s}^{j}$ (except to renumber its index) but is collapsed by $z_{s}^{j}$ into one of the first $m$ many $z_{s}^{j}$-blocks. In this case, $\mu^{z_{s}^{j}}(a)=\mu^{x_{s}^{j}}(b)$ for some $b<s$ and we have set $x_{s}^{j+1}(u)=b$. It is straightforward to check (as in the proof of Theorem (3.8) that $x_{s}^{j+1}$ is a coarsening of $x_{s}^{j}$ and that $\sigma_{j}^{+} \circ x_{s}^{j+1}=z_{s}^{j}$.

To complete the proof, we verify that the restriction of $\cup_{i<\ell} C_{i}$ to $(x)^{k}$ is $(m-1)$ reduced. Fix $p \in(x)^{k}$ and we show the color of $p$ depends only on $p \upharpoonright \mu^{p}(m-1)$.

Let $q \in(\omega)^{\omega}$ be the unique element with $p=q \circ x$, and let $\sigma=q \upharpoonright \mu^{q}(m-1)$. Then $\sigma^{\frown}(m-1) \in(s)^{m}$ for some $s$, and

$$
p \upharpoonright \mu^{p}(m-1)=\sigma \circ\left(x \upharpoonright \mu^{x}(s-1)\right)
$$

During stage $s$ and afterward, the first $s$ blocks of $x_{s}$ are always kept separate. Therefore, the above equation remains true when $x$ is replaced with $x_{s}^{j+1}$, where $j$ is the unique index such that $\sigma_{j}=\sigma$. Therefore, $p$ is a coarsening of $\sigma_{j}^{+} \circ x_{j}^{s+1}=z_{s}^{j}$ and $p$ keeps the first $m$ blocks of $z_{s}^{j}$ separate. Therefore, the color of $p$ is $i_{s}^{j}$, the homogeneous color obtained when we applied $\operatorname{CSL}(m, \ell)$ to obtain $z_{s}^{j}$. This completes the proof that the restriction of $\cup_{i<\ell} C_{i}$ to $(x)^{k}$ is $(m-1)$-reduced because the indices $s$ and $j$ in $z_{s}^{j}$ are determined only by $p \upharpoonright \mu^{p}(m-1)$.

We end this section with the proof of $\operatorname{CDRT}_{\ell}^{k}$.
Theorem 3.18. For all for $k \geq 2$ and all $\ell, \mathrm{CDRT}_{\ell}^{k}$ holds.
Proof. For $k=2, \mathrm{CDRT}_{\ell}^{k}$ follows from the pigeonhole principle as in Proposition 3.10. Now assume $k \geq 3$. Consider $\mathrm{CDRT}_{\ell}^{k}$ in the form given in Proposition 3.6 Let $y \in(\omega)^{\omega}$ and $(y)^{k}=\cup_{i<\ell} O_{i}$ be an open reduced coloring. These satisfy the assumptions of Lemma 3.17 with $m=k-1$. After $k-2$ applications of Lemma 3.17, we obtain $x \in(y)^{\omega}$ such that the restriction of $\cup_{i<\ell} O_{i}$ to $(x)^{k}$ is 1-reduced and hence the color of $p \in(x)^{k}$ depends only on $p \upharpoonright \mu^{p}(1)$. Since the numbers $n<\mu^{p}(1)$ must lie in $p^{-1}(0)$, the color of $p$ is determined by the value of $\mu^{p}(1)$. By the pigeonhole principle, there is an infinite set $X \subseteq\left\{\mu^{x}(a): a \geq 1\right\}$ and a color $i$ such that for all $p \in(x)^{k}$, if $\mu^{p}(1) \in X$, then $p \in C_{i}$. It follows that for any $z \in(x)^{\omega}$ such that $\mu^{z}(a) \in X$ for all $a \geq 1,(z)^{k} \subseteq C_{i}$ as required.

It is interesting to note that the only non-constructive steps in this proof are the $\omega \cdot(k-2)$ nested applications of the Carlson-Simpson Lemma.

## 4. The Borel Dual Ramsey Theorem for $k \geq 3$

In the next two sections we consider the Borel Dual Ramsey Theorem from the perspective of effective mathematics. We define Borel codes for topologically $\boldsymbol{\Sigma}_{\alpha}^{0}$ subsets of $(\omega)^{k}$ by induction on the ordinals below $\omega_{1}$. Let $L$ be some countable set of labels which effectively code for the clopen sets $\emptyset,(\omega)^{k}$ and $[\sigma]$ and $\overline{[\sigma]}$ for $\sigma \in(<\omega)^{k}$.

Definition 4.1. We define a Borel code for a $\boldsymbol{\Sigma}_{\alpha}^{0}$ or $\boldsymbol{\Pi}_{\alpha}^{0}$ set.

- A Borel code for a $\boldsymbol{\Sigma}_{0}^{0}$ or a $\boldsymbol{\Pi}_{0}^{0}$ set is a labeled tree $T$ consisting of just a root $\lambda$ in which the root is labeled by a clopen set from $L$. The Borel code represents that clopen set.
- For $\alpha \geq 1$, a Borel code for a $\boldsymbol{\Sigma}_{\alpha}^{0}$ set is a labeled tree with a root labeled by $\cup$ and attached subtrees at level 1 , each of which is a Borel code for a $\boldsymbol{\Sigma}_{\beta_{n}}^{0}$ or $\boldsymbol{\Pi}_{\beta_{n}}^{0}$ set $A_{n}$ for some $\beta_{n}<\alpha$. The code represents the set $\cup_{n} A_{n}$.
- For $\alpha \geq 1$, a Borel code for a $\boldsymbol{\Pi}_{\alpha}^{0}$ set is the same, except the root is labeled $\cap$. The code represents the set $\cap_{n} A_{n}$.
For $\alpha \geq 1$, a Borel code for a $\boldsymbol{\Delta}_{\alpha}^{0}$ set is a pair of labeled trees which encode the same set, where one encodes it as a $\boldsymbol{\Sigma}_{\alpha}^{0}$ set and the other encodes it as a $\boldsymbol{\Pi}_{\alpha}^{0}$ set.

The codes are faithful to the Borel hierarchy in the sense that every code for a $\boldsymbol{\Sigma}_{\alpha}^{0}$ set represents a $\boldsymbol{\Sigma}_{\alpha}^{0}$ set and every $\boldsymbol{\Sigma}_{\alpha}^{0}$ set is represented by a Borel code for a $\boldsymbol{\Sigma}_{\alpha}^{0}$ set. There is a uniform procedure to transform a Borel code $B$ for a $\boldsymbol{\Sigma}_{\alpha}^{0}$ set $A$ into a Borel code $\bar{B}$ for a $\Pi_{\alpha}^{0}$ set $\bar{A}$ : leave the underlying tree structure the same, swap the $\cup$ and $\cap$ labels and replace the leaf labels by their complements.

Observe also that a code for a $\Sigma_{1}^{0}$ set essentially agrees with the definition of a code for an open set in Definition 3.1 (up to a primitive recursive translation mapping elements of $\omega \times(<\omega)^{k}$ to leaves of a $\boldsymbol{\Sigma}_{1}^{0}$ code, and mapping each leaf of a $\boldsymbol{\Sigma}_{1}^{0}$ code to an element or sequence of elements of $\left.\omega \times(<\omega)^{k}\right)$. The one difference is that we must include a leaf label of $\emptyset$ in the definition of a Borel code, so that the empty set has a $\boldsymbol{\Sigma}_{1}^{0}$ code. Having included $\emptyset$ as a label, we also include $(\omega)^{k}$ to keep complementation effective.

We recall some notation from hyperarithmetic theory. Let $\mathcal{O}$ denote Kleene's set of computable ordinal notations. The ordinal represented by $a \in \mathcal{O}$ is denoted $|a|_{\mathcal{O}}$, with $|1|_{\mathcal{O}}=0,\left|2^{a}\right|_{\mathcal{O}}=|a|_{\mathcal{O}}+1$, and $\left|3 \cdot 5^{e}\right|_{\mathcal{O}}=\sup _{j}\left|\varphi_{e}(j)\right|_{\mathcal{O}}$. The $H$-sets are defined by effective transfinite recursion on $\mathcal{O}$ as follows: $H_{1}=\emptyset, H_{2^{a}}=H_{a}^{\prime}$ and $H_{3.5^{a}}=\left\{\langle i, j\rangle \mid i \in H_{\varphi_{a}(j)}\right\}$. The reader is referred to Sacks [16] for more details. To use oracles that line up better than the $H$-sets do with the levels of the Borel hierarchy, define

$$
\emptyset_{(a)}= \begin{cases}H_{a} & \text { if }|a|_{\mathcal{O}}<\omega \\ H_{2^{a}} & \text { otherwise }\end{cases}
$$

If $|a|_{\mathcal{O}}=|b|_{\mathcal{O}}=\alpha$, then $\emptyset_{(a)} \equiv_{1} \emptyset_{(b)}$, so we sometimes just write $\emptyset_{(\alpha)}$ in that situation. As usual, $\omega_{1}^{C K}$ denotes the least noncomputable ordinal.

Recall the standard effectivizations of the notions described above. We say that a Borel code $B$ is computable if it is computable as a labeled subtree of $\omega^{<\omega}$. We say $B$ is effectively $\Sigma_{\alpha}^{0}$ (respectively effectively $\Pi_{\alpha}^{0}$ ) if the root is labeled $\cup$ (respectively $\cap$ ) and additionally there is $a \in \mathcal{O}$ with $|a|_{\mathcal{O}}=\alpha$, and a computable labeling of the nodes of $B$ with notations from $\left\{b: b \leq_{\mathcal{O}} a\right\}$, such that the root is labeled with $a$ and each node has a label strictly greater than all its extensions.

It is well-known that an open set of high hyperarithmetic complexity can be represented by a computable Borel code for a $\Sigma_{\alpha}^{0}$ set, where $\alpha$ is an appropriate computable ordinal. In the following proposition, we use a standard technique to make this correspondence explicit. Fix an effective 1-to-1 enumeration $\tau_{n}$ for the strings $\tau \in(<\omega)^{k}$.

Proposition 4.2. There is a partial computable function $p(x, y)$ such that $p(a, e)$ is defined for all $a \in \mathcal{O}$ and $e \in \omega$ and such that if $a \in \mathcal{O}$ and $R=\bigcup\left\{\left[\tau_{n}\right]: n \in W_{e}^{\emptyset_{(a)}}\right\}$, then $\Phi_{p(a, e)}$ is a computable Borel code for $R$ as a $\Sigma_{\alpha+1}^{0}$ set, where $\alpha=|a|_{\mathcal{O}}$.

Proof. We define $p(a, e)$ for all $e$ by effective transfinite recursion on $a \in \mathcal{O}$. Let $\Phi_{p(1, e)}$ be a Borel code for the open set $R=\bigcup\left\{\left[\tau_{n}\right]: n \in W_{e}\right\}$.

For the successor step, consider $R=\bigcup\left\{\left[\tau_{n}\right]: n \in W_{e}^{\emptyset\left(2^{a}\right)}\right\}$. Each set which is $\Sigma_{1}^{0}$ in $\emptyset_{\left(2^{a}\right)}$ is $\Sigma_{2}^{0}$ in $\emptyset_{(a)}$ and for such sets, we can effectively pass from a $\Sigma_{1}^{0, \emptyset_{\left(2^{a}\right)}}$ index to a $\Sigma_{2}^{0, \emptyset_{(a)}}$ description. Specifically, uniformly in $e$, we compute an index $e^{\prime}$ such that for all oracles $X, \Phi_{e^{\prime}}^{X}(x, y)$ is a total $\{0,1\}$-valued function and

$$
n \in W_{e}^{X^{\prime}} \text { if and only if } \exists t \forall s \geq t\left(\Phi_{e^{\prime}}^{X}(n, s)=1\right)
$$

Let $R_{t}=\bigcup\left\{\left[\tau_{n}\right]: \exists s \geq t\left(\Phi_{e^{\prime}}^{\emptyset_{(a)}}(n, s)=0\right)\right\} . R_{0} \supseteq R_{1} \supseteq \cdots$ is a decreasing sequence of sets such that $x \notin R$ if and only if $\forall t\left(x \in R_{t}\right)$. Therefore, $R=\cup_{t} \overline{R_{t}}$. Each set $R_{t}$ can be represented as $R_{t}=\bigcup\left\{\left[\tau_{n}\right]: n \in W_{e_{t}}^{\emptyset_{(a)}}\right\}$, where $e_{t}$ is uniformly computable from $e$ and $t$. Applying the induction hypothesis, we define $p\left(2^{a}, e\right)$ to encode a tree whose root is labeled by a union and whose $t$-th subtree at level 1 is the Borel code representing the complement of $\Phi_{p\left(a, e_{t}\right)}$.

For the limit step, consider $R=\bigcup\left\{\left[\tau_{n}\right]: n \in W_{e}^{\emptyset\left(3.5^{d}\right)}\right\}$. Uniformly in $e$, we construct a sequence of indices $e_{t}$ for $t \in \omega$ such that for all oracles $X, \Phi_{e_{t}}^{X}(x)$ converges if and only if $\Phi_{e}^{X}(x)$ converges and only asks oracle questions about numbers in the first $t$ many columns of $X$. Let $R_{t}=\bigcup\left\{\left[\tau_{n}\right]: n \in W_{e_{t}}^{\oplus_{i \leq t} \emptyset_{\left(\varphi_{d}(i)\right)}}\right\}$ and note that $R=\cup_{t} R_{t}$. We can effectively pass to a sequence of indices $e_{t}^{\prime}$ such that $R_{t}=\bigcup\left\{\left[\tau_{n}\right]: n \in W_{e_{t}^{\prime}}^{\left.\emptyset_{\left(\varphi_{d}(t)\right)}\right\} \text {. By induction, each } p\left(\varphi_{d}(t), e_{t}^{\prime}\right) \text { is the index for }}\right.$ a computable Borel code for $R_{t}$ as a $\Sigma_{2^{\varphi_{d}(t)}}^{0}$ set, so we may define $p\left(3 \cdot 5^{d}, e\right)$ to be the index of a tree which has $\cup$ at the root and $\Phi_{p\left(\varphi_{d}(t), e_{t}^{\prime}\right)}$ as its subtrees. Since $2^{\varphi_{d}(t)}<_{\mathcal{O}} 3 \cdot 5^{d}$ for all $t$, the resulting Borel code has the required height.

To force the Dual Ramsey Theorem to output computationally powerful homogeneous sets, we use the following definition and a result of Jockusch 10 .

Definition 4.3. For functions $f, g: \omega \rightarrow \omega$, we say $g$ dominates $f$, and write $g \succeq f$, if $f(n) \leq g(n)$ for all but finitely many $n$.

Theorem 4.4 (Jockusch [10, see also [15, Exercise 16-98]). For each computable ordinal $\alpha$, there is a function $f_{\alpha}$ such that $f_{\alpha} \equiv_{T} \emptyset_{(\alpha)}$ and for every $g \succeq f_{\alpha}$, we have $\emptyset_{(\alpha)} \leq_{T} g$.

In Theorem 4.7, we use these functions $f_{\alpha}$ to show that for every computable ordinal $\alpha$, there is a computable Borel code for a set $R \subseteq(\omega)^{3}$ such that any homogeneous partition $p \in(\omega)^{\omega}$ for the coloring $(\omega)^{3}=R \cup \bar{R}$ computes $\emptyset_{(\alpha)}$.

Theorem 4.5. Let $A$ be a set and $f_{A}$ be a function such that $A \equiv_{T} f_{A}$ and for every $g \succeq f_{A}$, we have $A \leq_{T} g$. There is an A-computable clopen coloring $(\omega)^{3}=R \cup \bar{R}$ for which every homogeneous partition $p$ satisfies $p \geq_{T} A$.
Proof. Fix $A$ and $f_{A}$ as in the statement of the theorem. Without loss of generality, we assume that if $n<m$, then $f_{A}(n)<f_{A}(m)$. For $x \in(\omega)^{3}$, let $a_{x}=\mu^{x}(1)$ and $b_{x}=\mu^{x}(2)$. Let $O_{a, b}=\left\{x \in(\omega)^{3}: a_{x}=a \wedge b_{x}=b\right\}$. Set $R=\left\{x \in(\omega)^{3}: f_{A}\left(a_{x}\right) \leq\right.$ $\left.b_{x}\right\}$. Since $R=\bigcup\left\{O_{n, m} \mid f_{A}(n) \leq m\right\}$ and $\bar{R}=\bigcup\left\{O_{n, m} \mid f_{A}(n)>m\right\}$ both $R$ and $\bar{R}$ are $A$-computable open sets.

Claim. If $p \in(\omega)^{\omega}$ is homogeneous, then $(p)^{3} \subseteq R$.
It suffices to show that there is an $x \in(p)^{3}$ with $x \in R$. Let $u=\mu^{p}(1)$. Because $p$ has infinitely many blocks, there must be some $i$ with $\mu^{p}(i) \geq f(u)$. Consider the partition $x=w \circ p$, where $w(1)=1, w(i)=2$, and $w(m)=0$ for all other $m$. Then since $a_{x}=u$ and $b_{x} \geq f(u)$, we have $x \in(p)^{3}$ with $f\left(a_{x}\right) \leq b_{x}$, so $x \in R$.
Claim. If $p \in(\omega)^{\omega}$ is homogeneous, then $A \leq_{T} p$.
Fix $p$ and let $g(n)=\mu^{p}(n+2)$. Since $g$ is $p$-computable, it suffices to show $g \succeq f_{A}$. Because $n<\mu^{p}(n+1)$ and $f_{A}$ is increasing, we have $f_{A}(n)<f_{A}\left(\mu^{p}(n+1)\right)$. Therefore, to show $g \succeq f_{A}$, it suffices to show $f_{A}\left(\mu^{p}(n+1)\right) \leq \mu^{p}(n+2)=g(n)$.

Let $x_{n} \in(p)^{3}$ be defined by $x_{n}=w_{n} \circ p$, where $w_{n}(n+1)=1, w_{n}(n+2)=2$, and $w_{n}(m)=0$ for all other $m$. Note that $a_{x_{n}}=\mu^{p}(n+1)$ and $b_{x_{n}}=\mu^{p}(n+2)$. By the previous claim, $x_{n} \in R$, so $f_{A}\left(a_{x_{n}}\right) \leq b_{x_{n}}$. In other words, $f_{A}\left(\mu^{p}(n+1)\right) \leq \mu^{p}(n+2)$ as required.

Corollary 4.6. For each $k \geq 3$ and each recursive ordinal $\alpha$, there is an $\emptyset_{(\alpha)}$ computable clopen set $R \subseteq(\omega)^{k}$ such that if $p \in(\omega)^{\omega}$ is homogeneous for $(\omega)^{k}=$ $R \cup \bar{R}$, then $\emptyset_{(\alpha)} \leq_{T} p$.
Proof. For $k=3$, this corollary follows from Theorems 4.4 and 4.5. For $k>3$, use similiar definitions for $R$ and $\bar{R}$ ignoring what happens after the first three blocks of the partition.

Theorem 4.7. For every recursive ordinal $\alpha$, and every $k \geq 3$, there is a computable Borel code for a $\Delta_{\alpha+1}^{0}$ set $R \subseteq(\omega)^{k}$ such that every $p \in(\omega)^{\omega}$ homogeneous for the coloring $(\omega)^{k}=R \cup \bar{R}$ computes $\emptyset_{(\alpha)}$.

Proof. Let $R, \bar{R}$ be the $\emptyset_{(\alpha)}$-computable clopen sets from the previous corollary. By Proposition 4.2, both $R$ and $\bar{R}$ have computable Borel codes as $\Sigma_{\alpha+1}^{0}$ subsets of $(\omega)^{k}$. Therefore, $R$ has a computable Borel code as $\Delta_{\alpha+1}^{0}$ set. By the previous corollary, if $p$ is homogeneous for $(\omega)^{k}=R \cup \bar{R}$, then $p \geq_{T} \emptyset_{(\alpha)}$, as required.

For $\alpha=1$, Theorem 4.7 says there is a $\Delta_{2}^{0}$ clopen set $R \subseteq(\omega)^{3}$ such that $R$ and $\bar{R}$ have computable Borel codes as $\Sigma_{2}^{0}$ sets (and hence as $\Delta_{2}^{0}$ sets) and any homogeneous partition for $(\omega)^{3}=R \cup \bar{R}$ computes $\emptyset^{\prime}$.

## 5. The Borel Dual Ramsey Theorem for $k=2$

5.1. Effective Analysis. We consider the complexity of finding infinite homogeneous partitions for colorings $(\omega)^{2}=R \cup \bar{R}$ as a function of the descriptive complexity of $R$ and/or $\bar{R}$. We begin by showing that if $R$ is a computable open set, there is a computable homogeneous partition.

Theorem 5.1. Let $R$ be a computable code for an open set in $(\omega)^{2}$. There is a computable $p \in(\omega)^{\omega}$ such that $(p)^{2} \subseteq R$ or $(p)^{2} \subseteq \bar{R}$.

Proof. If there is an $n \geq 1$ such that $\left[0^{n}\right] \cap R=\emptyset$, then the partition $x \in(\omega)^{\omega}$ with blocks $\{0,1, \ldots, n\},\{n+1\},\{n+2\}, \ldots$ satisfies $(x)^{2} \subseteq \bar{R}$. Otherwise, for arbitrarily large $n$ there are $\tau \succ 0^{n} 1$ with $[\tau] \subseteq R$, and hence there is a computable sequence $\tau_{1}, \tau_{2}, \ldots$ of such $\tau$ with $0^{i} \prec \tau_{i}$. Computably thin this sequence so that for each $i, 0^{\left|\tau_{i}\right|} \prec \tau_{i+1}$. The partition $x$ with blocks $x^{-1}(i)=\left\{j: \tau_{i}(j)=1\right\}$ for $i>0$ satisfies $(x)^{2} \subseteq R$.

To extend to sets coded at higher finite levels of the Borel hierarchy, we will need the following generalization of the previous result.

Theorem 5.2. Let $R$ be a computable code for an open set in $(\omega)^{2}$ such that $R \cap\left[0^{n}\right] \neq \emptyset$ for all $n$. Let $\left\{D_{i}\right\}_{i<\omega}$ be a uniform sequence of computable codes for open sets such that each $D_{i}$ is dense in $R$. There is a computable $x \in(\omega)^{\omega}$ such that $(x)^{2} \subseteq R \cap\left(\cap_{i} D_{i}\right)$.
Proof. We build $x$ as the limit of an effective sequence $\tau_{0} \prec \tau_{1} \prec \cdots$ with $\tau_{s} \in(<$ $\omega)^{s+1}$. We define the strings $\tau_{s}$ in stages starting with $\tau_{0}=\langle 0\rangle$ which puts $x(0)=0$. For $s \geq 1$, we ensure that at the start of stage $s+1$, we have $\left[\sigma \circ \tau_{s}\right] \subseteq R$ for all $\sigma \in(s+1)^{2}$. That is, the open sets in $(\omega)^{2}$ determined by each way of coarsening the $s+1$ many blocks of $\tau_{s}$ to two blocks is contained in $R$.

At stage $s+1$, assume we have defined $\tau_{s} \in(<\omega)^{s+1}$. If $s \geq 1$, assume that for all $\sigma \in(s+1)^{2},\left[\sigma \circ \tau_{s}\right] \subseteq R$. Let $\sigma_{0}, \ldots, \sigma_{M_{s}-1}$ list the strings $\sigma \in(s+2)^{2}$. We define a sequence of strings $\tau_{s}^{0} \prec \cdots \prec \tau_{s}^{M_{s}}$ and set $\tau_{s+1}=\tau_{s}^{M_{s}}$.

We define $\tau_{s}^{0}$ to start a new block as follows. Since $\left[0^{\left|\tau_{s}\right|}\right] \cap R \neq \emptyset$, we effectively search for $\gamma_{s} \in(<\omega)^{2}$ such that $0^{\left|\tau_{s}\right|} \prec \gamma_{s}$ and $\left[\gamma_{s}\right] \subseteq R$. Since $\gamma_{s} \in(<\omega)^{2}$, there is at least one $m<\left|\gamma_{s}\right|$ such that $\gamma_{s}(m)=1$. Define $\tau_{s}^{0}$ with $\left|\tau_{s}^{0}\right|=\left|\gamma_{s}\right|$ by

$$
\tau_{s}^{0}(m)= \begin{cases}\tau_{s}(m) & \text { if } m<\left|\tau_{s}\right| \\ s+1 & \text { if } \gamma_{s}(m)=1\left(\text { and hence } m \geq\left|\tau_{s}\right|\right) \\ 0 & \text { if } m \geq\left|\tau_{s}\right| \text { and } \gamma_{s}(m)=0\end{cases}
$$

Note that $\tau_{s} \prec \tau_{s}^{0}$, and that $\left[\sigma \circ \tau_{s}^{0}\right] \subseteq R$ for all $\sigma \in(s+2)^{2}$. To see the latter, let $j$ be least such that $\sigma(j)=1$ and consider two cases. If $j<s+1$, then $\sigma \upharpoonright s+1 \in(s+1)^{2}$ and the conclusion follows by the induction hypothesis. If $j=s+1$, then $\sigma \circ \tau_{s}^{0}=\gamma_{s}$.

We continue to define the $\tau_{s}^{j}$ strings by induction. Assume that $\tau_{s}^{j}$ has been defined and consider the $j$-th string $\sigma_{j}$ enumerated above describing how to collapse $(s+2)$ many blocks into 2 blocks. Since $\tau_{s}^{0} \preceq \tau_{s}^{j}$, we have $\sigma_{j} \circ \tau_{s}^{0} \preceq \sigma_{j} \circ \tau_{s}^{j}$ and hence $\left[\sigma_{j} \circ \tau_{s}^{j}\right] \subseteq R$. Because $\cap_{n<s+1} D_{n}$ is dense in $R$, we can effectively search for a string $\delta_{s}^{j} \in(<\omega)^{2}$ such that $\sigma_{j} \circ \tau_{s}^{j} \preceq \delta_{s}^{j}$ and $\left[\delta_{s}^{j}\right] \subseteq \cap_{n<s+1} D_{n}$. To define $\tau_{s}^{j+1}$, we uncollapse $\delta_{s}^{j}$. Let $j^{*}$ be the least number such that $\sigma_{j}\left(j^{*}\right)=1$. Define

$$
\tau_{s}^{j+1}(m)= \begin{cases}\tau_{s}^{j}(m) & \text { if } m<\left|\tau_{s}^{j}\right| \\ j^{*} & \text { if } m \geq\left|\tau_{s}^{j}\right| \text { and } \delta_{s}^{j}(m)=1 \\ 0 & \text { if } m \geq\left|\tau_{s}^{j}\right| \text { and } \delta_{s}^{j}(m)=0\end{cases}
$$

It is straightforward to check that $\tau_{s}^{j} \preceq \tau_{s}^{j+1}$ and that $\sigma_{j} \circ \tau_{s}^{j+1}=\delta_{s}^{j}$. This completes the construction of the sequence $\tau_{s}^{0} \preceq \cdots \preceq \tau_{s}^{M_{s}}$ and of the computable partition $x$. It remains to show that if $p \in(x)^{2}$, then $p \in R$ and $p \in \cap_{n \in \omega} D_{n}$. Fix $p \in(x)^{2}$ and let $w \in(\omega)^{\omega}$ be such that $\omega \circ x=p$. Let $s_{0}$ be least such that $w\left(s_{0}+1\right)=1$.

Claim. $p \in R$.
Let $\sigma=\left(0^{s_{0}}\right)^{\wedge} 1$, so that $\sigma \prec w$. At stage $s_{0}+1$, we defined $\tau_{s_{0}}^{0} \prec x$ with the property that $\left[\sigma \circ \tau_{s_{0}}^{0}\right] \subseteq R$. Since $\sigma \circ \tau_{s}^{0} \prec p$, we have $p \in R$.

Claim. $p \in \cap_{n<\omega} D_{n}$.
Fix $k \in \omega$ and we show $p \in D_{k}$. Let $s=\max \left\{k, s_{0}\right\}$. Consider the action during stage $s+1$ of the construction. Let $\sigma=w \upharpoonright(s+2)$. Then $\sigma \in(s+2)^{2}$, so let $j$ be such that $\sigma_{j}=\sigma$. We defined $\delta_{s}^{j}$ and $\tau_{s}^{j+1}$ such that $\sigma_{j} \circ \tau_{s}^{j+1}=\delta_{s}^{j}$ and $\left[\delta_{s}^{j}\right] \subseteq \cap_{n<s+1} D_{n}$, so in particular, $\left[\delta_{j}^{s}\right] \subseteq D_{k}$. Since $\tau_{s}^{j+1} \prec x$, we have $\delta_{s}^{j}=\sigma_{j} \circ \tau_{s}^{j+1} \prec p$, so $p \in D_{k}$ as required.

The next proposition is standard, but we present the proof because some details will be relevant to Theorem 5.4. In the proof, we use codes for open sets as in Definition 3.1 .

Proposition 5.3. Let $n \in \omega$ and let $A \subseteq 2^{\omega}$ be defined by a $\Sigma_{n+1}^{0}$ predicate. There are a $\Delta_{n+1}^{0}$ code $U$ for an open set in $(\omega)^{2}$, a $\Delta_{n+2}^{0}$ code $V$ for an open set in $(\omega)^{2}$ and a uniformly $\Delta_{n+1}^{0}$ sequence $\left\langle D_{i}: i \in \omega\right\rangle$ of codes for dense open sets such that $U \cup V$ is dense and for all $p \in \cap_{i \in \omega} D_{i}$, if $p \in U$, then $p \in A$ and if $p \in V$ then $p \notin A$. Furthermore, the $\Delta_{n+1}^{0}$ and $\Delta_{n+2}^{0}$ indices for $U, V$ and $\left\langle D_{i}: i \in \omega\right\rangle$ can be obtained uniformly from a $\Sigma_{n+1}^{0}$ index for $A$.

Proof. We proceed by induction on $n$. Throughout this proof, $\sigma, \tau, \rho$ and $\delta$ denote elements of $(<\omega)^{2}$. In addition to the properties stated in the proposition, we ensure that if $\langle m, \sigma\rangle \in U$ (or $V$ ) and $\tau \succeq \sigma$, then there is a $k$ such that $\langle k, \tau\rangle \in U$ (or $V$ respectively). Thus, if $U \cap[\sigma] \neq \emptyset$, then there is $\langle k, \tau\rangle \in U$ with $\sigma \preceq \tau$.

For $n=0$, we have $X \in A \Leftrightarrow \exists k \exists m P(m, X \upharpoonright k)$ where $P(x, y)$ is a $\Pi_{0}^{0}$ predicate. Without loss of generality, we assume that if $P(m, X \upharpoonright k)$ holds, then $P\left(m^{\prime}, Y \upharpoonright k^{\prime}\right)$ holds for all $k^{\prime} \geq k, m^{\prime} \geq m$ and $Y \in 2^{\omega}$ such that $Y \upharpoonright k=X \upharpoonright k$. Let $U=\{\langle n, \sigma\rangle: P(\sigma, n)\}, V=\{\langle 0, \sigma\rangle: \forall x \forall \tau \succeq \sigma(\neg P(\tau, x))\}$ and $D_{i}=(<\omega)^{2}$ for $i \in \omega$. It is straightforward to check these codes have the required properties.

For the induction case, let $A \subseteq 2^{\omega}$ be defined by a $\Sigma_{n+2}^{0}$ predicate, so $X \in A \Leftrightarrow$ $\exists k P(X, k)$ where $P$ is a $\Pi_{n+1}^{0}$ predicate. For $k \in \omega$, let $A_{k}=\{X: \neg P(X, k)\}$. Apply the induction hypothesis to $A_{k}$ to fix indices (uniformly in $k$ ) for the $\Delta_{n+1}^{0}$ codes $U_{k}$ and $\left\langle D_{i, k}: i \in \omega\right\rangle$ and for the $\Delta_{n+2}^{0}$ code $V_{k}$ so that if $p \in \cap_{i \in \omega} D_{i, k}$, then $p \in U_{k}$ implies $\neg P(k, p)$ and $p \in V_{k}$ implies $P(k, p)$. Let

$$
\begin{gathered}
U=\left\{\langle\langle k, m\rangle, \sigma\rangle:\langle m, \sigma\rangle \in V_{k}\right\} \text { and } \\
V=\left\{\langle 0, \sigma\rangle: \forall k \forall \tau \succeq \sigma \exists m \exists \rho \succeq \tau\langle m, \rho\rangle \in U_{k}\right\} .
\end{gathered}
$$

$U$ is a $\Delta_{n+2}^{0}$ code for $\cup_{k} V_{k}$, and $V$ is a $\Delta_{n+3}^{0}$ code such that $\langle m, \sigma\rangle \in V$ if and only if every $U_{k}$ is dense in $[\sigma]$. We claim that $U \cup V$ is dense. Fix $\sigma$ and assume $U \cap[\sigma]=\emptyset$, so $V_{k} \cap[\sigma]=\emptyset$ for all $k$. Since $U_{k} \cup V_{k}$ is dense, $U_{k} \cap[\tau] \neq \emptyset$ for all $\tau \succeq \sigma$ and all $k$, so $\langle 0, \sigma\rangle \in V$.

For $i=\left\langle a_{i}, b_{i}\right\rangle$, define $D_{i}=D_{a_{i}, b_{i}} \cap\left(U_{i} \cup V_{i}\right) . D_{i}$ has a $\Delta_{n+2}^{0}$ code as a dense open set and the index can be uniformly computed from the indices for $U_{i}, V_{i}$ and $D_{a_{i}, b_{i}}$. Furthermore, if $p \in \cap_{i} D_{i}$ then $p \in \cap_{i, k} D_{i, k}$ and $p \in \cap_{k}\left(U_{k} \cup V_{k}\right)$.

Assume that $p \in \cap_{i} D_{i}$. First, we show that if $p \in U$, then $p \in A$. Suppose $p \in U=\cup_{k} V_{k}$ and fix $k$ such that $p \in V_{k}$. Since $p \in \cap_{i} D_{i, k}$ for this fixed $k, p \notin A_{k}$ by the induction hypothesis. Therefore, $P(k, p)$ holds and hence $p \in A$.

Second, we show that if $p \in V$ then $p \notin A$. Assume $p \in V$ and fix $\langle 0, \sigma\rangle \in V$ such that $\sigma \prec p$. It suffices to show $\neg P(k, p)$ holds for an arbitrary $k \in \omega$. Since $p \in \cap_{i} D_{i}$, we have $p \in U_{k} \cup V_{k}$ and $p \in \cap_{i} D_{i, k}$. If $p \in U_{k}$, then $\neg P(k, p)$ holds by induction and we are done. Therefore, suppose for a contradiction that $p \in V_{k}$. Fix $\langle 0, \tau\rangle \in V_{k}$ such that $\sigma \preceq \tau$ and $\tau \prec p$. Since $\langle 0, \sigma\rangle \in V$ and $\sigma \preceq \tau$, there are $\rho \succeq \tau$ and $m$ such that $\langle m, \rho\rangle \in U_{k}$, and therefore $[\rho] \subseteq U_{k} \cap V_{k}$. This containment is the desired contradiction because $q \in[\rho] \cap \cap_{i} D_{i, k}$ would satisfy $q \in A_{k}$ and $q \notin A_{k}$.

Theorem 5.4. For every coloring $(\omega)^{2}=R \cup \bar{R}$ such that $R$ is a computable code for a $\Sigma_{n+2}^{0}$ set, there is either a $\emptyset^{(n)}$-computable $x \in(\omega)^{\omega}$ which is homogeneous for $\bar{R}$ or a $\emptyset^{(n+1)}$-computable $x \in(\omega)^{\omega}$ which is homogeneous for $R$.

Proof. Fix $R$ and fix a $\Pi_{n+1}^{0}$ predicate $P(k, y)$ such that for $y \in(\omega)^{2}, y \in R \Leftrightarrow$ $\exists k P(k, y)$. Let $U_{k}, V_{k}$ and $\left\langle D_{i, k}: i \in \omega\right\rangle$ be the codes from Proposition 5.3 for $R_{k}=\{y: \neg P(y, k)\}$. Let $U=\cup_{k} V_{k}, V=\cup\left\{[\sigma]: \forall k U_{k}\right.$ is dense in $\left.[\sigma]\right\}$ and $D_{i}$, $i \in \omega$, be the corresponding codes for $R$. We split non-uniformly into cases.

Case 1: Assume $V$ is dense in $\left[0^{\ell}\right]$ for some fixed $\ell$. We make two observations. First, $U$ is disjoint from $\left[0^{\ell}\right]$. Therefore, each $V_{k}$ is disjoint from $\left[0^{\ell}\right]$ and hence each $U_{k}$ is dense in $\left[0^{\ell}\right]$. Second, suppose $y \in\left(\bigcap_{i, k} D_{i, k}\right) \cap\left(\bigcap_{k} U_{k}\right)$. For each $k$ we have $y \in \cap_{i} D_{i, k}$ and $y \in U_{k}$, so $\forall k \neg P(k, y)$ holds and hence $y \in \bar{R}$.

We apply Theorem 5.2 relativized to $\emptyset^{(n)}$ to the computable open set $O=\left[0^{\ell}\right]$ (which has nonempty intersection with $\left[0^{j}\right]$ for every $j$ ) and the $\emptyset^{(n)}$-computable sequence of codes $D_{i, k}$ and $U_{k}$ for $i, k<\omega$. By the first observation, each coded set in this sequence is dense in $O$. Therefore, there is a $\emptyset^{(n)}$-computable $x \in(\omega)^{\omega}$ such that $(x)^{2} \subseteq\left[0^{\ell}\right] \cap\left(\bigcap_{i, k} D_{i, k}\right) \cap\left(\bigcap_{k} U_{k}\right)$. By the second observation, $(x)^{2} \subseteq \bar{R}$ as required.

Case 2: Assume $V$ is not dense in $\left[0^{m}\right]$ for any $m$. In this case, since $U \cup V$ is dense, we have $U \cap\left[0^{m}\right] \neq \emptyset$ for all $m$. We apply Theorem 5.2 relativized to $\emptyset^{(n+1)}$ to the $\emptyset^{(n+1)}$-computable open set $U$ and the $\emptyset^{(n+1)}$-computable sequence of dense sets $D_{i}$ for $i \in \omega$ to obtain an $\emptyset^{(n+1)}$-computable $x$ with $(x)^{2} \subseteq U \cap\left(\bigcap_{i} D_{i}\right) \subseteq R$ as required.

We end this section by showing that the non-uniformity in the proof of Theorem 5.1 is necessary.

Theorem 5.5. For every Turing functional $\Delta$, there are computable codes $R_{0}$ and $R_{1}$ for complementary open sets in $(\omega)^{2}$ such that $\Delta^{R_{0} \oplus R_{1}}$ is not an infinite homogeneous partition for the reduced coloring $(\omega)^{2}=R_{0} \cup R_{1}$.

Proof. Fix $\Delta$. We define $R_{0}$ and $R_{1}$ in stages as $R_{0, s}$ and $R_{1, s}$. Our construction proceeds in a basic module while we wait for $\Delta_{s}^{R_{0, s} \oplus R_{1, s}}$ to provide appropriate computations. If these computations appear, we immediately diagonalize and complete the construction.

For the basic module at stage $s$, put $0^{2 s+1} 1 \in R_{0, s}$ and $0^{2 s+2} 1 \in R_{1, s}$. Check whether there is a $0<k<s$ such that $\Delta_{s}^{R_{0, s} \oplus R_{1, s}}(i)=0$ for all $i<k$ and $\Delta_{s}^{R_{0, s} \oplus R_{1, s}}(k)=1$. If there is no such $k$, then we proceed to stage $s+1$ and continue with the basic module.

If there is such a $k$, then we stop the basic module and fix $i<2$ such that $0^{k} 1 \in R_{i, s}$. (Since $k<s$, we have already enumerated $0^{k} 1$ into one of $R_{0, s}$ or $R_{1, s}$
depending on whether $k$ is even or odd.) We end the construction at this stage and define $R_{i}=R_{i, s}$ and $R_{1-i}=R_{1-i, s} \cup\left\{0^{t} 1 \mid 2 s+2<t\right\}$.

This completes the construction. It is clear that $R_{0}$ and $R_{1}$ are computable codes for complementary open sets and $(\omega)^{2}=R_{0} \cup R_{1}$ is a reduced coloring. If the construction never finds an appropriate value $k$, then $\Delta^{R_{0} \oplus R_{1}}$ is not an element of $(\omega)^{\omega}$ and we are done. Therefore, assume we find an appropriate value $k$ at stage $s$ in the construction. Fix $i$ such that $0^{k} 1 \in R_{i, s}$ and assume that $p=\Delta^{R_{0} \oplus R_{1}}$ is a element of $(\omega)^{\omega}$. We show $p$ is not homogeneous by giving elements $q_{0}, q_{1} \in(p)^{2}$ such that $q_{0} \in R_{i}$ and $q_{1} \in R_{1-i}$.

By construction, $0^{k} 1 \prec p$. Let $q_{0} \in(p)^{2}$ be any coarsening with $0^{k} 1 \prec q_{0}$ Then $q_{0} \in R_{i}$ because $\left[0^{k} 1\right] \subseteq R_{i}$.

On the other hand, since $p \in(\omega)^{\omega}$, there are infinitely many $p$-blocks. Let $n$ be least with $\mu^{p}(n)>2 s+2$. Let $q_{1} \in(p)^{2}$ be any coarsening for which $q_{1} \in\left[0^{\mu^{p}(n)} 1\right]$. Since $\mu^{p}(n)>2 s+2$, we put $0^{\mu^{p}(n)} 1 \in R_{1-i}$, so $q_{1} \in R_{1-i}$ as required.
5.2. Strong reductions for reduced colorings. In this section, we think of Borel-DRT $2_{2}^{2}$ as an instance-solution problem. Such a problem consists of a collection of subsets of $\omega$ called the instances of this problem, and for each instance, a collection of subsets of $\omega$ called the solutions to this instance (for this problem). A problem P is strongly Weihrauch reducible to a problem Q if there are fixed Turing functionals $\Phi$ and $\Psi$ such that given any instance $A$ of $\mathrm{P}, \Phi^{A}$ is an instance of Q , and given any solution $B$ to $\Phi^{A}$ in $\mathrm{Q}, \Psi^{B}$ is a solution to $A$ in P . There are a number of variations on this reducibility and we refer to the reader to [6] and 9 for background on these reductions and for connections to reverse mathematics. In this paper, we will only be interested in problems arising out of $\Pi_{2}^{1}$ statements of second order arithmetic. Any such statement can be put in the form $\forall X(\varphi(X) \rightarrow \exists Y \psi(X, Y))$, where $\phi$ and $\psi$ are arithmetical. We can then regard this as a problem, with instances being all $X$ such that $\varphi(X)$, and the solutions to $X$ being all $Y$ such that $\psi(X, Y)$. Note that while the choice of $\varphi$ and $\psi$ is not unique, we always have a fixed such choice in mind for a given $\Pi_{2}^{1}$ statement, and so also a fixed assignment of instances and solutions.

A reduced coloring $(\omega)^{2}=R_{0} \cup R_{1}$ is classically open and the color of $p \in$ $(\omega)^{2}$ depends only on $\mu^{p}(1)$. When $R_{0}$ and $R_{1}$ are codes for open sets, there is a homogeneous partition computable in $R_{0} \oplus R_{1}$, although by Theorem 5.5, not uniformly. We consider the case when the open sets $R_{0}$ and $R_{1}$ are represented by Borel codes for $\Sigma_{n}^{0}$ sets with $n \geq 2$.
$\Delta_{n}^{0}-\mathrm{rDRT}{ }_{2}^{2}$ is the statement that for each reduced coloring $(\omega)^{2}=R_{0} \cup R_{1}$ where $R_{0}$ and $R_{1}$ are Borel codes for $\Sigma_{n}^{0}$ sets, there exists an $x \in(\omega)^{\omega}$ and an $i<2$ such that $(x)^{2} \subseteq R_{i}$. In effective algebra, this statement is clear, but in $\mathrm{RCA}_{0}$, we need to specify how to handle these codes.

Recall that a Borel code for a $\boldsymbol{\Sigma}_{n}^{0}$ set is a labeled subtree of $\omega^{<n+1}$ which we write as $(B, \varphi)$ to specify the labeling function $\varphi$. For a leaf $\sigma$ and a partition $p$, we write $p \in \varphi(\sigma)$ if $p$ is an element of the clopen set coded by $\varphi(\sigma)$, and we write $\varphi(\sigma)=[\tau]$ to avoid specifying a coding scheme.

In reverse mathematics there are two ways that membership in a $\boldsymbol{\Sigma}_{\alpha}^{0}$ set could be discussed. The evaluation map method works for arbitrary $\alpha$ and requires a strong base theory. This method will be discussed in the next section. The virtual method works only for finite $\alpha$. For each $n<\omega$, there is a fixed $\Sigma_{n}^{0}$ formula $\eta(B, \varphi, p)$ such
that if $(B, \varphi)$ is a Borel code for a $\boldsymbol{\Sigma}_{n}^{0}$ set and $p \in(\omega)^{2}$, then $\eta(B, \varphi, p)$ says $p$ is in the set coded by $(B, \varphi)$. In this section we use only the virtual method.

The formula is defined as follows. We begin by defining formulas $\beta_{k}(\sigma, B, \varphi, p)$ for $1 \leq k \leq n$ by downward induction on $k$. For $\sigma \in B$ with $|\sigma|=k, \beta_{k}(\sigma, B, \varphi, p)$ says that $p$ is in the set coded by the labeled subtree of $(B, \varphi)$ above $\sigma$. Since any $\sigma \in B$ with $|\sigma|=n$ is a leaf, $\beta_{n}(\sigma, B, \varphi, p)$ is the formula $p \in \varphi(\sigma)$. For $1 \leq k<n$, $\beta_{k}(\sigma, B, \varphi, p)$ is the formula

$$
\begin{gathered}
\left(\varphi(\sigma)=\cup \rightarrow \alpha_{k}^{\cup}\right) \wedge\left(\varphi(\sigma)=\cap \rightarrow \alpha_{k}^{\cap}\right) \wedge\left(\varphi(\sigma) \in L \rightarrow \alpha_{k}^{L}\right), \text { where } \\
\alpha_{k}^{\cup}(\sigma, B, \varphi, p) \text { is } \exists \tau \in B\left(\sigma \prec \tau \wedge|\tau|=k+1 \wedge \beta_{k+1}(\tau, B, \varphi, p)\right) \\
\alpha_{k}^{\cap}(\sigma, B, \varphi, p) \text { is } \forall \tau \in B\left((\sigma \prec \tau \wedge|\tau|=k+1) \rightarrow \beta_{k+1}(\tau, B, \varphi, p)\right) \\
\text { and } \alpha_{k}^{L}(\sigma, B, \varphi, p) \text { is } p \in \varphi(\sigma) .
\end{gathered}
$$

The formula $\eta(B, \varphi, p)$ is $\exists \sigma \in B\left(|\sigma|=1 \wedge \beta_{1}(\sigma, B, \varphi, p)\right)$. In $\mathrm{RCA}_{0}$, we write $p \in B$ for $\eta(B, \varphi, p)$. The statement $\Delta_{n}^{0}-\mathrm{rDRT}{ }_{2}^{2}$ now has the obvious translation in $\mathrm{RCA}_{0}$.

A Borel code $(B, \varphi)$ for a $\boldsymbol{\Sigma}_{n}^{0}$ set is in normal form if $B=\omega^{<n+1}$ and for every $\sigma$ with $|\sigma|<n$, if $|\sigma|$ is even, then $\varphi(\sigma)=\cup$, and if $|\sigma|$ is odd, then $\varphi(\sigma)=\cap$. In $\mathrm{RCA}_{0}$, for every $(B, \varphi)$, there is a $(\widehat{B}, \widehat{\varphi})$ in normal form such that for all $p \in(\omega)^{2}$, $p \in B$ if and only if $p \in \widehat{B}$. Moreover, the transformation from $(B, \varphi)$ to $(\widehat{B}, \widehat{\varphi})$ is uniformly computable in $(B, \varphi)$. We describe the transformation when $(B, \varphi)$ is a Borel code for a $\Sigma_{2}^{0}$ set. The case for a $\Sigma_{n}^{0}$ set is similar.

Let $(B, \varphi)$ be a Borel code for a $\Sigma_{2}^{0}$ set. By definition, $\lambda \in B$ with $\varphi(\lambda)=\cup$. Each $\sigma \in B$ with $|\sigma|=1$ is the root of a subtree coding a $\Sigma_{0}^{0}$ set (if $\varphi(\sigma) \in L$ ), a $\Sigma_{1}^{0}$ set (if $\varphi(\sigma)=\cup$ ) or a $\Pi_{1}^{0}$ set (if $\varphi(\sigma)=\cap$ ). Consider the following sequence of transformations.

- To form $\left(B_{1}, \varphi_{1}\right)$, for each $\sigma \in B$ with $|\sigma|=1$ and $\varphi(\sigma)=\cup$, remove the subtree of $B$ above $\sigma$ (including $\sigma$ ). For each $\tau \in B$ with $\tau \succ \sigma$, add a new node $\tau^{\prime}$ to $B_{1}$ with $\left|\tau^{\prime}\right|=1$ and $\varphi_{1}\left(\tau^{\prime}\right)=\varphi(\tau) \in L$.
- To form $\left(B_{2}, \varphi_{2}\right)$, for each leaf $\sigma \in B_{1}$ with $|\sigma|=1$, relabel $\sigma$ by $\varphi_{2}(\sigma)=\cap$ and add a new successor $\tau$ to $\sigma$ with label $\varphi_{2}(\tau)=\varphi_{1}(\sigma)$.
- To form $\left(B_{3}, \varphi_{3}\right)$, for each $\sigma \in B_{2}$ with $|\sigma|=1$, let $\tau_{\sigma} \in B_{1}$ be the first successor of $\sigma$. Add infinite many new nodes $\delta \succ \sigma$ to $B_{3}$ with $\varphi_{3}(\delta)=$ $\varphi_{2}\left(\tau_{\sigma}\right)$.
- To form $\left(B_{4}, \varphi_{4}\right)$, let $\sigma$ be the first node of $B_{3}$ at level 1. Add infinitely many copies of the subtree above $\sigma$ to $B_{4}$ with the same labels as in $B_{3}$.
In $\left(B_{4}, \varphi_{4}\right)$, the leaves are at level 2 , every interior node is infinitely branching and $\varphi_{4}(\sigma)=\cap$ when $|\sigma|=1$. There is a uniform procedure to define a bijection $f: B_{4} \rightarrow \omega^{<3}$. We define $(\widehat{B}, \widehat{\varphi})$ by $\widehat{B}=\omega^{<3}$ and $\widehat{\varphi}(\sigma)=\varphi_{4}\left(f^{-1}(\sigma)\right)$. In $\mathrm{RCA}_{0}$, for all $p \in(\omega)^{2}, \eta(B, \varphi, p)$ holds if and only if $\eta(\widehat{B}, \widehat{\varphi}, p)$ holds.

When $(B, \varphi)$ is a Borel code for a $\Sigma_{n}^{0}$ set in normal form, $\eta(B, \varphi, p)$ is equivalent to $\exists x_{0} \forall x_{1} \cdots \mathrm{Q}_{n-1} x_{n-1}\left(p \in \varphi\left(\left\langle x_{0}, x_{1}, \ldots, x_{n-1}\right\rangle\right)\right)$ where $\mathrm{Q}_{n-1}$ is $\forall$ or $\exists$ depending on whether $n-1$ is odd or even. We have analogous definitions for Borel codes for $\Pi_{n}^{0}$ sets in normal form.

To define $D_{2}^{n}$, let $[\omega]^{n}$ denote the set of $n$ element subsets of $\omega$. We view the elements of $[\omega]^{n}$ as strictly increasing sequences $s_{0}<s_{1}<\cdots<s_{n-1}$.
Definition 5.6. A coloring $c:[\omega]^{n} \rightarrow 2$ is stable if for all $k$, the limit

$$
\lim _{s_{1}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, \ldots, s_{n-1}\right)
$$

exists. $L \subseteq \omega$ is limit-homogeneous for a stable coloring $c$ if there is an $i<2$ such that for each $k \in L$,

$$
\lim _{s_{1}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, \ldots, s_{n-1}\right)=i
$$

$\mathrm{D}_{2}^{n}$ is the statement that each stable coloring $c:[\omega]^{n} \rightarrow 2$ has an infinite limithomogeneous set.

Below, the proof of Theorem 5.7(2) is a formalization of the proof of Theorem 5.7(1), and the additional induction used is a consequence of this formalization. We do not know if its use is necessary; that is, we do now if $\mathrm{RCA}_{0}+I \Sigma_{n-1}^{0}$ can be replaced simply by $\mathrm{RCA}_{0}$ when $n>2$.
Theorem 5.7. Fix $n \geq 2$.
(1) $\Delta_{n}^{0}-\mathrm{rDRT}{ }_{2}^{2} \equiv_{\mathrm{sW}} \mathrm{D}_{2}^{n}$.
(2) Over $\mathrm{RCA}_{0}+\mathrm{I} \Sigma_{n-1}^{0}, \Delta_{n}^{0}-\mathrm{rDRT}{ }_{2}^{2}$ is equivalent to $\mathrm{D}_{2}^{n}$.

Corollary 5.8. $\Delta_{2}^{0}-\mathrm{rDRT} T_{2}^{2}$ is equivalent to $\mathrm{SRT}_{2}^{2}$ over $\mathrm{RCA}_{0}$.
Proof. $\mathrm{D}_{2}^{2}$ is equivalent to $\mathrm{SRT}_{2}^{2}$ over $\mathrm{RCA}_{0}$ by Chong, Lempp, and Yang [5].
Corollary 5.9. $\Delta_{2}^{0}-\mathrm{rDRT}{ }_{2}^{2}<_{\mathrm{sW}} \mathrm{SRT}_{2}^{2}$.
Proof. $\mathrm{D}_{2}^{2}<_{\mathrm{sW}} \mathrm{SRT}_{2}^{2}$ by Dzhafarov [6, Corollary 3.3]. (It also follows immediately that $\Delta_{2}^{0}-\mathrm{rDRT}{ }_{2}^{2} \equiv \mathrm{~W}_{\mathrm{W}} \mathrm{D}_{2}^{2}<\mathrm{W} \mathrm{SRT}_{2}^{2}$.)

Proof of Theorem 5.7. We prove the two parts simultaneously, remarking, where needed, how to formalize the argument in $\mathrm{RCA}_{0}+\mathrm{I} \Sigma_{n-1}^{0}$.

To show that $\Delta_{n}^{0}-\mathrm{rDRT} T_{2}^{2} \leq_{\mathrm{sW}} \mathrm{D}_{2}^{n}$, and that $\Delta_{n}^{0}-\mathrm{rDRT}_{2}^{2}$ is implied by $\mathrm{D}_{2}^{n}$ over $\mathrm{RCA}_{0}+I \Sigma_{n-1}^{0}$, fix an instance $(\omega)^{2}=R_{0} \cup R_{1}$ of $\Delta_{n}^{0}-\mathrm{rDRT}{ }_{2}^{2}$ where each $R_{i}$ is a Borel code for a $\Sigma_{n}^{0}$ set. Without loss of generality, $R_{0}$ and $R_{1}$ are in normal form. For each $k \geq 1$, fix the partition $p_{k}=\chi_{\{k\}}$ (that is, $p_{k}$ has blocks $\omega \backslash\{k\}$ and $\{k\}$ ).

For $m<n$, we let $R_{i}\left(t_{0}, \ldots, t_{m}\right)$ denote the Borel set coded by the subtree of $R_{i}$ above $\left\langle t_{0}, \ldots, t_{m}\right\rangle$. Since $\left\langle t_{0}, \ldots, t_{n-1}\right\rangle$ is a leaf, $R_{i}\left(t_{0}, \ldots, t_{n-1}\right)$ is the clopen set $\varphi_{i}\left(\left\langle t_{0}, \ldots, t_{n-1}\right\rangle\right)$. If $m<n-1$, then $R_{i}\left(t_{0}, \ldots, t_{m}\right)$ is a code for a $\Sigma_{n-(m+1)}^{0}$ set (if $m$ is odd) or a $\Pi_{n-(m+1)}^{0}$ set (if $m$ is even) in normal form.

We define a coloring $c:[\omega]^{n} \rightarrow 2$ as follows. Let $c\left(0, s_{1}, \ldots, s_{n-1}\right)=0$ for all $s_{1}<\cdots<s_{n-1}$. For $m \leq n$, let $\mathrm{Q}_{m}$ stand for $\exists$ or $\forall$, depending as $m$ is even or odd, respectively. Given $1 \leq k<s_{1}<\ldots<s_{n-1}$, define

$$
c\left(k, s_{1}, \ldots, s_{n-1}\right)=1
$$

if and only if there is a $t_{0} \leq s_{1}$ such that

$$
\left(\forall t_{1} \leq s_{1}\right) \cdots\left(\mathrm{Q}_{m} t_{m} \leq s_{m}\right) \cdots\left(\mathrm{Q}_{n-1} t_{n-1} \leq s_{n-1}\right) p_{k} \in \varphi_{0}\left(\left\langle t_{0}, \ldots, t_{n-1}\right\rangle\right)
$$

and for which there is no $u_{0}<t_{0}$ such that

$$
\left(\forall u_{1} \leq s_{1}\right) \cdots\left(\mathrm{Q}_{m} u_{m} \leq s_{m}\right) \cdots\left(\mathrm{Q}_{n-1} u_{n-1} \leq s_{n-1}\right) p_{k} \in \varphi_{1}\left(\left\langle u_{0}, \ldots u_{n-1}\right\rangle\right)
$$

(Note that $s_{1}$ bounds $t_{0}, t_{1}$ and $u_{1}$, whereas the other $s_{m}$ bound only $t_{m}$ and $u_{m}$. ) The coloring $c$ is uniformly computable in $\left(R_{0}, \varphi_{0}\right)$ and $\left(R_{1}, \varphi_{1}\right)$ and is definable in RCA $A_{0}$ as a total function since all the quantification is bounded.

We claim that for each $k \geq 1$,

$$
\lim _{s_{1}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, \ldots, s_{n-1}\right)
$$

exists. Furthermore, if this limit equals 1 , then $p_{k} \in R_{0}$, and if this limit equals 0 , then $p_{k} \in R_{1}$. We break this claim into two halves.

First, for $1 \leq m \leq n-1$, we claim that for all fixed $1 \leq k<s_{1}<\ldots<s_{m}$,

$$
\lim _{s_{m+1}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, \ldots, s_{m}, s_{m+1}, \ldots, s_{n-1}\right)
$$

exists, and the limit equals 1 if and only if there is a $t_{0} \leq s_{1}$ such that

$$
\begin{equation*}
\left(\forall t_{1} \leq s_{1}\right) \cdots\left(\mathrm{Q}_{m} t_{m} \leq s_{m}\right) p_{k} \in R_{0}\left(t_{0}, \ldots, t_{m}\right) \tag{2}
\end{equation*}
$$

and there is no $u_{0}<t_{0}$ such that

$$
\begin{equation*}
\left(\forall u_{1} \leq s_{1}\right) \cdots\left(\mathrm{Q}_{m} u_{m} \leq s_{m}\right) p_{k} \in R_{1}\left(u_{0}, \ldots, u_{m}\right) \tag{3}
\end{equation*}
$$

The proof is by downward induction on $m$. (In $\mathrm{RCA}_{0}$, the induction is performed externally, so we do not need to consider its complexity.) For $m=n-1$, there are no limits involved and the values of $c$ are correct by definition.

Assume the result is true for $m+1$ and we show it remains true for $m$. By the definition of $R_{0}\left(t_{0}, \ldots, t_{m}\right)$, $t_{0}$ satisfies (2) if and only if

$$
\left(\forall t_{1} \leq s_{1}\right) \cdots\left(\mathrm{Q}_{m} t_{m} \leq s_{m}\right)\left(\mathrm{Q}_{m+1} t_{m+1}\right) p_{k} \in R_{0}\left(t_{0}, \ldots, t_{m}, t_{m+1}\right)
$$

which in turn holds if and only if there is a bound $v$ such that for all $s_{m+1} \geq v$,

$$
\left(\forall t_{1} \leq s_{1}\right) \cdots\left(\mathrm{Q}_{m} t_{m} \leq s_{m}\right)\left(\mathrm{Q}_{m+1} t_{m+1} \leq s_{m+1}\right) p_{k} \in R_{0}\left(t_{0}, \ldots, t_{m}, t_{m+1}\right)
$$

If $\mathrm{Q}_{m+1}$ is $\exists$, then over $\mathrm{RCA}_{0}$, this equivalence requires a bounding principle. Since $p_{k} \in R_{0}\left(t_{0}, \ldots, t_{m+1}\right)$ is a $\Pi_{n-(m+2)}^{0}$ predicate and $m+2 \geq 3$, we need at most $\mathrm{B} \Pi_{n-3}^{0}$ which follows from $I \Sigma_{n-1}^{0}$. An analogous analysis applies to numbers $u_{0}$ satisfying (3). Thus, we can fix a common bound $v$ that works for all $t_{0} \leq s_{1}$ in (2) and all $u_{0}<t_{0} \leq s_{1}$ in (3).

Suppose there is a $t_{0} \leq s_{1}$ satisfying (2) for which there is no $u_{0}<t_{0}$ satisfying (3). Then, for all $s_{m+1} \geq v, t_{0}$ satisfies the version of (2) for $m+1$, and there is no $u_{0}<t_{0}$ satisfying the version of (3) for $m+1$. Therefore, by induction

$$
\exists v \forall s_{m+1} \geq v\left(\lim _{s_{m+2}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, \ldots, s_{n-1}\right)=1\right)
$$

and hence $\lim _{s_{m+1}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, \ldots, s_{n-1}\right)=1$ as required.
On the other hand, suppose that there is no $t_{0} \leq s_{1}$ satisfying (2), or that for every $t_{0} \leq s_{1}$ satisfying (2), there is a $u_{0}<t_{0}$ satsifying (3). Then, for all $s_{m+1} \geq v$, we have the analogous condition for $m+1$ and the induction hypothesis gives $\lim _{s_{m+1}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, \ldots, s_{n-1}\right)=0$. This completes the first part of the claim.

We can now prove the rest of the claim. For each $k \geq 1$, we have $p_{k} \in R_{0}$ or $p_{k} \in R_{1}$. Let $t_{0}$ be least such that $p_{k} \in R_{0}\left(t_{0}\right)$ or $p_{k} \in R_{1}\left(t_{0}\right)$. Since $p_{k} \in R_{i}(t)$ is a $\Pi_{n-1}^{0}$ statement, we use $I \Sigma_{n-1}^{0}$ to fix this value in $\mathrm{RCA}_{0}$.

Suppose $p_{k} \in R_{0}\left(t_{0}\right)$, so for all $u_{0}<t_{0}$, it is not the case that $p_{k} \in R_{1}\left(u_{0}\right)$. By the first half of the claim with $m=1$, we have for every $s_{1} \geq t_{0}$

$$
\lim _{s_{2}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, s_{2}, \ldots, s_{n-1}\right)=1
$$

and therefore $\lim _{s_{1}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, \ldots, s_{n-1}\right)=1$.
Suppose $p_{k} \notin R_{0}\left(t_{0}\right)$, and hence $p_{k} \in R_{1}\left(t_{0}\right)$. Again, by the first half of the claim with $m=1$, we have for every $s_{1} \geq t_{0}$

$$
\lim _{s_{2}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, s_{2}, \ldots, s_{n-1}\right)=0
$$

so $\lim _{s_{1}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, \ldots, s_{n-1}\right)=0$. This completes the proof of the claim.
Since $c$ is an instance of $\mathrm{D}_{2}^{n}$, fix $i<2$ and an infinite limit-homogeneous set $L$ for $c$ with color $i$. By the claim, $p_{k} \in R_{1-i}$ for all $k \in L$. List the non-zero elements of $L$ as $k_{0}<k_{1}<\cdots$, and let $p \in(\omega)^{\omega}$ be the partition whose blocks are $\left[0, k_{0}\right)$ and $\left[k_{m}, k_{m+1}\right)$ for $m \in \omega$. Each $x \in(p)^{2}$ satisfies $\mu^{x}(1)=k_{m}$ for some $m$. Since $R_{0} \cup R_{1}$ is a reduced coloring, $x$ and $p_{k_{m}}$ have the same color, which is $R_{1-i}$. Since $x$ was arbitrary, $(p)^{2} \subseteq R_{1-i}$ as required to complete this half of the theorem.

Next, we show that $\mathrm{D}_{2}^{n} \leq_{\mathrm{sW}} \Delta_{n}^{0}-\mathrm{rDRT}{ }_{2}^{2}$, and that $\mathrm{D}_{2}^{n}$ is implied by $\Delta_{n}^{0}-\mathrm{rDRT} T_{2}^{2}$ over $\mathrm{RCA}_{0}$. (No extra induction is necessary for this implication.) Fix an instance $c:[\omega]^{n} \rightarrow 2$ of $\mathrm{D}_{2}^{n}$, and define a partition $R_{0} \cup R_{1}$ of $(\omega)^{2}$ as follows. For $x \in(\omega)^{2}$ with $\mu^{x}(1)=k, x \in R_{i}$ for the unique $i$ such that

$$
\lim _{s_{1}} \cdots \lim _{s_{n-1}} c\left(k, s_{1}, \ldots, s_{n-1}\right)=i
$$

Since each of the iterated limits is assumed to exist over what follows on the right, we may express these limits by alternating $\Sigma_{2}^{0}$ and $\Pi_{2}^{0}$ definitions, as

$$
\left(\exists t_{1} \forall s_{1} \geq t_{1}\right)\left(\forall t_{2} \geq s_{1} \exists s_{2} \geq t_{2}\right) \cdots c\left(k, s_{1}, \ldots, s_{n-1}\right)=i
$$

Thus, $R_{0}$ and $R_{1}$ are $\Sigma_{n}^{0}$-definable open subsets of $(\omega)^{2}$. By standard techniques, there are Borel codes for $R_{0}$ and $R_{1}$ as $\Sigma_{n}^{0}$ sets uniformly computable in $c$ and in $\mathrm{RCA}_{0}$. (Below, we illustrate this process for $\mathrm{D}_{2}^{3}$.)

By definition, $(\omega)^{2}=R_{0} \cup R_{1}$ is a reduced coloring and hence is an instance of $\Delta_{n}^{0}$ $\mathrm{rDRT}_{2}^{2}$. Let $p \in(\omega)^{\omega}$ be a solution to this instance, say with color $i<2$. Thus, for every $x \in(p)^{2}$, the limit color of $k=\mu^{x}(1)$ is $i$. Define $L=\left\{\mu^{p}(m): m \geq 1\right\}$. Since for each $k \in L$, there is an $x \in(p)^{2}$ such that $\mu^{x}(1)=k, L$ is limit-homogeneous for $c$ with color $i$.

We end this proof by illustrating how to define the Borel codes for $R_{0}$ and $R_{1}$ as $\Sigma_{3}^{0}$ sets from a stable coloring $c\left(k, s_{1}, s_{2}\right)$. In this case, we have

$$
\lim _{s_{1}} \lim _{s_{2}} c\left(k, s_{1}, s_{2}\right)=i \Leftrightarrow \exists t_{1}\left(\forall s_{1} \geq t_{1} \forall t_{2} \geq s_{1}\right)\left(\exists s_{2} \geq t_{2}\right) c\left(k, s_{1}, s_{2}\right)=i
$$

The nodes in each $R_{i}$ are the initial segments of the strings $\left\langle\left\langle k, t_{1}\right\rangle,\left\langle s_{1}, t_{2}\right\rangle, s_{2}\right\rangle$ for $k \leq t_{1}<s_{1} \leq t_{2}<s_{2}$ and the labeling functions are $\varphi_{i}(\sigma)=\cup$ if $|\sigma| \in\{0,2\}$, $\varphi_{i}(\sigma)=\cap$ if $|\sigma|=1$ and $\varphi_{i}\left(\left\langle\left\langle k, t_{1}\right\rangle,\left\langle s_{1}, t_{2}\right\rangle, s_{2}\right\rangle\right)=\left[0^{k} 1\right]$ if $c\left(k, s_{1}, s_{2}\right)=i$ and is equal to $\emptyset$ if $c\left(k, s_{1}, s_{2}\right)=1-i$. It is straightforward to check in $\mathrm{RCA}_{0}$ that $R_{i}$ represents the union of clopen sets $\left[0^{k} 1\right]$ such that the limit color of $k$ is $i$.

## 6. Reverse math and Borel codes

6.1. Equivalence of the Borel and Baire versions over $A T R_{0}$. In this subsection we show that over the base theory $\mathrm{ATR}_{0}$, the Baire and Borel versions of the Dual Ramsey Theorem are equivalent. We make the following definition in reverse mathematics.

Definition $6.1\left(\mathrm{RCA}_{0}\right)$. A Borel code is a pair $(B, \varphi)$, where $B \subseteq \omega^{<\omega}$ is wellfounded and $\varphi$ is a labeling function as in Definition 4.1.

This definition differs slightly from the definition of a Borel code which is found in the standard reference [17]. In that treatment, there is no labeling function, but certain conventions on the strings in $B$ determine the labels. Because there is no labeling function, the set of leaves of $B$ may not be guaranteed to exist in weak theories. In [17, the base theory for anything to do with Borel sets is $\mathrm{ATR}_{0}$, so this
distinction is never used. We would like to consider weaker base theories. When the base theory is weaker, a constructive presentation of a Borel code should include knowledge of which nodes are leaves. For example, this leaf-knowledge was used in the proof of Theorem 5.7. This is the reason for including the labeling function in our definition.

In Section 5.2 we diverged from the standard definition in a second way, by ascertaining membership in a $\boldsymbol{\Sigma}_{n}^{0}$ set virtually. The standard method, which we use in this section, is via evaluation maps.

Definition 6.2 $\left(\mathrm{RCA}_{0}\right)$. Let $(B, \varphi)$ be a Borel code and $x \in(\omega)^{k}$. An evaluation map for $B$ at $x$ is a function $f: B \rightarrow\{0,1\}$ such that

- For leaves $\sigma \in B, f(\sigma)=1$ if and only if $x \in \varphi(\sigma)$.
- If $\varphi(\sigma)=\cup, f(\sigma)=1$ if and only if there exists $n$ such that $\sigma^{\wedge} n \in B$ and $f\left(\sigma^{\wedge} n\right)=1$.
- If $\varphi(\sigma)=\cap, f(\sigma)=1$ if and only if for all $n$ such that $\sigma^{\wedge} n \in B, f\left(\sigma^{\wedge} n\right)=$ 1.

We say $x \in B$ if there is an evaluation map with value 1 at the root, and we say $x \notin B$ if there is an evaluation map with value 0 at the root.

Observe that both $x \in B$ and $x \notin B$ are $\Sigma_{1}^{1}$ statements. In general, $\mathrm{ATR}_{0}$ is required to show that evaluation maps exist. Similarly, $(\omega)^{k}=C_{0} \cup \ldots \cup C_{\ell-1}$ is the $\Pi_{2}^{1}$ statement that for every $x \in(\omega)^{k}$ and $i<\ell$, there is an evaluation map for $C_{i}$ at $x$ and for some $i<\ell, x \in C_{i}$.
Definition $6.3\left(\mathrm{RCA}_{0}\right)$. Let $B$ be a Borel (or open or closed) code for subset of $(\omega)^{k}$. A Baire code for $B$ consists of open sets $U$ and $V$ and a sequence $\left\langle D_{n}: n \in \omega\right\rangle$ of dense open sets such that $U \cup V$ is dense and for every $p \in \cap_{n \in \omega} D_{n}$, if $p \in U$ then $p \in B$ and if $p \in V$ then $p \notin B$.

Definition $6.4\left(\mathrm{RCA}_{0}\right)$. A Baire code for a Borel coloring $(\omega)^{k}=C_{0} \cup \cdots \cup C_{\ell-1}$ consists of open sets $O_{i}, i<\ell$, and a sequence $\left\langle D_{n}: n \in \omega\right\rangle$ of dense open sets such that $\cup_{i<\ell} O_{i}$ is dense and for every $p \in \cap_{n \in \omega} D_{n}$ and $i<\ell$, if $p \in O_{i}$ then $p \in C_{i}$.

We confirm that ATR $_{0}$ proves that every Borel set has the property of Baire. This is just an effectivization of the usual proof.

Proposition 6.5 $\left(\mathrm{ATR}_{0}\right)$. Every Borel code for a subset of $(\omega)^{k}$ has a Baire code.
Proof. Fix a Borel code $B$. For $\sigma \in B$, let $B_{\sigma}=\{\tau \in B: \tau$ is comparable to $\sigma\}$. $B_{\sigma}$ is a Borel code for the set coded coded by the subtree of $B$ above $\sigma$ in the following sense. Let $f$ be an evaluation map for $B$ at $x$. The function $g: B_{\sigma} \rightarrow 2$ defined by $g(\tau)=f(\tau)$ for $\tau \succeq \sigma$ and $g(\tau)=f(\sigma)$ for $\tau \prec \sigma$ is an evaluation map for $B_{\sigma}$ at $x$ which witnesses $x \in B_{\sigma}$ if and only if $f(\sigma)=1$. We denote this function $g$ by $f_{\sigma, x}$.

Formally, our proof proceeds in two steps. First, by arithmetic transfinite recursion on the Kleene-Brouwer order $K B(B)$, we construct open sets $U_{\sigma}, V_{\sigma}$ and $D_{n, \sigma}, n \in \omega$, which are intended to form a Baire code for $B_{\sigma}$. This construction is essentially identical to the proof of Proposition 5.3. Second, for any $x \in(\omega)^{k}$ and evaluation map $f$ for $B$ at $x$, we show by arithmetic transfinite induction on $K B(B)$ that if $x \in \cap_{n \in \omega} D_{n, \sigma}$, then $x \in U_{\sigma}$ implies $x \in B_{\sigma}$ via $f_{\sigma, x}$ and $x \in V_{\sigma}$ implies $x \notin B_{\sigma}$ via $f_{\sigma, x}$. For ease of presentation, we combine these two steps.

Since $A^{\prime} R_{0}$ suffices to construct evaluation maps, we treat Borel codes as sets in a naive manner and suppress explicit mention of the evaluation maps.

If $\sigma$ is a leaf coding a basic clopen set $[\tau]$, we set $U_{\sigma}=[\tau], V_{\sigma}=\overline{[\tau]}$ and $D_{n, \sigma}=(\omega)^{k}$. Similarly, if $\sigma$ codes $\overline{[\tau]}$, we switch the values of $U_{\sigma}$ and $V_{\sigma}$. In either case, it is clear that these open sets form a Baire code for $B_{\sigma}$.

Suppose $\sigma$ is an internal node coding a union, so $B_{\sigma}$ is the union of $B_{\sigma \sim k}$ for $\sigma^{\wedge} k \in B$. We define $U_{\sigma}$ to be the union of $U_{\sigma \wedge k}$ for $\sigma^{\wedge} k \in B$ and $V_{\sigma}$ to be the union of $[\tau]$ such that $V_{\sigma \sim k}$ is dense in $[\tau]$ for all $\sigma^{\wedge} k \in B$. The sequence $D_{n, \sigma}$ is the sequence of all open sets of the form $D_{n, \sigma{ }^{\prime}} \cap\left(U_{\sigma{ }^{\prime} k} \cup V_{\sigma \wedge k}\right)$ for $n \in \omega$ and $\sigma^{\wedge} k \in B$. As in the proof of Proposition 5.3, $U_{\sigma} \cup V_{\sigma}$ and each $D_{n, \sigma}$ are dense.

Let $x \in \cap_{n \in \omega} D_{n, \sigma}$. Suppose $x \in U_{\sigma}$ and we show $x \in B_{\sigma}$. By the definition of $U_{\sigma}$, fix $\sigma^{\wedge} k \in B$ such that $x \in U_{\sigma^{\wedge} k}$. Since $x \in \cap_{n \in \omega} D_{n, \sigma^{\wedge}}$, we have by induction that $x \in B_{\sigma^{\wedge}}$ and hence $x \in B_{\sigma}$. On the other hand, suppose $x \in V_{\sigma}$ and we show $x \notin B_{\sigma}$. Fix $\tau$ such that $\tau \prec x$ and $[\tau] \subseteq V_{\sigma}$, and fix $k$ such that $\sigma^{\wedge} k \in B$. Since $x \in \cap_{n \in \omega} D_{n, \sigma}, x \in U_{\sigma \wedge k} \cup V_{\sigma \wedge k}$. However, $V_{\sigma \sim k}$ is dense in [ $\left.\tau\right]$. Therefore, $x \notin U_{\sigma^{\wedge}}$ (because $U_{\sigma^{\wedge}}$ and $V_{\sigma^{\wedge}}$ must be disjoint as in the proof of Proposition 5.3), so $x \in V_{\sigma \wedge k}$. Since $x \in \cap_{n \in \omega} D_{n, \sigma^{\wedge}}$, we have by induction that $x \notin B_{\sigma^{\wedge}}$. Because this holds for every $\sigma^{\wedge} k \in B$, it follows that $x \notin B_{\sigma}$, completing the case for unions.

The case for an interior node coding an intersection is similar with the roles of $U_{\sigma}$ and $V_{\sigma}$ switched.

Proposition $6.6\left(\mathrm{ATR}_{0}\right)$. Baire-DRT ${ }_{\ell}^{k}$ implies Borel-DRT ${ }_{\ell}^{k}$.
Proof. By Proposition 6.5 fix Baire codes $U_{i}, V_{i}$ and $D_{n, i}$ for each $C_{i}$. We claim that the open sets $U_{i}$ for $i<\ell$ and the sequence of dense open sets $D_{n, i}$ for $i<\ell$ and $n<\omega$ form a Baire code for this coloring. Note that if $i<\ell$ and $x \in \cap_{n, i} D_{n, i}$, then $x \in U_{i}$ implies $x \in C_{i}$. Therefore, it suffices to show that $\cup_{i<\ell} U_{i}$ is dense.

Suppose not. Then there is $\tau$ such that $[\tau] \cap U_{i}=\emptyset$ for all $i$. Because each set $U_{i} \cup V_{i}$ is open and dense, by the Baire Category Theorem there is $x \in[\tau]$ such that $x \in \cap_{n \in \omega, i<\ell} D_{n, i}$ and $x \in \cap_{i<\ell}\left(U_{i} \cup V_{i}\right)$. Since $x$ is not in any $U_{i}$, we have $x \in V_{i}$ for each $i$. Therefore, for each $i, x \notin C_{i}$, contradicting that $(\omega)^{k}=C_{0} \cup \cdots \cup C_{\ell-1}$.

Lemma $6.7\left(\mathrm{RCA}_{0}\right)$. For every code $O$ for an open set, there is a Borel code $B$ such that $(\omega)^{k}=B \cup \bar{B}$ and for all $x \in(\omega)^{k}, x \in B$ if and only if $x \in O$.

Proof. The content here lies in the proof that $(\omega)^{k}=B \cup \bar{B}$. That is, we need to show that in the obvious Borel code, every $x \in(\omega)^{k}$ has an evaluation map.

Fix $O$. Let $(B, \varphi)$ be the Borel code consisting of a root and a single leaf for each $\langle s, \tau\rangle \in O$, where the leaf is labeled with $[\tau]$.

We claim that for every $x \in(\omega)^{k}$, there is a unique evaluation map $f$ for $B$ at $x$, and $f(\lambda)=1$ if and only if $x \in O$. To prove this claim, we define two potential evaluation maps, $f_{0}$ and $f_{1}$. Let $f_{0}(\lambda)=0$ and $f_{1}(\lambda)=1$. Then for each $i \in\{0,1\}$ and each leaf $\sigma$ with label $\tau$, define $f_{i}(\sigma)=1$ if and only if $x \in[\tau]$. Both these functions have $\Delta_{1}^{0}(x, B, \varphi)$ definitions, and exactly one of them satisfies the condition to be an evaluation map. Clearly, this condition implies that $x \in B$ if and only if $x \in O$.

Corollary $6.8\left(\mathrm{RCA}_{0}\right)$. Borel-DRT ${ }_{\ell}^{k}$ implies Baire-DRT ${ }_{\ell}^{k}$.

Proof. The previous proposition shows that Borel-DRT ${ }_{\ell}^{k}$ implies $\mathrm{ODRT}_{\ell}^{k}$ and hence implies Baire-DRT ${ }_{\ell}^{k}$.
6.2. The strength of "Every Borel set has the property of Baire". We have just seen that over ATR $_{0}$, the Borel and Baire versions of the Dual Ramsey Theorem are equivalent. But only one direction used $\mathrm{ATR}_{0}$, in order to assert that every Borel set has the property of Baire. In this section, we ask if this principle really requires $A T R_{0}$. We find that it does, but the reason is unsatisfactory, because it depends on a technicality in the standard definition of a Borel set. Some of the authors of the present paper removed that technicality in the later-researched but earlier-appearing paper [2]. When the technicality is removed, a principle strictly weaker than ATR $_{0}$ emerges. We refer the reader to [2] for details.

In this section we show:
Theorem $6.9\left(\mathrm{RCA}_{0}\right)$. The following are equivalent.
(1) $\mathrm{ATR}_{0}$.
(2) For every Borel code $B$ for a subset of $(\omega)^{k}$, there is an $x \in(\omega)^{k}$ such that $x \in B$ or $x \notin B$.
(3) Every Borel code $B$ for a subset of $(\omega)^{k}$ has a Baire code.

In fact, the implication from (2) to (1) can be witnessed using only trivial Borel codes, which we define as follows.

Definition $6.10\left(\mathrm{RCA}_{0}\right)$. A Borel code $(B, \varphi)$ for a subset of $(\omega)^{k}$ is trivial if every leaf is labeled with either $\emptyset$ or $(\omega)^{k}$.

If $B$ is a trivial Borel code, then an evaluation map for $B$ at $p$ is independent of $p$, so we can refer to an evaluation map $f$ for $B$. Because we work with trivial Borel codes, the underlying topological space does not matter as long as Borel codes are defined in a manner similar to Definitions 6.1 and 6.2, For example, Theorem 6.9 holds for Borel codes of subsets of $2^{\omega}$ or $\omega^{\omega}$ as defined in Simpson 17]. (The fact that the leaves are labeled in Definition 6.1 does not affect any of the arguments in this section.)

The main ideas in the proof that (2) implies (1) use effective transfinite recursion and are similar to those in Section 7.7 of Ash and Knight [1].

Proposition $6.11\left(\mathrm{RCA}_{0}\right)$. The statement "every trivial Borel code has an evaluation map" implies ACA $_{0}$.
Proof. Fix $g: \omega \rightarrow \omega$ and we show range $(g)$ exists. Let $B$ be the trivial Borel code consisting of the initial segments of $\langle n, m, 1\rangle$ for $g(m)=n$ and $\langle n, m, 0\rangle$ for $g(m) \neq n$. Label all leaves which end in 0 with $\emptyset$, and label all leaves which end in 1 with the entire space. Label all interior nodes with $\cup$. Let $f$ be an evaluation map for $B$. Then $f(\langle n\rangle)=1$ if and only if there is an $m$ such that $g(m)=n$.

In order to strengthen this result to imply $\mathrm{ATR}_{0}$, we need to verify that effective transfinite recursion works in $\mathrm{ACA}_{0}$. Let $L O(X)$ and $W O(X)$ be the standard formulas in second order arithmetic saying $X$ is a linear order and $X$ is a well order. We abuse notation and write $x \in X$ in place of $x \in$ field $(X)$. For a formula $\varphi(n, X)$, $H_{\varphi}(X, Y)$ is the formula stating $L O(X)$ and $Y=\left\{\langle n, j\rangle: j \in X \wedge \varphi\left(n, Y^{j}\right)\right\}$ where $Y^{j}=\left\{\langle m, a\rangle: a<_{X} j \wedge\langle m, a\rangle \in Y\right\}$. When $\varphi$ is arithmetic, $H_{\varphi}(X, Y)$ is arithmetic and $\mathrm{ACA}_{0}$ proves that if $W O(X)$, then there is at most one $Y$ such that $H_{\varphi}(X, Y)$. We define our formal version of effective transfinite recursion.

Definition 6.12. ETR is the axiom scheme

$$
\forall X\left[(W O(X) \wedge \forall Y \forall n(\varphi(n, Y) \leftrightarrow \neg \psi(n, Y))) \rightarrow \exists Y H_{\varphi}(X, Y)\right]
$$

where $\varphi$ and $\psi$ range over $\Sigma_{1}^{0}$ formulas.
We show that ETR is provable in $\mathrm{ACA}_{0}$. Following Simpson [17], we avoid using the recursion theorem and note that the only place the proof goes beyond $\mathrm{RCA}_{0}$ is in the use of transfinite induction for $\Pi_{2}^{0}$ formulas, which holds is $A C A_{0}$ and is equivalent to transfinite induction for $\Sigma_{1}^{0}$ formulas. Greenberg and Montalbán [8] point out that ETR can also be proved using the recursion theorem, although this proof also uses $\Sigma_{1}^{0}$ transfinite induction.

Proposition 6.13. ETR is provable in $\mathrm{ACA}_{0}$.
Proof. Fix a well order $X$ and $\Sigma_{1}^{0}$ formulas $\varphi$ and $\psi$. Throughout this proof, we let $f, g$ and $h$ be variables denoting finite partial functions from $\omega$ to $\{0,1\}$ coded in the canonical way as finite sets of ordered pairs. We write $f \preceq g$ (or $f \prec X$ ) if $f \subseteq g$ (or $f \subseteq \chi_{X}$ ) as sets of ordered pairs. By the usual normal form results (e.g. Theorem II.2.7 in Simpson), we fix a $\Sigma_{0}^{0}$ formula $\varphi_{0}$ such that

$$
\forall Y \forall n\left(\varphi(n, Y) \leftrightarrow \exists f\left(f \prec Y \wedge \varphi_{0}(n, f)\right)\right)
$$

and such that if $\varphi_{0}(n, f)$ and $f \prec g$, then $\varphi_{0}(n, g)$. We fix a formula $\psi_{0}$ related to $\psi$ in the same manner. Since $\varphi(n, Y) \leftrightarrow \neg \psi(n, Y)$, we cannot have compatible $f$ and $g$ such that $\varphi_{0}(n, f)$ and $\psi_{0}(n, g)$.

Our goal is to use partial functions $f$ as approximations to a set $Y$ such that $H_{\varphi}(X, Y)$. Therefore, we view $\operatorname{dom}(f)$ as consisting of coded pairs $\langle n, a\rangle$. For $f$ to be a suitable approximation to $Y$, we need that if $\langle n, a\rangle \in \operatorname{dom}(f)$ and $a \notin X$, then $f(\langle n, a\rangle)=0$. Similarly, if $f$ is an approximation to $Y^{j}$, we need that $f(\langle n, a\rangle)=0$ whenever $\langle n, a\rangle \in \operatorname{dom}(f)$ and $a \geq_{X} j$. These observations motivate the following definitions.

Let $f$ be a finite partial function and let $i \in X$. We define

$$
f^{i}=f \upharpoonright\left\{\langle n, a\rangle: n \in \omega \wedge a<_{X} i\right\} .
$$

We say $g \succeq f$ is an $i$-extension of $f$ if for all $\langle n, a\rangle \in \operatorname{dom}(g)-\operatorname{dom}(f), g(\langle n, a\rangle)=0$ and either $a \notin X$ or $i \leq_{X} a$.

For $j \in X, f$ is a $j$-approximation if the following conditions hold.

- If $\langle n, a\rangle \in \operatorname{dom}(f)$ with $a \notin X$ or $j \leq_{X} a$, then $f(\langle n, a\rangle)=0$.
- If $\langle n, a\rangle \in \operatorname{dom}(f)$ and $a<_{X} j$, then
- if $f(\langle n, a\rangle)=1$, then there is an $a$-extension $h$ of $f^{a}$ such that $\varphi_{0}(n, h)$, and
- if $f(\langle n, a\rangle)=0$, then there is an $a$-extension $h$ of $f^{a}$ such that $\psi_{0}(n, h)$.

Note that if $f$ is a $j$-approximation and $i<_{X} j$, then $f^{i}$ is an $i$-approximation. Also, if $f$ is a $j$-approximation and $g$ is a $j$-extension of $f$, then $g$ is a $j$-approximation.

Claim. For all $j \in X$, there do not exist $m \in \omega$ and $j$-approximations $f$ and $g$ such that $\varphi_{0}(m, f)$ and $\psi_{0}(m, g)$.

The proof is by transfinite induction on $j$. Fix the least $j \in X$ for which this property fails and fix witnesses $m, f$ and $g$. To derive a contradiction, it suffices to show that $f$ and $g$ are compatible. Fix $\langle k, a\rangle$ such that both $f(\langle k, a\rangle)$ and $g(\langle k, a\rangle)$ are defined. If $a \notin X$ or $j \leq_{X} a$, then $f(\langle k, a\rangle)=g(\langle k, a\rangle)=0$.

Suppose for a contradiction that $a<_{X} j$ and $f(\langle k, a\rangle) \neq g(\langle k, a\rangle)$. Without loss of generality, $f(\langle k, a\rangle)=1$ and $g(\langle k, a\rangle)=0$. Fix $a$-extensions $h$ and $h^{\prime}$ of $f^{a}$ and $g^{a}$ respectively such that $\varphi_{0}(k, h)$ and $\psi_{0}\left(k, h^{\prime}\right)$. Since $f$ is a $j$-approximation, $f^{a}$ is an $a$-approximation, and since $h$ is an $a$-extension of $f^{a}, h$ is also an $a$ approximation. Similarly, $h^{\prime}$ is an $a$-approximation. Therefore, we have $k \in \omega$, $a<_{X} j$ and $a$-approximation $h$ and $h^{\prime}$ such that $\varphi_{0}(k, h)$ and $\psi_{0}\left(k, h^{\prime}\right)$ contradicting the minimality of $j$.
Claim. For any $j$-approximation $f$ and any $m \in \omega$, there is a $j$-approximation $g \succeq f$ such that either $\varphi_{0}(m, g)$ or $\psi_{0}(m, g)$.

The proof is again by transfinite induction on $j$. Fix the least $j$ for which this property fails and fix witnesses $f$ and $m$. Let $\left\langle n_{s}, i_{s}\right\rangle$ enumerate the pairs not in the domain of $f$. Below, we define a sequence $f=f_{0} \preceq f_{1} \preceq \cdots$ of $j$-approximations such that $f_{s+1}\left(\left\langle n_{s}, i_{s}\right\rangle\right)$ is defined. Let $Y$ be the set with $\chi_{Y}=\cup_{s} f_{s}$. Either $\varphi(m, Y)$ or $\psi(m, Y)$ holds, and so there is a $g \prec Y$ such that $\varphi_{0}(m, g)$ or $\psi_{0}(m, g)$ holds. Fixing $s$ such that $g \preceq f_{s}$ shows that either $\varphi_{0}\left(m, f_{s}\right)$ or $\psi_{0}\left(m, f_{s}\right)$ holds for the desired contradiction.

To define $f_{s+1}$, we need to extend $f_{s}$ to a $j$-approximation $f_{s+1}$ with $\left\langle n_{s}, i_{s}\right\rangle \in$ $\operatorname{dom}\left(f_{s+1}\right)$. We break into several cases. If $f_{s}\left(\left\langle n_{s}, i_{s}\right\rangle\right)$ is already defined, let $f_{s+1}=$ $f_{s}$. Otherwise, if $i_{s} \notin X$ or $j \leq_{X} i_{s}$, set $f_{s+1}\left(\left\langle n_{s}, i_{s}\right\rangle\right)=0$ and leave the remaining values as in $f_{s}$. In both cases, it is clear that $f_{s+1}$ is a $j$-approximation.

Finally, if $i_{s}<_{X} j$ and $f_{s}\left(\left\langle n_{s}, i_{s}\right\rangle\right)$ is undefined, we apply the induction hypothesis to the $i_{s}$-approximation $f_{s}^{i_{s}}$ to get an $i_{s}$-approximation $g \succeq f_{s}^{i_{s}}$ such that either $\varphi_{0}\left(n_{s}, g\right)$ holds or $\psi_{0}\left(n_{s}, g\right)$ holds. Define $f_{s+1}$ as follows.

- For $\langle m, a\rangle \in \operatorname{dom}(g)$ with $a<_{X} i_{s}$, set $f_{s+1}(\langle m, a\rangle)=g(\langle m, a\rangle)$.
- For $\langle m, a\rangle \in \operatorname{dom}\left(f_{s}\right)$ with $i_{s} \leq_{X} a$ or $a \notin X$, set $f_{s+1}(\langle m, a\rangle)=f_{s}(\langle m, a\rangle)$.
- Set $f_{s+1}\left(\left\langle n_{s}, i_{s}\right\rangle\right)=1$ if $\varphi_{0}\left(n_{s}, g\right)$ holds and $f_{s+1}\left(\left\langle n_{s}, i_{s}\right\rangle\right)=0$ if $\psi_{0}\left(n_{s}, g\right)$ holds.
It is straightforward to verify that $f_{s} \prec f_{s+1}, g$ is an $i_{s}$-extension of $f_{s+1}^{i_{s}}$ and $f_{s+1}$ is a $j$-approximation, completing the proof of the claim.

We define the set $Y$ for which we will show $H_{\varphi}(X, Y)$ holds by $\langle m, j\rangle \in Y$ if and only if $j \in X$ and there is a $j$-approximation $f$ such that $\varphi_{0}(m, f)$. It follows from the claims above that $\langle m, j\rangle \notin Y$ if and only if either $j \notin X$ or there is a $j$-approximation $f$ such that $\psi_{0}(m, f)$. Therefore, $Y$ has a $\Delta_{1}^{0}$ definition. The next two claims show that $H_{\varphi}(X, Y)$ holds, completing our proof.

Claim. If $f$ is a $j$-approximation, then $f \prec Y^{j}$.
Consider $\langle m, a\rangle \in \operatorname{dom}(f)$. If $a \notin X$ or $j \leq_{X} a$, then $f(\langle m, a\rangle)=Y^{j}(\langle m, a\rangle)=0$. Suppose $a<_{X} j$. If $f(\langle m, a\rangle)=1$, then there is an $a$-extension $g$ of $f^{a}$ such that $\varphi_{0}(m, g)$. Since $f^{a}$ is an $a$-approximation and $g$ is an $a$-extension of $f^{a}, g$ is an $a$-approximation. Therefore, $\langle m, a\rangle \in Y$ by definition and hence $\langle m, a\rangle \in Y^{j}$. By similar reasoning, if $f(\langle m, a\rangle)=0$, then $\langle m, a\rangle \notin Y$ and hence $\langle m, a\rangle \notin Y^{j}$.
Claim. $\langle m, j\rangle \in Y$ if and only if $\varphi\left(m, Y^{j}\right)$.
Assume that $\langle m, j\rangle \in Y$ and fix a $j$-approximation $f$ such that $\varphi_{0}(m, f)$. Since $f \prec Y^{j}, \varphi\left(m, Y^{j}\right)$. For the other direction, assume that $\varphi\left(m, Y^{j}\right)$. Fix a $j$ approximation $f$ such that either $\varphi_{0}(m, f)$ or $\psi_{0}(m, f)$. Since $f \prec Y^{j}$ and $\varphi\left(m, Y^{j}\right)$, we must have $\varphi_{0}(m, f)$ and therefore $\langle m, j\rangle \in Y$ by definition.

We recall some notation and facts from Simpson [17] to state the equivalence of $\mathrm{ATR}_{0}$ we will prove. We let $T J(X)$ denote the Turing jump in $\mathrm{ACA}_{0}$ given by fixing a universal $\Pi_{1}^{0}$ formula. We use the standard recursion theoretic notations $\Phi_{e}^{X}$ and $\Phi_{e, s}^{X}$ with the understanding that they are defined by this fixed universal formula.
$\mathcal{O}_{+}(a, X)$ is the arithmetic statement that $a=\langle e, i\rangle, e$ is an $X$-recursive index of an $X$-recursive linear order $\leq_{e}^{X}$ and $i \in \operatorname{field}\left(\leq_{e}^{X}\right) . \mathcal{O}_{+}^{X}=\left\{a: \mathcal{O}_{+}(a, X)\right\}$ exists in $\mathrm{ACA}_{0}$. For $a, b \in \mathcal{O}_{+}^{X}$, we write $b<_{\mathcal{O}}^{X} a$ if $a=\langle e, i\rangle, b=\langle e, j\rangle$ and $j<_{e}^{X} i$. For $a \in \mathcal{O}_{+}^{X}$, the set $\left\{b: b<_{\mathcal{O}}^{X} a\right\}$ exists in $\mathrm{ACA}_{0}$.
$\mathcal{O}(a, X)$ is the $\Pi_{1}^{1}$ statement $\mathcal{O}_{+}(a, X) \wedge W O\left(\left\{b: b<_{\mathcal{O}}^{X} a\right\}\right)$. Intuitively, $\mathcal{O}(a, X)$ says that $a=\langle e, i\rangle$ is an $X$-recursive ordinal notation for the well ordering given by the restriction of $\leq_{e}^{X}$ to $\left\{j: j<_{e}^{X} i\right\}$. In $\mathrm{ATR}_{0}$, if $\mathcal{O}(a, X)$, then the set

$$
H_{a}^{X}=\{\langle y, 0\rangle: y \in X\} \cup\left\{\langle y, b+1\rangle: b<_{\mathcal{O}}^{X} a \wedge y \in T J\left(H_{b}^{X}\right)\right\}
$$

exists. In $\mathrm{ACA}_{0}$, there is an arithmetic formula $H(a, X, Y)$ which, under the assumption that $\mathcal{O}(a, X)$, holds if and only if $Y=H_{a}^{X}$.

By Theorem VIII.3.15 in Simpson [17], ATR ${ }_{0}$ is equivalent over ACA $_{0}$ to

$$
\forall X \forall a\left(\mathcal{O}(a, X) \rightarrow H_{a}^{X}\right. \text { exists). }
$$

If $\mathcal{O}(a, X)$ with $a=\langle e, i\rangle$, then we can assume without loss of generality that there are $a^{\prime}$ and $a^{\prime \prime}$ such that $\mathcal{O}\left(a^{\prime}, X\right), \mathcal{O}\left(a^{\prime \prime}, X\right)$ and $a<_{\mathcal{O}}^{X} a^{\prime}<_{\mathcal{O}}^{X} a^{\prime \prime}$ by adding two new successors of $i$ in $\leq_{e}^{X}$ if necessary. Therefore, to prove ATR $_{0}$, it suffices to fix $a$ and $X$ such that $\mathcal{O}(a, X)$ and prove $\forall c<_{\mathcal{O}}^{X} b\left(H_{c}^{X}\right.$ exists) for each $b<_{\mathcal{O}}^{X} a$.
Theorem 6.14 $\left(\mathrm{ACA}_{0}\right)$. The statement "every trivial Borel code has an evaluation map" implies $\mathrm{ATR}_{0}$.
Proof. Fix $a$ and $X$ such that $\mathcal{O}(a, X)$, so the restriction of $<_{\mathcal{O}}^{X}$ to $\left\{b: b<_{\mathcal{O}}^{X} a\right\}$ is a well order. Using ETR, we define trivial Borel codes $B_{x, b}$ for $x \in \omega$ by transfinite recursion on $b<\mathcal{O}_{\mathcal{O}}^{X} a$. We explain the intuitive construction before the formal definition.

Let $b<_{\mathcal{O}}^{X} a$ and $x \in \omega$. We want to define a trivial Borel code $B_{x, b}$ such that if $f$ is an evaluation map for $B_{x, b}$, then $f(\lambda)=1$ if and only if $x \in T J\left(H_{b}^{X}\right)$. We label $\lambda$ with $\cup$. For each binary string $\sigma$ such that $\Phi_{x,|\sigma|}^{\sigma}(x)$ converges, we add a successor $\left\langle n_{\sigma}\right\rangle$. Here $\sigma \mapsto n_{\sigma}$ is just some primitive recursive bijection between $2^{<\omega}$ and $\omega$. It follows that $f(\lambda)=1$ if and only if there is a $\sigma$ such that $\Phi_{x,|\sigma|}^{\sigma}(x)$ converges and $f\left(\left\langle n_{\sigma}\right\rangle\right)=1$. (In case $\Phi_{x,|\sigma|}^{\sigma}(x)$ always diverges, we may also add a leaf $\langle n\rangle$ which is labeled with $\emptyset$. In this case, $f(\lambda)=f(\langle n\rangle)=0$ and $x \notin T J\left(H_{b}^{X}\right)$ which is what we want.)

Next, we want to ensure $f\left(\left\langle n_{\sigma}\right\rangle\right)=1$ if and only if $\sigma \prec H_{b}^{X}$. We label $\left\langle n_{\sigma}\right\rangle$ with $\cap$, and for each $k<|\sigma|$, we add a successor $\left\langle n_{\sigma}, k\right\rangle$. We want $f\left(\left\langle n_{\sigma}, k\right\rangle\right)=1$ if and only if $\sigma(k)=H_{b}^{X}(k)$. We break into cases to determine the extensions of $\left\langle n_{\sigma}, k\right\rangle$.

For the first case, suppose $k=\langle y, 0\rangle$. We want $f\left(\left\langle n_{\sigma}, k\right\rangle\right)=1$ if and only if $y \in X$. If $\sigma(k)=X(y)$, we label this node with the entire space, and if $\sigma(k) \neq X(y)$, we label this node with $\emptyset$. In either case, the successor nodes will be leaves so we have $f\left(\left\langle n_{\sigma}, k\right\rangle\right)=1$ if and only if $k \in H_{b}^{X}$.

For the second case, suppose $k=\langle y, c+1\rangle$ and $c<_{\mathcal{O}}^{X} b$. By the induction hypothesis, we have defined the trivial Borel code $B_{y, c}$ already. If $\sigma(k)=1$, then we label $\left\langle n_{\sigma}, k\right\rangle$ with $\cup$, and attach to it a copy of $B_{y, c}$, treating $\left\langle n_{\sigma}, k\right\rangle$ as the root of $B_{y, c}$. The map $f$ restricted to the subtree above $\left\langle n_{\sigma}, k\right\rangle$ is an evaluation map for
$B_{y, c}$ and hence by the inductive hypothesis

$$
f\left(\left\langle n_{\sigma}, k\right\rangle\right)=1 \Leftrightarrow y \in T J\left(H_{c}^{X}\right) \Leftrightarrow k \in H_{b}^{X} \Leftrightarrow \sigma(k)=H_{b}^{X}(k) .
$$

On the other hand, if $\sigma(k)=0$, then we label $\left\langle n_{\sigma}, k\right\rangle$ with $\cap$ and extend it by a copy of $\bar{B}_{y, c}$. By the inductive hypothesis, we have

$$
f\left(\left\langle n_{\sigma}, k\right\rangle\right)=1 \Leftrightarrow y \notin T J\left(H_{c}^{X}\right) \Leftrightarrow k \notin H_{b}^{X} \Leftrightarrow \sigma(k)=H_{b}^{X}(k) .
$$

For the third case, suppose that $k=\langle y, c+1\rangle$ and $c \not_{\mathcal{O}}^{X} b$. In this case, we know $H_{b}^{X}(k)=0$. If $\sigma(k)=0$, we label $\left\langle n_{\sigma}, k\right\rangle$ with the entire space, and if $\sigma(k)=1$ we label it with $\emptyset$.

The formal construction follows this outline. To simplify the notation, for a trivial Borel code $B$, we let $B^{1}=B$ and $B^{0}=\bar{B}$. Since " $\Phi_{x,|\sigma|}^{\sigma}(x)$ converges" is a bounded quantifier statement and $c<_{\mathcal{O}}^{X} b$ is a $\Delta_{1}^{0}$ statement with parameter $X$, the following recursion on $b<_{\mathcal{O}}^{X} a$ can be done with ETR. For each $x \in \omega$, we put $\lambda$ in $B_{x, b}$ with label $\cup$. For each $\sigma$ such that $\Phi_{x,|\sigma|}^{\sigma}(x)$ converges, we put $\left\langle n_{\sigma}\right\rangle$ and $\left\langle n_{\sigma}, k\right\rangle$ in $B_{x, b}$ for all $k<|\sigma|$. We label $\left\langle n_{\sigma}\right\rangle$ with $\cap$. We extend $\left\langle n_{\sigma}, k\right\rangle$ as follows.

- For $k=\langle y, 0\rangle$ : if $\sigma(k)=X(y)$, then $\left\langle n_{\sigma}, k\right\rangle$ is labeled with the whole space, and if $\sigma(k) \neq X(y)$, then it is labeled with $\emptyset$.
- For $k=\langle y, c+1\rangle$ with $c<_{\mathcal{O}}^{X} b,\left\langle n_{\sigma}, k\right\rangle \sim \tau \in B_{x, b}$ for all $\tau \in B_{y, c}^{\sigma(k)}$, with labels inherited from $B_{y, c}^{\sigma(k)}$.
- For $k=\langle y, c+1\rangle$ with $c \nless \mathcal{O}_{X}^{X} b,\left\langle n_{\sigma}, k\right\rangle$ gets labeled with the whole set if $\sigma(k)=0$ and labeled with $\emptyset$ if $\sigma(k)=1$.
This completes the construction of the trivial Borel codes $B_{x, b}$ for $b<_{\mathcal{O}}^{X} a$ by ETR. To complete the proof, we fix an arbitrary $b<_{\mathcal{O}}^{X} a$ and show that $\forall c<_{\mathcal{O}}^{X}$ $b$ ( $H_{c}^{X}$ exists).

Fix an index $x$ and $s \in \omega$ such that $\Phi_{x, s}^{1^{s}}(x)$ converges. Let $N$ be the least value of $s$ witnessing this convergence so $\Phi_{x, s}^{1^{s}}(x)$ converges for all $s \geq N$. Let $f$ be an evaluation map for $B_{x, b}$.

For $c<_{\mathcal{O}}^{X} b$ and $y \in \omega$, let $\sigma=1^{N+k}$ where $k=\langle y, c+1\rangle$. Define $f_{y, c}(\tau)=$ $f\left(\left\langle n_{\sigma}, k\right\rangle{ }^{\wedge} \tau\right)$. We claim $f_{y, c}$ is an evaluation map for $B_{y, c}$. By the choice of $x$, $\Phi_{x,|\sigma|}^{\sigma}(x)$ converges. Since $c<_{\mathcal{O}}^{X} b$ and $\sigma(k)=1$, we have $\left\langle n_{\sigma}, k\right\rangle^{\wedge} \tau \in B_{x, b}$ if and only if $\tau \in B_{y, c}$. Therefore, $f_{y, c}$ is defined on $B_{y, c}$ and it satisfies the conditions for an evaluation map because $f$ does.

Recall that $H(x, X, Y)$ is a fixed arithmetic formula such that if $\mathcal{O}(x, X)$, then $H(x, X, Y)$ holds if and only if $Y=H_{x}^{X}$. Define

$$
Z=\{\langle y, 0\rangle: y \in X\} \cup\left\{k: k=\langle y, c+1\rangle \wedge c<_{\mathcal{O}}^{X} b \wedge f\left(\left\langle n_{\sigma}, k\right\rangle\right)=1\right\} .
$$

For $c<_{\mathcal{O}}^{X} b$, let $Z^{c}=\left\{\langle y, r\rangle \in Z: r=0 \vee r-1<_{\mathcal{O}}^{X} c\right\}$. We show the following properties by simultaneous arithmetic induction on $c<_{\mathcal{O}}^{X} b$.
(1) $H\left(c, X, Z^{c}\right)$ holds. That is, $Z^{c}=H_{c}^{X}$.
(2) For all $y, f_{y, c}(\lambda)=1$ if and only if $y \in T J\left(Z^{c}\right)=T J\left(H_{c}^{X}\right)$.

These properties imply $\forall c<_{\mathcal{O}}^{X} b\left(H_{c}^{X}\right.$ exists) completing our proof.
Fix $c<_{\mathcal{O}}^{X} b$ and assume (1) and (2) hold for $d<_{\mathcal{O}}^{X} c$. To see (1) holds for $c$, fix $k$. If $k=\langle y, 0\rangle$, then $k \in Z^{c} \Leftrightarrow y \in X \Leftrightarrow k \in H_{c}^{X}$. Suppose $k=\langle y, d+1\rangle$. If $d \not \chi_{\mathcal{O}}^{X} c$, then $k \notin H_{c}^{X}$ and $k \notin Z^{c}$. If $d<_{\mathcal{O}}^{X} c$, then

$$
k \in Z^{c} \Leftrightarrow f\left(\left\langle n_{\sigma}, k\right\rangle\right)=1 \Leftrightarrow f_{y, d}(\lambda)=1 .
$$

By the induction hypothesis, $k \in Z^{c}$ if and only if $y \in T J\left(Z^{d}\right)=T J\left(H_{d}^{X}\right)$, which holds if and only if $k \in H_{c}^{X}$, completing the proof of (1).

To prove (2), fix $y$ and let $k=\langle y, c+1\rangle$. By definition,

$$
k \in Z^{c} \Leftrightarrow f_{y, c}(\lambda)=f\left(\left\langle n_{\sigma}, k\right\rangle\right)=1
$$

and $y \in T J\left(Z^{c}\right)=T J\left(H_{c}^{X}\right)$ if and only if there is a $\sigma$ such that $\Phi_{y,|\sigma|}^{\sigma}(y)$ converges and $\sigma \prec Z^{c}=H_{c}^{X}$.

Suppose there are no $\sigma$ such that $\Phi_{y,|\sigma|}^{\sigma}(y)$ converges. In this case, $y \notin T J\left(H_{c}^{X}\right)$ and $f_{y, c}(\lambda)=0$. Therefore $f_{y, c}(\lambda)=1$ if and only if $y \in T J\left(H_{c}^{X}\right)$ as required.

Suppose $\Phi_{y,|\sigma|}^{\sigma}(y)$ converges for some $\sigma$. For any such $\sigma,\left\langle n_{\sigma}, k\right\rangle \in B_{y, c}$ for all $k<|\sigma|$. By the induction hypothesis and the case analysis in the intuitive explanation of the construction, we have $f_{y, c}\left(\left\langle n_{\sigma}\right\rangle\right)=1$ if and only if $\sigma \prec H_{c}^{X}=Z^{c}$, and therefore, $f_{y, c}(\lambda)=1$ if and only if there is a $\sigma$ such that $\Phi_{y,|\sigma|}^{\sigma}(y)$ converges and $\sigma \prec H_{c}^{X}$, completing the proof of (2) and of the theorem.

We conclude with a proof of Theorem 6.9.
Proof. Lemma V.3.3 in Simpson [17] shows (1) implies (2) in the space $2^{\omega}$ and the proof translates immediately to $(\omega)^{k}$. By Proposition 6.5, (1) implies (3). It follows from Theorem 6.14 that (2) implies (1). We show (3) implies (2). Let $B$ be a Borel code. Fix a Baire code $U, V$ and $D_{n}$ for $B$. Since each $D_{n}$ and $U \cup V$ is a dense open set, there is an $x \in(U \cup V) \cap \cap_{n \in \omega} D_{n}$. If $x \in U$, then by the definition of a Baire code, $x \in B$, and similarly, if $x \in V$, then $x \notin B$. Therefore, we have a partition $x$ such that $x \in B$ or $x \notin B$ as required.

## 7. Open Questions

While Figure 1.1 summarizes the known implications among the studied principles, in most cases it is not known whether the results are optimal. It is particularly dissatisfying that the best upper bound for these principles remains $\Pi_{1}^{1}-\mathrm{CA}_{0}$. Observe that, on the basis of the proof of $\mathrm{CDRT}_{\ell}^{k}$ given in Theorem 3.18, any upper bound on the strength of the Carlson-Simpson $\operatorname{Lemma} \operatorname{CSL}(k-1, \ell)$ would also imply a related upper bound on the strength of $\mathrm{CDRT}_{\ell}^{k}$. Therefore, it would be interesting to know the following:
Question 7.1. For any $k \geq 3$, does $\operatorname{CSL}(k, \ell)$ follow from $\mathrm{ATR}_{0}$ ?
The best known upper bound for $\operatorname{CSL}(2, \ell)$ is $\mathrm{ACA}_{0}$; it is shown in 11 that the stronger principle $\operatorname{OVW}(2, \ell)$ follows from $\mathrm{ACA}_{0}$.

Turning attention now to lower bounds, the principles $\mathrm{CDRT}_{\ell}^{k}$ for $k \geq 4$ are not obviously implied by HT or $\mathrm{ACA}_{0}^{+}$. We wonder whether an implication might go the other way.
Question 7.2. For any $k \geq 4$, does $\mathrm{CDRT}_{\ell}^{k}$ imply HT or $\mathrm{ACA}_{0}^{+}$?
When $k \geq 4$, it is known that $\mathrm{CDRT}_{\ell}^{k}$ implies $\mathrm{ACA}_{0}$ (this was proved for $\mathrm{ODRT}_{\ell}^{k}$ in [12]). On the other hand, while $\mathrm{CDRT}_{2}^{3}$ is provable from Hindman's Theorem, the best lower bound we have on $\mathrm{CDRT}_{2}^{3}$ is $\mathrm{RT}_{2}^{2}$. Furthermore, nothing about the relationship of $\mathrm{CDRT}_{2}^{3}$ and $\mathrm{ACA}_{0}$ is known.
Question 7.3. Is $\mathrm{CDRT}_{2}^{3}$ comparable to $\mathrm{ACA}_{0}$ ?
For the $k=2$ case, can Theorem 5.7 be strengthened in the following way?

## Question 7.4. Is $\Delta_{n}^{0}$ - $\mathrm{DRT}_{2}^{2} \equiv_{\mathrm{sW}} D_{2}^{n}$ ?

These are just a few of the many questions that remain concerning these principles.

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