A CHOICE-FREE CARDINAL EQUALITY

GUOZHEN SHEN

Abstract. For a cardinal $\mathfrak{a}$, let $\mathrm{fin}(\mathfrak{a})$ be the cardinality of the set of all finite subsets of a set which is of cardinality $\mathfrak{a}$. It is proved without the aid of the axiom of choice that for all infinite cardinals $\mathfrak{a}$ and all natural numbers n,
$$2^{\mathrm{fin}(\mathfrak{a})^n} = 2^{[\mathrm{fin}(\mathfrak{a})]^n}.$$
On the other hand, it is proved that the following statement is consistent with ZF: there exists an infinite cardinal $\mathfrak{a}$ such that
$$2^{\mathrm{fin}(\mathfrak{a})} < 2^{\mathrm{fin}(\mathfrak{a})^2} < 2^{\mathrm{fin}(\mathfrak{a})^3} < \cdots < 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}.$$

arXiv:1912.12435v1 [math.LO] 28 Dec 2019

1. Introduction

For a cardinal $\mathfrak{a}$, let $\mathrm{fin}(\mathfrak{a})$ be the cardinality of the set of all finite subsets of a set which is of cardinality $\mathfrak{a}$. The axiom of choice implies that $\mathrm{fin}(\mathfrak{a}) = \mathfrak{a}$ for any infinite cardinal $\mathfrak{a}$. However, in the absence of the axiom of choice, this is no longer the case. In fact, in the ordered Mostowski model (cf. [2, pp. 198–202]), the cardinality $\mathfrak{a}$ of the set of atoms satisfies

$$\begin{aligned}\mathrm{fin}(\mathfrak{a}) < [\mathrm{fin}(\mathfrak{a})]^2 < \mathrm{fin}(\mathfrak{a})^2 < [\mathrm{fin}(\mathfrak{a})]^3 < \mathrm{fin}(\mathfrak{a})^3 < \cdots \\ < \mathrm{fin}(\mathrm{fin}(\mathfrak{a})) < \mathrm{fin}(\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))) < \cdots < \aleph_0 \cdot \mathrm{fin}(\mathfrak{a}).\end{aligned} \quad (1)$$

It is natural to ask which relationships between the *powers* of the cardinals in (1) for an arbitrary infinite cardinal $\mathfrak{a}$ can be proved without the aid of the axiom of choice.

The first result of this kind is Läuchli's lemma (cf. [3] or [2, Lemma 5.27]), which states that for all infinite cardinals $\mathfrak{a}$,

$$2^{\aleph_0 \cdot \mathrm{fin}(\mathfrak{a})} = 2^{\mathrm{fin}(\mathfrak{a})}.$$

Läuchli's lemma implies that, in the ordered Mostowski model, the powers of the cardinals in (1) are all equal, where $\mathfrak{a}$ is the cardinality of the set of atoms.

In this paper, we give a complete answer to the above question. We first prove in ZF that for all infinite cardinals $\mathfrak{a}$,

$$2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))} = 2^{\mathrm{fin}(\mathrm{fin}(\mathrm{fin}(\mathfrak{a})))} = 2^{\mathrm{fin}(\mathrm{fin}(\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))))} = \cdots.$$

2010 *Mathematics Subject Classification.* Primary 03E10, 03E25.
Key words and phrases. ZF, cardinal, finite subsets, axiom of choice.

Then, as our main result, we prove in ZF that for all infinite cardinals $\mathfrak{a}$ and all natural numbers n,

$$2^{\mathrm{fin}(\mathfrak{a})^n} = 2^{[\mathrm{fin}(\mathfrak{a})]^n}.$$

Finally, we prove that the following statement is consistent with ZF: there exists an infinite cardinal $\mathfrak{a}$ such that

$$2^{\mathrm{fin}(\mathfrak{a})} < 2^{\mathrm{fin}(\mathfrak{a})^2} < 2^{\mathrm{fin}(\mathfrak{a})^3} < \cdots < 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}.$$

2. Basic notions and facts

Throughout this paper, we shall work in ZF. In this section, we indicate briefly our use of some terminology and notation. The cardinality of x, which we denote by $|x|$, is the least ordinal α equinumerous to x, if x is well-orderable, and the set of all sets y of least rank which are equinumerous to x, otherwise. We shall use lower case German letters $\mathfrak{a}, \mathfrak{b}$ for cardinals.

For a function f, we shall use $\mathrm{dom}(f)$ for the domain of f, $\mathrm{ran}(f)$ for the range of f, $f[x]$ for the image of x under f, $f^{-1}[x]$ for the inverse image of x under f, and $f{\restriction}x$ for the restriction of f to x. For functions f and g, we use $g \circ f$ for the composition of g and f.

Definition 2.1. Let x, y be arbitrary sets, let $\mathfrak{a} = |x|$, and let $\mathfrak{b} = |y|$.

(1) $x \preccurlyeq y$ means that there exists an injection from x into y; $\mathfrak{a} \leqslant \mathfrak{b}$ means that $x \preccurlyeq y$.
(2) $x \preccurlyeq^* y$ means that there exists a surjection from a subset of y onto x; $\mathfrak{a} \leqslant^* \mathfrak{b}$ means that $x \preccurlyeq^* y$.
(3) $\mathfrak{a} \nleqslant \mathfrak{b}$ ($\mathfrak{a} \nleqslant^* \mathfrak{b}$) denotes the negation of $\mathfrak{a} \leqslant \mathfrak{b}$ ($\mathfrak{a} \leqslant^* \mathfrak{b}$).
(4) $\mathfrak{a} < \mathfrak{b}$ means that $\mathfrak{a} \leqslant \mathfrak{b}$ and $\mathfrak{b} \nleqslant \mathfrak{a}$.
(5) $\mathfrak{a} =^* \mathfrak{b}$ means that $\mathfrak{a} \leqslant^* \mathfrak{b}$ and $\mathfrak{b} \leqslant^* \mathfrak{a}$.

It follows from the Cantor–Bernstein theorem that if $\mathfrak{a} \leqslant \mathfrak{b}$ and $\mathfrak{b} \leqslant \mathfrak{a}$ then $\mathfrak{a} = \mathfrak{b}$. Clearly, if $\mathfrak{a} \leqslant \mathfrak{b}$ then $\mathfrak{a} \leqslant^* \mathfrak{b}$, and if $\mathfrak{a} \leqslant^* \mathfrak{b}$ then $2^{\mathfrak{a}} \leqslant 2^{\mathfrak{b}}$. Thus $\mathfrak{a} =^* \mathfrak{b}$ implies that $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$.

Definition 2.2. Let x, y be arbitrary sets, let $\mathfrak{a} = |x|$, and let $\mathfrak{b} = |y|$.

(1) x^y is the set of all functions from y into x; $\mathfrak{a}^{\mathfrak{b}} = |x^y|$.
(2) $x^{\underline{y}}$ is the set of all injections from y into x; $\mathfrak{a}^{\underline{\mathfrak{b}}} = |x^{\underline{y}}|$.
(3) $[x]^y$ is the set of all subsets of x which have the same cardinality as y; $[\mathfrak{a}]^{\mathfrak{b}} = |[x]^y|$.
(4) $\mathrm{seq}(x) = \bigcup_{n\in\omega} x^n$; $\mathrm{seq}(\mathfrak{a}) = |\mathrm{seq}(x)|$.
(5) $\mathrm{seq}^{1\text{-}1}(x) = \bigcup_{n\in\omega} x^{\underline{n}}$; $\mathrm{seq}^{1\text{-}1}(\mathfrak{a}) = |\mathrm{seq}^{1\text{-}1}(x)|$.
(6) $\mathrm{fin}(x) = \bigcup_{n\in\omega} [x]^n$; $\mathrm{fin}(\mathfrak{a}) = |\mathrm{fin}(x)|$.

Below we list some basic properties of these cardinals. We first note that $\mathrm{fin}(\mathfrak{a}) \leqslant^* \mathrm{seq}^{1\text{-}1}(\mathfrak{a}) \leqslant \mathrm{seq}(\mathfrak{a})$.

Fact 2.3. *For all cardinals* $\mathfrak{a}$, $\mathrm{seq}^{1\text{-}1}(\mathfrak{a}) \leqslant \mathrm{fin}(\mathrm{fin}(\mathfrak{a}))$.

Proof. For every set x, the function f defined on $\mathrm{seq}^{1\text{-}1}(x)$ given by $f(t) = \{t[n] \mid n \leqslant \mathrm{dom}(t)\}$ is an injection from $\mathrm{seq}^{1\text{-}1}(x)$ into $\mathrm{fin}(\mathrm{fin}(x))$. □

Lemma 2.4. *For all non-zero cardinals* $\mathfrak{a}$, $\mathrm{seq}(\mathrm{seq}(\mathfrak{a})) = \mathrm{seq}(\mathfrak{a})$.

Proof. Cf. [1, Lemma 2]. □

Lemma 2.5. *For all non-zero cardinals* $\mathfrak{a}$, $\mathrm{seq}(\mathfrak{a}) = \aleph_0 \cdot \mathrm{seq}^{1\text{-}1}(\mathfrak{a})$.

Proof. Cf. [4, Lemma 2.22]. □

Lemma 2.6. *For all infinite cardinals* $\mathfrak{a}$, $\aleph_0 \cdot \mathrm{seq}^{1\text{-}1}(\mathfrak{a}) \leqslant^* \mathrm{seq}^{1\text{-}1}(\mathfrak{a})$.

Proof. Let x be an infinite set. Let p be a bijection from $\omega \times \omega$ onto ω such that $n \leqslant p(m, n)$ for any $m, n \in \omega$. Let f be the function defined on $\mathrm{seq}^{1\text{-}1}(x)$ given by

$$f(t) = (m, t{\upharpoonright}n),$$

where $m, n \in \omega$ are such that $\mathrm{dom}(t) = p(m, n)$. It is easy to see that f is a surjection from $\mathrm{seq}^{1\text{-}1}(x)$ onto $\omega \times \mathrm{seq}^{1\text{-}1}(x)$. □

Proposition 2.7. *For all infinite cardinals* $\mathfrak{a}$,

$$\mathrm{seq}^{1\text{-}1}(\mathfrak{a}) =^* \mathrm{fin}(\mathrm{fin}(\mathfrak{a})) =^* \mathrm{fin}(\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))) =^* \cdots =^* \mathrm{seq}(\mathfrak{a}).$$

Proof. Immediately follows from Fact 2.3 and Lemmata 2.4, 2.5 and 2.6. □

Corollary 2.8. *For all infinite cardinals* $\mathfrak{a}$,

$$2^{\mathrm{seq}^{1\text{-}1}(\mathfrak{a})} = 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))} = 2^{\mathrm{fin}(\mathrm{fin}(\mathrm{fin}(\mathfrak{a})))} = \cdots = 2^{\mathrm{seq}(\mathfrak{a})}.$$

Proof. Immediately follows from Proposition 2.7. □

The following lemma will be used in Section 4.

Lemma 2.9. *For all cardinals* $\mathfrak{a}$ *and all* $n \in \omega$, $\mathfrak{a}^{\underline{2^n}} \leqslant \mathrm{fin}(\mathfrak{a})^{n+1}$.

Proof. Let x be an arbitrary set and let $n \in \omega$. Let f be the function defined on $x^{\underline{\wp(n)}}$ such that for all $t \in x^{\underline{\wp(n)}}$, $f(t)$ is the function on $n+1$ given by

$$f(t)(k) = \begin{cases} \{t(\varnothing)\}, & \text{if } k = n; \\ \{t(a) \mid a \subseteq n \text{ and } k \in a\}, & \text{otherwise.} \end{cases}$$

Clearly, $\operatorname{ran}(f) \subseteq \operatorname{fin}(x)^{n+1}$. It is easy to verify that for all $t \in x^{\underline{\wp(n)}}$, t is the function defined on $\wp(n)$ given by

$$t(a) = \begin{cases} \bigcup f(t)(n), & \text{if } a = \varnothing; \\ \bigcup\big(\bigcap_{k\in a} f(t)(k) \setminus \bigcup_{k\in n\setminus a} f(t)(k)\big), & \text{otherwise.} \end{cases}$$

Hence, f is an injection from $x^{\underline{\wp(n)}}$ into $\operatorname{fin}(x)^{n+1}$. □

3. The main theorem

In this section, we prove our main result which states that for all infinite cardinals $\mathfrak{a}$ and all natural numbers n,

$$2^{\operatorname{fin}(\mathfrak{a})^n} = 2^{[\operatorname{fin}(\mathfrak{a})]^n}.$$

The main idea of the proof is originally from [3].

Fix an arbitrary infinite set A and a non-zero natural number n. For a finite sequence $\langle x_1, \dots, x_n\rangle$ of length n, we write $\vec{x} = \langle x_1, \dots, x_n\rangle$ for short. For finite sequences $\vec{x} = \langle x_1, \dots, x_n\rangle$ and $\vec{y} = \langle y_1, \dots, y_n\rangle$, we introduce the following abbreviations: $\vec{x} \sqsubseteq \vec{y}$ means that $x_i \subseteq y_i$ for any $i = 1, \dots, n$; $\vec{x} \sqsubset \vec{y}$ means that $\vec{x} \sqsubseteq \vec{y}$ but $\vec{x} \neq \vec{y}$; $\vec{x} \sqcup \vec{y}$ denotes the finite sequence $\langle x_1 \cup y_1, \dots, x_n \cup y_n\rangle$; $\vec{x} \sqcap \vec{y}$ denotes the finite sequence $\langle x_1 \cap y_1, \dots, x_n \cap y_n\rangle$; $\vec{\varnothing}$ denotes the finite sequence $\langle \varnothing, \dots, \varnothing\rangle$ of length n. For an operator H and an $m \in \omega$, we write $H^{(m)}(X)$ for $H(H(\cdots H(X)\cdots))$ (m times), and if $m = 0$ then $H^{(0)}(X)$ is X itself.

Definition 3.1. For all natural numbers $k_1, \dots, k_n$ and $l_1, \dots, l_n$ such that $k_i \leqslant l_i$ for any $i = 1, \dots, n$, we introduce the following three functions:

(1) $F_{n,\vec{k},\vec{l}}$ is the function defined on $\wp([A]^{k_1} \times \cdots \times [A]^{k_n})$ given by
$$F_{n,\vec{k},\vec{l}}(X) = \big\{\vec{y} \in [A]^{l_1} \times \cdots \times [A]^{l_n} \bigm| \vec{x} \sqsubseteq \vec{y} \text{ for some } \vec{x} \in X\big\};$$

(2) $G_{n,\vec{k},\vec{l}}$ is the function defined on $\wp([A]^{k_1} \times \cdots \times [A]^{k_n})$ given by
$$G_{n,\vec{k},\vec{l}}(X) = \left\{\vec{x} \in [A]^{k_1} \times \cdots \times [A]^{k_n} \;\middle|\; \begin{array}{l} \text{for all } \vec{y} \in [A]^{l_1} \times \cdots \times [A]^{l_n} \\ \text{if } \vec{x} \sqsubseteq \vec{y} \text{ then } \vec{y} \in F_{n,\vec{k},\vec{l}}(X) \end{array}\right\};$$

(3) $H_{n,\vec{k},\vec{l}}$ is the function defined on $\wp([A]^{k_1} \times \cdots \times [A]^{k_n})$ given by
$$H_{n,\vec{k},\vec{l}}(X) = G_{n,\vec{k},\vec{l}}(X) \setminus X.$$

The proof of the following fact is easy and will be omitted.

Fact 3.2. *Let $k_1, \dots, k_n$ and $l_1, \dots, l_n$ be natural numbers such that $k_i \leqslant l_i$ for any $i = 1, \dots, n$.*

(i) *If $X \subseteq Y \subseteq [A]^{k_1} \times \cdots \times [A]^{k_n}$ then $F_{n,\vec{k},\vec{l}}(X) \subseteq F_{n,\vec{k},\vec{l}}(Y)$.*
(ii) *If $X \subseteq [A]^{k_1} \times \cdots \times [A]^{k_n}$ then $X \subseteq G_{n,\vec{k},\vec{l}}(X)$.*

(iii) *If* $X \subseteq Y \subseteq [A]^{k_1} \times \cdots \times [A]^{k_n}$ *then* $G_{n,\vec{k},\vec{l}}(X) \subseteq G_{n,\vec{k},\vec{l}}(Y)$.
(iv) *If* $X \subseteq [A]^{k_1} \times \cdots \times [A]^{k_n}$ *then* $G_{n,\vec{k},\vec{l}}(G_{n,\vec{k},\vec{l}}(X)) = G_{n,\vec{k},\vec{l}}(X)$.
(v) *If* $X \subseteq [A]^{k_1} \times \cdots \times [A]^{k_n}$ *then* $F_{n,\vec{k},\vec{l}}(G_{n,\vec{k},\vec{l}}(X)) = F_{n,\vec{k},\vec{l}}(X)$.
(vi) $F_{n,\vec{k},\vec{l}}$ *is injective on* $\{X \subseteq [A]^{k_1} \times \cdots \times [A]^{k_n} \mid G_{n,\vec{k},\vec{l}}(X) = X\}$.
(vii) *If* $X \subseteq [A]^{k_1} \times \cdots \times [A]^{k_n}$ *and* $m \in \omega$ *then*
$$H^{(m)}_{n,\vec{k},\vec{l}}(X) = G_{n,\vec{k},\vec{l}}(H^{(m)}_{n,\vec{k},\vec{l}}(X)) \setminus H^{(m+1)}_{n,\vec{k},\vec{l}}(X).$$
(viii) *Let* $l'_1, \dots, l'_n$ *be natural numbers such that* $l_i \leqslant l'_i$ *for any* $i = 1, \dots, n$. *If* $X \subseteq [A]^{k_1} \times \cdots \times [A]^{k_n}$ *then* $G_{n,\vec{k},\vec{l}}(X) \subseteq G_{n,\vec{k},\vec{l'}}(X)$, *and hence* $G_{n,\vec{k},\vec{l'}}(X) = X$ *implies that* $G_{n,\vec{k},\vec{l}}(X) = X$.

The key step of our proof is the following lemma.

Lemma 3.3. *For all natural numbers* $k_1, \dots, k_n$ *and* $l_1, \dots, l_n$ *such that* $k_i \leqslant l_i$ *for any* $i = 1, \dots, n$, *if* $X \subseteq [A]^{k_1} \times \cdots \times [A]^{k_n}$ *then*
$$H^{(k_1+\cdots+k_n+1)}_{n,\vec{k},\vec{l}}(X) = \varnothing.$$

Before we prove Lemma 3.3, we use it to prove our main theorem.

Theorem 3.4. *For all infinite cardinals* $\mathfrak{a}$ *and all natural numbers* n,
$$2^{\mathrm{fin}(\mathfrak{a})^n} = 2^{[\mathrm{fin}(\mathfrak{a})]^n}.$$

Proof. Let A be an infinite set such that $|A| = \mathfrak{a}$. The case $n = 0$ is obvious. So assume that n is a non-zero natural number. For all natural numbers $k_1, \dots, k_n, m$, let $s(\vec{k}, m)$ be the finite sequence
$$\langle p_1^{k_1} \cdots p_n^{k_n} p_{n+1}^m p_{n+2}^i \rangle_{1 \leqslant i \leqslant n}$$
where p_j is the j-th prime number, and let $t(\vec{k}) = s(\vec{k}, k_1 + \cdots + k_n)$.

For all $X \subseteq \mathrm{fin}(A)^n$ and all natural numbers $k_1, \dots, k_n, m$, we define
$$X_{\vec{k}} = X \cap ([A]^{k_1} \times \cdots \times [A]^{k_n});$$
$$Y_{\vec{k},m} = G_{n,\vec{k},t(\vec{k})}(H^{(m)}_{n,\vec{k},t(\vec{k})}(X_{\vec{k}}));$$
$$Z_{\vec{k},m} = F_{n,\vec{k},s(\vec{k},m)}(Y_{\vec{k},m}).$$
Notice that for any finite sequence $\vec{x} = \langle x_1, \dots, x_n \rangle$, $\mathrm{ran}(\vec{x}) = \{x_1, \dots, x_n\}$. Now, let Φ be the function defined on $\wp(\mathrm{fin}(A)^n)$ given by
$$\Phi(X) = \big\{\mathrm{ran}(\vec{y}) \bigm| \exists k_1, \dots, k_n, m \in \omega \big(m \leqslant k_1 + \cdots + k_n \text{ and } \vec{y} \in Z_{\vec{k},m}\big)\big\}.$$
We claim that Φ is an injection from $\wp(\mathrm{fin}(A)^n)$ into $\wp([\mathrm{fin}(A)]^n)$.

Let $X \subseteq \mathrm{fin}(A)^n$. For all $\vec{y} = \langle y_1, \dots, y_n \rangle \in Z_{\vec{k},m}$, it is easy to see that $|y_i| = p_1^{k_1} \cdots p_n^{k_n} p_{n+1}^m p_{n+2}^i$ for any $i = 1, \dots, n$, and thus $|y_1| < \cdots < |y_n|$, which implies that $\mathrm{ran}(\vec{y}) \in [\mathrm{fin}(A)]^n$. Hence $\Phi(X) \subseteq [\mathrm{fin}(A)]^n$. Moreover, X is uniquely determined by $\Phi(X)$ in the following way:

First, for all natural numbers $k_1,\dots,k_n,m$ such that $m\leqslant k_1+\dots+k_n$, $Z_{\vec{k},m}$ is uniquely determined by $\Phi(X)$:

$$Z_{\vec{k},m}=\bigl\{\vec{y}\in[A]^{l_1}\times\dots\times[A]^{l_n}\bigm|\operatorname{ran}(\vec{y})\in\Phi(X)\bigr\},$$

where $l_i=p_1^{k_1}\cdots p_n^{k_n}p_{n+1}^{m}p_{n+2}^{i}$ for any $i=1,\dots,n$.

Then, for all natural numbers $k_1,\dots,k_n,m$ such that $m\leqslant k_1+\dots+k_n$, by Fact 3.2(iv)(vi)(viii), $Y_{\vec{k},m}$ is the unique subset of $[A]^{k_1}\times\dots\times[A]^{k_n}$ such that $G_{n,\vec{k},t(\vec{k})}(Y_{\vec{k},m})=Y_{\vec{k},m}$ and $F_{n,\vec{k},s(\vec{k},m)}(Y_{\vec{k},m})=Z_{\vec{k},m}$, which implies that $Y_{\vec{k},m}$ is uniquely determined by $\Phi(X)$.

Now, for all natural numbers $k_1,\dots,k_n$, it follows from Fact 3.2(vii) and Lemma 3.3 that

$$X_{\vec{k}}=Y_{\vec{k},0}\setminus(Y_{\vec{k},1}\setminus(\cdots(Y_{\vec{k},k_1+\dots+k_n-1}\setminus Y_{\vec{k},k_1+\dots+k_n})\cdots)),$$

and thus $X_{\vec{k}}$ is uniquely determined by $\Phi(X)$.

Finally, since

$$X=\bigcup_{k_1,\dots,k_n\in\omega}X_{\vec{k}},$$

it follows that X is also uniquely determined by $\Phi(X)$.

Hence, Φ is an injection from $\wp(\operatorname{fin}(A)^n)$ into $\wp([\operatorname{fin}(A)]^n)$, and thus $2^{\operatorname{fin}(\mathfrak{a})^n}\leqslant 2^{[\operatorname{fin}(\mathfrak{a})]^n}$. Since $[\operatorname{fin}(\mathfrak{a})]^n\leqslant^*\operatorname{fin}(\mathfrak{a})^n$, it follows that $2^{[\operatorname{fin}(\mathfrak{a})]^n}\leqslant 2^{\operatorname{fin}(\mathfrak{a})^n}$, and thus $2^{\operatorname{fin}(\mathfrak{a})^n}=2^{[\operatorname{fin}(\mathfrak{a})]^n}$ follows from the Cantor–Bernstein theorem. □

We still have to prove Lemma 3.3. To this end, we need the following version of Ramsey's theorem, whose proof will be omitted.

Lemma 3.5. *Let n be a non-zero natural number. There exists a function R defined on $\omega^n\times(\omega\setminus\{0\})\times\omega$ such that for all natural numbers $j_1,\dots,j_n,c,r$ with $c>0$ and all finite sets $S_1,\dots,S_n,Y_1,\dots,Y_c$, if $|S_i|\geqslant R(j_1,\dots,j_n,c,r)$ for any $i=1,\dots,n$ and*

$$[S_1]^{j_1}\times\dots\times[S_n]^{j_n}=Y_1\cup\dots\cup Y_c,$$

then for each $i=1,\dots,n$ there exist a $T_i\in[S_i]^r$ such that

$$[T_1]^{j_1}\times\dots\times[T_n]^{j_n}\subseteq Y_d$$

for some $d=1,\dots,c$.

Proof of Lemma 3.3. Let A be an arbitrary infinite set and n a non-zero natural number. Let $k_1,\dots,k_n$ and $l_1,\dots,l_n$ be natural numbers such that $k_i\leqslant l_i$ for any $i=1,\dots,n$. Since in this proof the natural numbers $n,k_1,\dots,k_n,l_1,\dots,l_n$ are fixed, we shall omit the subscripts in $F_{n,\vec{k},\vec{l}}$, $G_{n,\vec{k},\vec{l}}$ and $H_{n,\vec{k},\vec{l}}$ for convenience.

Consider the following two formulae:

$\phi(X,\vec{x},\vec{y})$**:** $X\subseteq[A]^{k_1}\times\cdots\times[A]^{k_n}$ and $\vec{x},\vec{y}\in\operatorname{fin}(A)^n$ are such that $|x_i|\leqslant k_i$ for any $i=1,\ldots,n$, such that $\vec{x}\sqcap\vec{y}=\vec{\varnothing}$, and such that $\vec{x}\sqcup\vec{z}\in X$ for any $\vec{z}\in[y_1]^{k_1-|x_1|}\times\cdots\times[y_n]^{k_n-|x_n|}$.

$\psi(X,\vec{x})$**:** For all $r\in\omega$ there exists a $\vec{y}\in([A]^r)^n$ such that $\phi(X,\vec{x},\vec{y})$.

We claim that for all $X\subseteq[A]^{k_1}\times\cdots\times[A]^{k_n}$ and all $\vec{x}\in\operatorname{fin}(A)^n$,

$$\text{if } \psi(H(X),\vec{x}) \text{ then } \psi(X,\vec{u}) \text{ for some } \vec{u}\sqsubset\vec{x}. \tag{2}$$

Once we prove (2), we finish the proof of Lemma 3.3 as follows. Assume towards a contradiction that $X\subseteq[A]^{k_1}\times\cdots\times[A]^{k_n}$ and there exists an $\vec{x}\in H^{(k_1+\cdots+k_n+1)}(X)$. It is obvious that $\psi(H^{(k_1+\cdots+k_n+1)}(X),\vec{x})$. Now, by repeatedly applying (2), we get a descending sequence

$$\vec{x}\sqsupset\vec{u}_1\sqsupset\cdots\sqsupset\vec{u}_{k_1+\cdots+k_n+1},$$

which is absurd, since $\vec{x}\in[A]^{k_1}\times\cdots\times[A]^{k_n}$.

Now, let us prove (2). Let $X\subseteq[A]^{k_1}\times\cdots\times[A]^{k_n}$ and let $\vec{x}\in\operatorname{fin}(A)^n$ be such that $\psi(H(X),\vec{x})$. It suffices to prove that

$$\forall r\geqslant l_1+\cdots+l_n\,\exists\vec{u}\sqsubset\vec{x}\,\exists\vec{y}\in([A]^r)^n\,\phi(X,\vec{u},\vec{y}), \tag{3}$$

since then there must be a $\vec{u}\sqsubset\vec{x}$ such that for infinitely many $r\in\omega$ there exists a $\vec{y}\in([A]^r)^n$ such that $\phi(X,\vec{u},\vec{y})$, and for this $\vec{u}$ we have $\psi(X,\vec{u})$.

We prove (3) as follows. Let $r\geqslant l_1+\cdots+l_n$. Let R be the function whose existence is asserted by Lemma 3.5. We define

$$\begin{aligned} r' &= \max\{R(j_1,\ldots,j_n,2,r)\mid j_i\leqslant k_i \text{ for any } i=1,\ldots,n\};\\ r'' &= R(l_1-|x_1|,\ldots,l_n-|x_n|,2^{|x_1|+\cdots+|x_n|},r'). \end{aligned}$$

Since $\psi(H(X),\vec{x})$, we can find an $\vec{S}=\langle S_1,\ldots,S_n\rangle\in([A]^{r''})^n$ such that $\phi(H(X),\vec{x},\vec{S})$. Notice that $\vec{x}\sqcap\vec{S}=\vec{\varnothing}$. For each $\vec{u}\sqsubseteq\vec{x}$, let

$$Y_{\vec{u}}=\big\{\vec{w}\in[S_1]^{l_1-|x_1|}\times\cdots\times[S_n]^{l_n-|x_n|}\;\big|\;\vec{u}\sqcup\vec{v}\in X \text{ for some } \vec{v}\sqsubseteq\vec{w}\big\}.$$

We claim that

$$[S_1]^{l_1-|x_1|}\times\cdots\times[S_n]^{l_n-|x_n|}=\bigcup\{Y_{\vec{u}}\mid\vec{u}\sqsubseteq\vec{x}\}. \tag{4}$$

Let $\vec{w}\in[S_1]^{l_1-|x_1|}\times\cdots\times[S_n]^{l_n-|x_n|}$. Take a $\vec{z}\in[S_1]^{k_1-|x_1|}\times\cdots\times[S_n]^{k_n-|x_n|}$ such that $\vec{z}\sqsubseteq\vec{w}$. Then it follows from $\phi(H(X),\vec{x},\vec{S})$ that $\vec{x}\sqcup\vec{z}\in H(X)$, and thus $\vec{x}\sqcup\vec{z}\in G(X)$. Since $\vec{x}\sqcup\vec{z}\sqsubseteq\vec{x}\sqcup\vec{w}\in[A]^{l_1}\times\cdots\times[A]^{l_n}$, it follows that $\vec{x}\sqcup\vec{w}\in F(X)$, and hence $\vec{a}\sqsubseteq\vec{x}\sqcup\vec{w}$ for some $\vec{a}\in X$. Now, if we take $\vec{u}=\vec{a}\sqcap\vec{x}$ and $\vec{v}=\vec{a}\sqcap\vec{w}$, then we have $\vec{u}\sqcup\vec{v}=\vec{a}\in X$ and hence $\vec{w}\in Y_{\vec{u}}$.

By (4) and Lemma 3.5, we can find a $\vec{u} = \langle u_1, \dots, u_n \rangle \sqsubseteq \vec{x}$ such that for each $i = 1, \dots, n$ there exist a $T_i \in [S_i]^{r'}$ such that

$$[T_1]^{l_1-|x_1|} \times \cdots \times [T_n]^{l_n-|x_n|} \subseteq Y_{\vec{u}}. \tag{5}$$

Let

$$Z = \big\{\vec{v} \in [T_1]^{k_1-|u_1|} \times \cdots \times [T_n]^{k_n-|u_n|} \bigm| \vec{u} \sqcup \vec{v} \in X\big\}.$$

Since $|T_i| = r' \geqslant R(k_1-|u_1|, \dots, k_n-|u_n|, 2, r)$ for any $i = 1, \dots, n$, it follows from Lemma 3.5 that we can find a $\vec{y} = \langle y_1, \dots, y_n \rangle$ such that $y_i \in [T_i]^r$ for any $i = 1, \dots, n$, and such that either

$$[y_1]^{k_1-|u_1|} \times \cdots \times [y_n]^{k_n-|u_n|} \subseteq Z \tag{6}$$

or

$$([y_1]^{k_1-|u_1|} \times \cdots \times [y_n]^{k_n-|u_n|}) \cap Z = \varnothing. \tag{7}$$

We claim that (7) is impossible. Since $|y_i| = r \geqslant l_i \geqslant l_i - |x_i|$ for any $i = 1, \dots, n$, there is a $\vec{w} \in [y_1]^{l_1-|x_1|} \times \cdots \times [y_n]^{l_n-|x_n|}$, and thus it follows from (5) that $\vec{w} \in Y_{\vec{u}}$, which implies that $\vec{u} \sqcup \vec{v} \in X$ for some $\vec{v} \sqsubseteq \vec{w}$ and such a $\vec{v}$ is in $([y_1]^{k_1-|u_1|} \times \cdots \times [y_n]^{k_n-|u_n|}) \cap Z$. Therefore (6) must hold, from which $\phi(X, \vec{u}, \vec{y})$ follows.

It remains to show that $\vec{u} \neq \vec{x}$. Since $\phi(H(X), \vec{x}, \vec{S})$ and $\vec{y} \sqsubseteq \vec{S}$, it follows that $\phi(H(X), \vec{x}, \vec{y})$. If $\vec{u} = \vec{x}$, then we also have $\phi(X, \vec{x}, \vec{y})$, which is impossible: Since $|y_i| = r \geqslant l_i \geqslant k_i \geqslant k_i - |x_i|$ for any $i = 1, \dots, n$, there is a $\vec{z} \in [y_1]^{k_1-|x_1|} \times \cdots \times [y_n]^{k_n-|x_n|}$, and for such a $\vec{z}$, we cannot have both $\vec{x} \sqcup \vec{z} \in H(X)$ and $\vec{x} \sqcup \vec{z} \in X$. □

4. Consistency results

In this section, we establish some consistency results by the method of permutation models. Permutation models are not models of ZF; they are models of ZFA (the Zermelo-Fraenkel set theory with atoms). Nevertheless, they indirectly give, via the Jech–Sochor theorem (cf. [2, Theorem 17.2]), models of ZF.

For our purpose, we only consider the basic Fraenkel model $\mathcal{V}_{\mathrm{F}}$ (cf. [2, pp. 195–196]). The set A of atoms of $\mathcal{V}_{\mathrm{F}}$ is denumerable, and $x \in \mathcal{V}_{\mathrm{F}}$ if and only if $x \subseteq \mathcal{V}_{\mathrm{F}}$ and x has a *finite support*, that is, a set $B \in \mathrm{fin}(A)$ such that every permutation of A fixing B pointwise also fixes x.

Lemma 4.1. *Let A be the set of atoms of $\mathcal{V}_{\mathrm{F}}$ and let $\mathfrak{a} = |A|$. In $\mathcal{V}_{\mathrm{F}}$,*

$$2^{\mathsf{fin}(\mathfrak{a})} < 2^{\mathsf{fin}(\mathfrak{a})^2} < 2^{\mathsf{fin}(\mathfrak{a})^3} < \cdots < 2^{\mathsf{fin}(\mathsf{fin}(\mathfrak{a}))}.$$

Proof. Let $n \in \omega$. We claim that in $\mathcal{V}_{\mathrm{F}}$,

$$2^{\mathfrak{a}^{\underline{2n}}} \nleqslant 2^{\mathrm{fin}(\mathfrak{a})^n}. \tag{8}$$

Assume towards a contradiction that there exists an injection $f \in \mathcal{V}_{\mathrm{F}}$ from $\wp(A^{\underline{2n}})$ into $\wp(\mathrm{fin}(A)^n)$. Let B be a finite support of f. Take an arbitrary $C \in [A \setminus B]^{2^n+1}$ and a $u \in C^{\underline{2n}}$. We say that a permutation π of A is *even* (*odd*) if π moves only elements of C and can be written as a product of an even (odd) number of transpositions. It is well-known that a permutation of A cannot be both even and odd. Now, let

$$\mathcal{E} = \{\pi(u) \mid \pi \text{ is an even permutation of } A\},$$

and let

$$\mathcal{O} = \{\sigma(u) \mid \sigma \text{ is an odd permutation of } A\}.$$

Clearly, $\{\mathcal{E}, \mathcal{O}\}$ is a partition of $C^{\underline{2n}}$, for all even permutations π of A we have $\pi(\mathcal{E}) = \mathcal{E}$, and for all odd permutations σ of A we have $\sigma(\mathcal{E}) = \mathcal{O}$. Now, let us consider $f(\mathcal{E})$. For each $t \in f(\mathcal{E})$, let $\sim_t$ be the equivalence relation on C such that for all $a, b \in C$,

$$a \sim_t b \quad \text{if and only if} \quad \forall k < n \big(a \in t(k) \leftrightarrow b \in t(k)\big).$$

For all even permutations π of A, since B is a finite support of f, it follows that $\pi(f) = f$, and thus $\pi(f(\mathcal{E})) = f(\mathcal{E})$. For all odd permutations σ of A and all $t \in f(\mathcal{E})$, since $|C/{\sim_t}| \leqslant 2^n$ and $|C| = 2^n + 1$, there are $a, b \in C$ such that $a \neq b$ and $a \sim_t b$, and therefore the transposition τ that swaps a and b fixes t, which implies that $\sigma(t) = (\sigma \circ \tau)(t) \in f(\mathcal{E})$ since $\sigma \circ \tau$ is even. Hence, for all odd permutations σ of A, $\sigma(f(\mathcal{E})) = f(\mathcal{E})$, which implies that $f(\mathcal{O}) = f(\sigma(\mathcal{E})) = \sigma(f(\mathcal{E})) = f(\mathcal{E})$, contradicting the injectivity of f.

Now, it follows from Lemma 2.9 that $\mathfrak{a}^{\underline{2n}} \leqslant \mathrm{fin}(\mathfrak{a})^{n+1}$, and therefore $2^{\mathfrak{a}^{\underline{2n}}} \leqslant 2^{\mathrm{fin}(\mathfrak{a})^{n+1}}$, which implies that $2^{\mathrm{fin}(\mathfrak{a})^n} < 2^{\mathrm{fin}(\mathfrak{a})^{n+1}}$ by (8). It follows from Theorem 3.4 that $2^{\mathrm{fin}(\mathfrak{a})^n} = 2^{[\mathrm{fin}(\mathfrak{a})]^n} \leqslant 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}$. Hence

$$2^{\mathrm{fin}(\mathfrak{a})} < 2^{\mathrm{fin}(\mathfrak{a})^2} < 2^{\mathrm{fin}(\mathfrak{a})^3} < \cdots < 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}. \qquad \square$$

Now the following proposition immediately follows from Lemma 4.1 and the Jech–Sochor theorem.

Proposition 4.2. *The following statement is consistent with* ZF*: there is an infinite cardinal* $\mathfrak{a}$ *such that*

$$2^{\mathrm{fin}(\mathfrak{a})} < 2^{\mathrm{fin}(\mathfrak{a})^2} < 2^{\mathrm{fin}(\mathfrak{a})^3} < \cdots < 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}.$$

It is natural to wonder whether the conclusion of Theorem 3.4 can be strengthened to $\mathrm{fin}(\mathfrak{a})^n \leqslant^* [\mathrm{fin}(\mathfrak{a})]^n$. We shall give a negative answer to this question. The case $n = 1$ of the following lemma is proved in [5].

Lemma 4.3. *Let A be the set of atoms of $\mathcal{V}_{\mathrm{F}}$. In $\mathcal{V}_{\mathrm{F}}$, for every $n \in \omega$, $\operatorname{fin}(A)^n$ is dually Dedekind finite; that is, every surjection from $\operatorname{fin}(A)^n$ onto $\operatorname{fin}(A)^n$ is injective.*

Proof. Let $n \in \omega$. Take an arbitrary surjection $f \in \mathcal{V}_{\mathrm{F}}$ from $\operatorname{fin}(A)^n$ onto $\operatorname{fin}(A)^n$. In order to prove the injectivity of f, it suffices to show that

$$\text{for all } t \in \operatorname{fin}(A)^n \text{ there is an } m > 0 \text{ such that } f^{(m)}(t) = t. \tag{9}$$

Let B be a finite support of f. For each $t \in \operatorname{fin}(A)^n$, let $\sim_t$ be the equivalence relation on $A \setminus B$ such that for all $a, b \in A \setminus B$,

$$a \sim_t b \quad \text{if and only if} \quad \forall k < n \bigl(a \in t(k) \leftrightarrow b \in t(k)\bigr).$$

Let $\sqsubseteq$ be the preorder on $\operatorname{fin}(A)^n$, such that for all $t, u \in \operatorname{fin}(A)^n$,

$$t \sqsubseteq u \quad \text{if and only if} \quad {\sim_u} \subseteq {\sim_t}.$$

Claim 4.4. *There is an $l \in \omega$ such that every $\sqsubseteq$-chain without repetition must have length less than l.*

Proof of Claim 4.4. We first prove that for all $u \in \operatorname{fin}(A)^n$,

$$\bigl|\bigl\{t \in \operatorname{fin}(A)^n \bigm| {\sim_t} = {\sim_u}\bigr\}\bigr| \leqslant 2^{(|B|+2^n)\cdot n}. \tag{10}$$

Let $u \in \operatorname{fin}(A)^n$. Let g be the function defined on $\operatorname{fin}(A)^n$ such that for all $t \in \operatorname{fin}(A)^n$, $g(t)$ is the function on n given by

$$g(t)(k) = \bigl(t(k) \cap B,\, \bigl\{w \in (A \setminus B)/{\sim_u} \bigm| w \subseteq t(k)\bigr\}\bigr).$$

Clearly, $\operatorname{ran}(g) \subseteq \bigl(\wp(B) \times \wp((A \setminus B)/{\sim_u})\bigr)^n$. It is also easy to see that $g{\restriction}\{t \in \operatorname{fin}(A)^n \mid {\sim_t} = {\sim_u}\}$ is injective. Since $|(A \setminus B)/{\sim_u}| \leqslant 2^n$, we have

$$\bigl|\bigl\{t \in \operatorname{fin}(A)^n \bigm| {\sim_t} = {\sim_u}\bigr\}\bigr| \leqslant \bigl|\bigl(\wp(B) \times \wp((A \setminus B)/{\sim_u})\bigr)^n\bigr| \leqslant 2^{(|B|+2^n)\cdot n}.$$

For each $t \in \operatorname{fin}(A)^n$, let $k_t = |(A \setminus B)/{\sim_t}|$. Clearly, for all $t, u \in \operatorname{fin}(A)^n$ such that $t \sqsubseteq u$, we have $0 < k_t \leqslant k_u \leqslant 2^n$, and if $k_t = k_u$ then ${\sim_t} = {\sim_u}$. Thus, by (10), every $\sqsubseteq$-chain without repetition must have length less than or equal to $2^{(|B|+2^n)\cdot n} \cdot 2^n$. Now, it suffices to take $l = 2^{(|B|+2^n+1)\cdot n} + 1$. □

Claim 4.5. *For all $u \in \operatorname{fin}(A)^n$ we have $f(u) \sqsubseteq u$.*

Proof of Claim 4.5. Assume towards a contradiction that ${\sim_u} \nsubseteq {\sim_{f(u)}}$ for some $u \in \operatorname{fin}(A)^n$. Let $a, b \in A \setminus B$ be such that $a \sim_u b$ but not $a \sim_{f(u)} b$. Clearly $a \neq b$. Let τ be the transposition that swaps a and b. Then $\tau(u) = u$ but $\tau(f(u)) \neq f(u)$, contradicting that B is a finite support of f. □

We prove (9) as follows. Let $t \in \mathrm{fin}(A)^n$. By Claim 4.4, there is an $l \in \omega$ such that every $\sqsubseteq$-chain without repetition must have length less than l. Let h be a function from l into $\mathrm{fin}(A)^n$, such that $h(0) = t$ and for all $i < l$ if $i + 1 < l$ then $h(i) = f(h(i + 1))$. Such an h exists since f is surjective. Clearly, for all $i < l$, $f^{(i)}(h(i)) = t$. By Claim 4.5, h is a $\sqsubseteq$-chain, and since the length of h is l, we can find $i, j < l$ such that $i < j$ and $h(i) = h(j)$. Now, if we take $m = j - i$, then we have $m > 0$ and

$$f^{(m)}(t) = f^{(j-i)}(t) = f^{(j-i)}(f^{(i)}(h(i))) = f^{(j)}(h(j)) = t. \qquad \square$$

Now the following proposition immediately follows from Lemma 4.3 and the Jech–Sochor theorem.

Proposition 4.6. *The following statement is consistent with* ZF*: there is an infinite set* A *such that* $\mathrm{fin}(A)^n$ *is dually Dedekind finite for any* $n \in \omega$*.*

Corollary 4.7. *The following statement is consistent with* ZF*: there exists an infinite cardinal* $\mathfrak{a}$ *such that* $\mathrm{fin}(\mathfrak{a})^n \nleqslant^* [\mathrm{fin}(\mathfrak{a})]^n$ *for any* $n \geqslant 2$*.*

Proof. Notice that for all infinite sets A and all natural numbers $n \geqslant 2$, there exists a non-injective surjection from $\mathrm{fin}(A)^n$ onto $[\mathrm{fin}(A)]^n$. Hence, this corollary follows from Proposition 4.6. $\square$

We conclude this paper with two open problems.

Question 4.8. *Is it provable in* ZF *that* $2^{2^{\mathrm{fin}(\mathfrak{a})}} = 2^{2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}}$ *for any infinite cardinal* $\mathfrak{a}$*?*

Notice that Proposition 4.2 shows that $2^{\mathrm{fin}(\mathfrak{a})} = 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}$ cannot be proved in ZF for an arbitrary infinite cardinal $\mathfrak{a}$.

Question 4.9. *Does* ZF *prove that* $2^{2^{\mathfrak{a}}} = 2^{2^{\mathfrak{a}+1}}$ *for any infinite cardinal* $\mathfrak{a}$*?*

Notice that for all Dedekind finite cardinals $\mathfrak{a}$ we have $\mathfrak{a} < \mathfrak{a} + 1$, and for all power Dedekind finite cardinals $\mathfrak{a}$ (i.e., cardinals $\mathfrak{a}$ such that $2^{\mathfrak{a}}$ is Dedekind finite) we have $2^{\mathfrak{a}} < 2^{\mathfrak{a}+1}$.

Question 4.9 is asked in [3] (cf. also [2, p. 132]). Notice that, in [3], Läuchli proves in ZF that for all infinite cardinals $\mathfrak{a}$,

$$2^{2^{\mathfrak{a}}} = 2^{2^{\mathfrak{a}}+1}.$$

Acknowledgements. I would like to give thanks to Professor Qi Feng for his advice and encouragement during the preparation of this paper.

References

[1] E. Ellentuck, *Generalized idempotence in cardinal arithmetic*, Fund. Math. 58 (1966), 241–258.

[2] L. Halbeisen, *Combinatorial Set Theory: With a Gentle Introduction to Forcing*, 2nd ed., Springer Monogr. Math., Springer, Cham, 2017.

[3] H. Läuchli, *Ein Beitrag zur Kardinalzahlarithmetik ohne Auswahlaxiom*, Z. Math. Log. Grundl. Math. 7 (1961), 141–145.

[4] G. Shen and J. Yuan, *Factorials of infinite cardinals in* ZF*, Part I:* ZF *results*, J. Symb. Log. (2019), to appear. https://doi.org/10.1017/jsl.2019.74

[5] J. Truss, *Classes of Dedekind finite cardinals*, Fund. Math. 84 (1974), 187–208.

Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China

School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People's Republic of China

E-mail address: shen_guozhen@outlook.com