arXiv:1201.5799v2 [math.LO] 28 Aug 2013

WITNESSING DP-RANK

ITAY KAPLAN AND PIERRE SIMON

Abstract. We prove that in NTP_2 theories if p is a dependent type with dp-rank $\geq \kappa$, then this can be witnessed by indiscernible sequences of tuples satisfying p. If p has dp-rank infinity, then this can be witnessed by singletons (in any theory).

1. Introduction

In this note we answer a question of Alf Onshuus and Alexander Usvyatsov, whether dp-minimality can be witnessed by indiscernible sequences of singletons. We prove two general theorems regarding dp-rank.

Let $Card$ denote the class of cardinals. We define $Card^*$ to be the class $Card$ to which we add an element κ_- for each infinite cardinal κ. We extend the linear order from $Card$ to $Card^*$ by setting $\mu < \kappa_- < \kappa$ whenever $\mu < \kappa$ are cardinals.

Definition 1.1. Let $p(x)$ be a partial (consistent) type over a set A (x is a finite tuple, here and throughout the paper). We define the *dp-rank* of $p(x)$ (which is an element of $Card^*$ or ∞) as follows:

- Let κ be a cardinal. We will say that $p(x)$ has dp-rank $< \kappa$ (which we write $\text{rk-dp}(p) < \kappa$) if given any realization a of p and any κ mutually indiscernible sequences over A, at least one of them is indiscernible over Aa.
- We say that p has dp-rank κ over A (or $\text{rk-dp}(p) = \kappa$) if it has dp-rank $< \mu$ for all $\mu > \kappa$, but it is not the case that $\text{rk-dp}(p) < \kappa$.
- If κ is an infinite cardinal, we say that p has dp-rank κ_- over A (or $\text{rk-dp}(p) = \kappa_-$) if it has dp-rank $< \kappa$, but for no $\mu < \kappa$ do we have $\text{rk-dp}(p) < \mu$.
- If $\text{rk-dp}(p) < \kappa$ holds for no cardinal κ, then we say that p has dp-rank ∞.
- We call p *dp-minimal* if it has dp-rank 1.
- We call p *dependent* if $\text{rk-dp}(p) < \infty$. This is equivalent to $\text{rk-dp}(p) < |T|^+$ (see Corollary 2.3).

Remark 1.2. It is easy to see that the set A does not matter, as long as p is defined over it. Indeed, for a set B over which p is defined, let us define for the sake of discussion $\text{rk-dp}(p, B)$ as the dp-rank of p over B similarly to the definition above but we add the requirement that the

The first author's research was partially supported by the SFB 878 grant.

sequences are mutually indiscernible over B. If $A \subseteq B$, and p is a type over A, then it is easy to see that $\text{rk-dp}(p, B) \leq \text{rk-dp}(p, A)$ while the other direction uses a standard application of Ramsey theorem, so $\text{rk-dp}(p, B) = \text{rk-dp}(p, A)$.

Note also that if $q(x)$ extends $p(x)$ then $\text{rk-dp}(p(x)) \geq \text{rk-dp}(q(x))$, so:

Remark 1.3. Any extension of a dependent type is dependent.

Recall:

Definition 1.4. A (complete, first order) theory T is *dp-minimal* if the type $\{x = x\}$ is dp-minimal. The theory T is *dependent* if the type $\{x = x\}$ is dependent.

Dp-rank and dependent types were originally defined in [Usv07] and further studied in [OU11]. Dp-rank is a simplification of the various ranks appearing in [She12]. We use a slightly different convention for it than those two papers which has the advantage of distinguishing between κ and κ_-. Yet another convention is used in [KOU11] which has the disadvantage of giving a different meaning to $\text{rk-dp}(p) = \kappa$ depending on whether κ is finite or infinite. Dp-minimality was first defined in [OU11]. It is shown in [Sim11] that the original definition of dp-minimality is equivalent to the definition given here.

Examples of dp-minimal theories include all o-minimal theories and C-minimal theories.

Note that the sequences that witness $\text{rk-dp}(p) \geq \kappa$ in Definition 1.1 can always be taken to be sequences of finite tuples, but can we bound the length?

Question. *(A. Onshuus, A. Usvyatsov) Can we assume in the definition of dp-minimality that the indiscernible sequences are sequences of singletons?*

We provide a positive answer in Corollary 1.7 below, but we need to add parameters to the base.

We prove the following two theorems:

Main Theorem A. *If p is a type over A which is independent (i.e. $\text{rk-dp}(p) = \infty$), then there is some $A' \supseteq A$ such that $|A' \backslash A|$ is finite, a realization $a \models p$ and A'-mutually indiscernible sequences of* <u>*singletons*</u> $\left\langle I_i \,\middle|\, i < |T|^+ + |A|^+ \right\rangle$ *such that I_i is not indiscernible over $A'a$ for all i.*

From this we will deduce:

Corollary 1.5. *To check whether a theory is dependent it is enough to check that for every indiscernible sequence of singletons $\left\langle a_i \,\middle|\, i < |T|^+ \right\rangle$ over some finite A, and for every singleton c, there is $\alpha < |T|^+$ such that $\langle a_i \,|\, i > \alpha \rangle$ is indiscernible over Ac.*

The second result is about dependent types, but to prove it we need to assume[1] that the theory is NTP_2.

Definition 1.6. A theory T is NTP_2 *(does not have the tree property of the second kind)* if there is no formula $\varphi(x, y)$ and array $\langle a_{i,j} \mid i, j < \omega \rangle$ such that for every $i < \omega$, $\{\varphi(x, a_{i,j}) \mid j < \omega\}$ is k-inconsistent (i.e. each subset of size k is inconsistent) and for every $\eta : \omega \to \omega$, the set $\left\{\varphi\left(x, a_{i,\eta(i)}\right) \mid i < \omega\right\}$ is consistent.

The class of NTP_2 theories contains both simple and dependent theories.

Main Theorem B. *Assume T is NTP_2, and that p is a dependent type over A with $\text{rk-dp}(p) \geq \kappa$. Then there is some $A' \supseteq A$, some $a \models p$ and A'-mutually indiscernible sequences $\{I_i \mid i < \kappa\}$ such that each of them is not indiscernible over $A'a$ <u>and</u> all tuples in each I_i satisfy p.*

Note that we may always choose A' so that $|A' \setminus A|$ is at most $\kappa + \aleph_0$ since, for each sequence I_i, we only need finitely many parameters from A' to witness that I_i is not indiscernible over $A'a$.

Now we can answer Question 1:

Corollary 1.7. *If T is not dp-minimal, then there is some finite set A', some singleton a and two A'-mutually indiscernible sequences $\{I, J\}$ <u>of singletons</u> such that both I and J are not indiscernible over $A'a$.*

Proof. Right to left is obvious. For the other direction, if T is dependent then we may use Main Theorem B (since there are only two sequences, only finitely many parameters from A' are needed to witness non-indiscernibility, so we may assume that A' is finite). But if T is not dependent, then by Main Theorem A there exists such a, A and infinitely many such sequences. $\square$

The following question remains open:

Question 1.8. *(J. Ramakrishnan) Can we assume in the definition of dp-rank that the indiscernible sequences are sequences of singletons by adding parameters to the base?*

Our results show that this is indeed the case when the type is independent or when it is the type of a singleton in an NTP_2 theory.

In Section 2 we prove Main Theorem A, and in Section 3 we prove Main Theorem B.

Question 1.9. *Are the extra parameters in the Main Theorems needed?*

Throughout the paper, $\mathfrak{C}$ will denote a monster model of the theory T (i.e. a very big saturated model).

---

[1] After the appearance of this note, Artem Chernikov has removed this assumption, see [Che12].

2. On dependent types and a proof of Main Theorem A

2.1. **On dependent types.** We start with the following easy observation (which is somewhat similar to [OU11, Observation 2.7]), with a very straightforward proof.

Claim 2.1. Suppose $p(x)$ is a partial type over A. Then the following are equivalent:

(1) There is $a \models p$ and A-mutually indiscernible sequences $\langle I_i \mid i < \omega \rangle$ such that the sequence $\langle I_i \mid i < \omega \rangle$ is indiscernible over Aa, and for each i, I_i is not indiscernible over Aa.

(2) p is independent.

(3) $\text{rk-dp}(p) \geq |T|^+ + |A|^+$.

(4) There is an A-indiscernible sequence $\langle a_i \mid i < \omega \rangle$ such that $a_i \models p$, a formula $\varphi(x, y)$ and some c such that $\varphi(a_i, c)$ holds iff i is even.

(5) There is an A-indiscernible sequence $\langle b_i \mid i < \omega \rangle$, a formula $\psi(y, x)$ and some $d \models p$ such that $\psi(b_i, d)$ holds iff i is even.

(6) There is a set $\{a_i \mid i < \omega\}$ of realizations of p and a formula $\varphi(x, y)$ such that for every $s \subseteq \omega$, there is some c_s such that $\varphi(a_i, c_s)$ holds iff $i \in s$.

(7) There is a set $\{b_i \mid i < \omega\}$ and a formula $\psi(y, x)$ such that for every $s \subseteq \omega$, there is some $d_s \models p$ such that $\psi(b_i, d_s)$ holds iff $i \in s$.

Proof. (1) implies (2) and (2) implies (3) are easy. Assume (3) and show (1). We can find $a \models p$ and A-mutually indiscernible sequences $\left\langle I_i \,\middle|\, i < |T|^+ + |A|^+ \right\rangle$ such that for all i, I_i is not indiscernible over Aa. We may assume that the order type of these sequences is ω. The fact that I_i is not indiscernible over Aa is witnessed by some formula over A and increasing tuples from I_i, so we may assume that for infinitely many i, the formula is the same, and the position of these tuples does not depend on i (maybe changing a). Then, by Ramsey and compactness, we may assume that $\langle I_i \mid i < \omega \rangle$ is indiscernible over Aa.

(5) follows from (1): Denote $I_i = \langle a_{i,j} \mid j < \omega \rangle$. There is a formula $\psi(x, y)$ over A and an increasing tuple $k_0 < \ldots < k_{n-1} < r_0 < \ldots < r_{n-1}$ such that, letting $a_{i,\bar{k}} = \left(a_{i,k_0}, \ldots, a_{i,k_{n-1}}\right)$ (and similarly we define $a_{i,\bar{r}}$), $\psi\left(a_{i,\bar{k}}, a\right) \wedge \neg\psi\left(a_{i,\bar{r}}, a\right)$ holds for all $i < \omega$. The sequence $\langle b_i \mid i < \omega \rangle$ defined by $b_i = a_{i,\bar{k}}$ when i is even and $b_i = b_{i,\bar{r}}$ when i is odd satisfies (5). The fact that ψ is over A is no problem — we can add the parameters to b_i.

(2) follows from (5) is easy by compactness.

(6) is equivalent to (4) and (7) is equivalent to (5) by a standard application of Ramsey.

(6) follows from (5): By indiscernibility, we may extend $\langle b_i \mid i < \omega \rangle$ to $\langle b_r \mid r \in \mathcal{P}(\omega) \rangle$ (with some ordering), and so, for every subset $s \subseteq \mathcal{P}(\omega)$, there is some $d_s \models p$ such that $\psi(b_r, d_s)$ iff $r \in s$. For $i < \omega$, let $d_i = d_{\{r \subseteq \omega : i \in r\}}$. Then for each subset $r \subseteq \omega$, $\psi(b_r, d_i)$ iff $i \in r$. This gives us (6). The same exact argument gives that (7) follows from (4). $\square$

Proposition 2.2. *If p is a dependent type over A, then there is $B \subseteq A$ of size $|B| \leq |T|$ such that $p|_B$ is dependent.*

Proof. By Claim 2.1 (6), it cannot be that there exists a formula $\varphi(x,y)$ and a set $\{a_i \mid i < \omega\}$ of realizations of p such that for each $s \subseteq \omega$, there is some c_s such that $\varphi(a_i, c_s)$ holds iff $i \in s$. By compactness, there is no formula $\varphi(x,y)$ such that for all finite $B \subseteq A$ we can find such a set $\{a_i \mid i < \omega\}$ of realizations of $p|_B$ and such c_s for $s \subseteq \omega$. So for each formula $\varphi(x,y)$, there is some finite $B_\varphi \subseteq A$ such that there is no such set. Let $B = \bigcup_\varphi B_\varphi$. Then $p|_B$ is easily seen to be dependent. □

Corollary 2.3. *The following are equivalent for a type $p(x)$ over A:*

(1) *$p(x)$ is independent.*
(2) $\text{rk-dp}(p) \geq |T|^+$.

Proof. If p is dependent, then there is some $B \subseteq A$ such that $p|_B$ is dependent and $|B| \leq |T|$. By Claim 2.1 (3), this means that $\text{rk-dp}(p|_B) < |T|^+$, so $\text{rk-dp}(p) < |T|^+$. □

In this section we show that some useful properties that are true in dependent theories are actually true in the local context as well.

Fact 2.4. [KOU11, Theorem 4.11] *If p is a dependent type over A, and $a_i \models p$ for $i < n < \omega$, then* $\text{tp}(a_0, \ldots, a_{n-1}/A)$ *is also dependent.*

Recall the notions of forking and dividing. All the definitions and properties we need can be found in [CK12].

Proposition 2.5. *If p is dependent type over a model M, and q is a global non-forking extension of p (i.e. an extension to $\mathfrak{C}$), then q is invariant over M.*

Proof. Suppose that $\varphi(x, c_0) \wedge \neg\varphi(x, c_1) \in q$ where $c_0 \equiv_M c_1$. Then using a standard technique, we can assume that c_0, c_1 start an indiscernible sequence $\langle c_0, c_1, \ldots \rangle$ over M. The set

$$p(x) \cup \left\{ \varphi(x, c_i)^{(i \text{ is even})} \mid i < \omega \right\}$$

is inconsistent by Claim 2.1. This means that for some formula $\psi(x) \in p$,

$$\{\psi(x) \wedge \varphi(x, c_{2i}) \wedge \neg\varphi(x, c_{2i+1}) \mid i < \omega\}$$

is inconsistent, and so $\psi(x) \wedge \varphi(x, c_0) \wedge \neg\varphi(x, c_1)$ divides over M — contradiction. □

Proposition 2.6. *(shrinking of indiscernibles) Suppose that $p(x)$ is a dependent type over A and that B is a set of realizations of p.*

If $I = \left\langle a_i \,\middle|\, i < |T|^+ + |B|^+ \right\rangle$ is an A-indiscernible sequence, then some end-segment is indiscernible over AB. Note that the size of A and the size of the tuple a_i do not matter.

Proof. We may assume that B is finite. The type $\operatorname{tp}(B/A)$ is dependent by Fact 2.4. The proof easily follows from Corollary 2.3. $\square$

2.2. **Proof of Main Theorem A.**

Definition 2.7. Let $p(x)$ be a type over A. We say that p is 1-*independent over* A if there is a realization $a \models p$ and A-mutually indiscernible sequences $\langle I_i \mid i < \omega \rangle$ of singletons such that the sequence $\langle I_i \mid i < \omega \rangle$ is indiscernible over Aa and for each $i < \omega$, I_i is not indiscernible over Aa.

We say that p is 1-*dependent over* A if it is not 1-independent over A. We say that p is 1-*dependent* if it is 1-dependent over any $A' \supseteq A$ such that $A' \backslash A$ is finite.

Observe that by Claim 2.1, if $p(x)$ is dependent then it is 1-dependent. Also, as in Remark 1.2, this definition does not depend on A.

Claim 2.8. If $p(x)$ is a type over A which is 1-dependent, then:

- For every $A' \supseteq A$ such that $A' \backslash A$ is finite, every A'-indiscernible sequence
$$\left\langle a_i \,\middle|\, i < |T|^+ + |A|^+ \right\rangle$$
of tuples satisfying p and singleton c, there is some $\alpha < |T|^+ + |A|^+$ such that the end-segment $\langle a_i \mid \alpha < i \rangle$ is indiscernible over $A'c$.

Proof. To simplify notations, assume $A = A' = \emptyset$. Towards a contradiction we find a formula $\varphi(\bar{x}, y)$ and an indiscernible sequence $\langle \bar{a}_i \mid i < \omega \rangle$ such that $\bar{a}_i$ is a tuple of length n of tuples satisfying p and $\varphi(\bar{a}_i, c)$ holds iff i is even. By the proof of Claim 2.1 (i.e. (5) implies (4), with $p = \operatorname{tp}(c)$), there is an indiscernible sequence $\left\langle c_{\bar{i}} \,\middle|\, \bar{i} \in \omega^{n+1} \right\rangle$ (ordered lexicographically) of singletons such that $\varphi(\bar{a}_0, c_{\bar{i}})$ holds iff the last number in $\bar{i}$ is even. We may also assume (by Ramsey) that the sequence $\langle \bar{c}_{\bar{i}} \mid \bar{i} \in \omega^n \rangle$ is indiscernible over $\bar{a}_0$, where $\bar{c}_{\bar{i}} = \left\langle c_{\bar{i}\frown j} \mid j < \omega \right\rangle$.

Suppose $\bar{a}_0 = (a_{0,0}, \ldots, a_{0,n-1})$ where $a_{0,i} \models p$. Since p is 1-dependent over $\emptyset$, there is some $i_0 < \omega$ such that $\left\langle c_{i_0 \frown \bar{i}} \mid \bar{i} \in \omega^n \right\rangle$ is indiscernible over $a_{0,0}$. By assumption, p is 1-dependent over $a_{0,0}$. Inductively, we can find $i_1, \ldots, i_{n-1} < \omega$ such that $\bar{c}_{(i_0, \ldots, i_{n-1})}$ is indiscernible over $\bar{a}_0$ — contradiction. $\square$

The following theorem implies Main Theorem A:

Theorem 2.9. *If $p(x)$ is a type over A which satisfies the conclusion of Claim 2.8, then it is dependent.*

Proof. Again, assume $A = \emptyset$. Suppose p is a counterexample. By Claim 2.1, there is an indiscernible sequence $\left\langle a_i \,\middle|\, i < |T|^+ \right\rangle$ such that $a_i \models p$, a formula $\varphi(x, y)$ and some tuple $c = (c_0, \ldots, c_{n-1})$ such that $\varphi(a_i, c)$ holds iff i is even. By assumption, there is some end-segment which is indiscernible over c_0. Applying the conclusion of Claim 2.8 again with $A' = \{c_0\}$, we get an end-segment which is indiscernible over c_0c_1. Continuing like this, we get an end-segment which is indiscernible over c — contradiction. $\square$

Since dependent implies 1-dependent, we get:

Corollary 2.10. *The type $p(x)$ is 1-dependent iff it is dependent iff it satisfies the conclusion of Claim 2.8.*

Corollary 1.5 follows:

Corollary 2.11. *A theory T is dependent iff for every indiscernible sequence of singletons*

$$\left\langle a_i \,\middle|\, i < |T|^+ \right\rangle$$

over some finite A, and for every singleton c, there is $\alpha < |T|^+$ such that $\langle a_i \mid \alpha < i \rangle$ is indiscernible over Ac.

Proof. Apply Corollary 2.10 with $p(x) = \{x = x\}$. $\square$

3. Proof of Main Theorem B

3.1. **Preliminaries on NTP_2 theories.** From here up to the end of the section, we assume that the theory is NTP_2.

In the study of forking in NTP_2 theories, it is sometimes useful to consider independence relations. For instance, we denote $a \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} C$ for $\mathrm{tp}(a/BC)$ does not fork over B. Similarly, $a \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{B} C$ means that there is a global extension (i.e. an extension to $\mathfrak{C}$) of $\mathrm{tp}(a/BC)$ which is Lascar invariant over B, meaning that if d and c have the same Lascar strong type over B then either both $\varphi(x,c)$ and $\varphi(x,d)$ are in this extension or neither of them is. We do not really need Lascar strong type in this section, because we only work over models. Over a model, Lascar invariance is the same as invariance.

In the proofs we shall only use the following facts about NTP_2 theories. These were proved in [CK12].

Definition 3.1. (strict invariance) We say that $\mathrm{tp}(a/Bb)$ is strictly invariant over B (denoted by $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{ist}}_{B} b$) if there is a global extension p, which is Lascar invariant over B (so $a \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{B} b$) and for any $C \supseteq Bb$, if $c \models p|_C$ then $C \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{B} c$.

Fact 3.2. *In NTP_2 theories*

1. *Forking equals dividing over models.*
2. *"Kim's Lemma": If $\varphi(x,a)$ divides over A, and $\langle b_i \,| i < \omega \rangle$ is a sequence satisfying $b_i \equiv_A a$ and $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{ist}_{A} b_{<i}$, then $\{\varphi(x,b_i) \,|\, i < \omega\}$ is inconsistent. In particular, if $\langle b_i \,| i < \omega \rangle$ is an indiscernible sequence then it witnesses dividing of $\varphi(x,a)$.*

Recall:

Definition 3.3. Suppose p is a global type which is invariant over a set A.

(1) We say that a sequence $\langle a_i \mid i < \alpha \rangle$ is a Morley sequence of a type p over $B \supseteq A$ if $a_0 \models p|_B$ and for all $i < \alpha$, $a_i \models p|_{Ba_{<i}}$. This is an indiscernible sequence over B.

(2) We let the type $p^{(\alpha)}$ be the union of $\operatorname{tp}(\langle a_i \mid i < \alpha \rangle / B)$ running over all $B \supseteq A$. This is again an A-invariant type.

(3) If q is also an A-invariant global type, we define $p \otimes q$ as the union of $\operatorname{tp}(a, b/B)$ running over all $B \supseteq A$ where $a \models p|_B$ and $b \models q|_{Ba}$. This is also an A-invariant global type.

(4) Similarly, given a sequence $\langle p_i \mid i < \alpha \rangle$ of A-invariant global types, we define $\bigotimes_{i<\alpha} p_i$ as the union of $\operatorname{tp}(\langle a_i \mid i < \alpha \rangle / B)$ running over all $B \supseteq A$, where $a_i \models p_i|_{Ba_{<i}}$. Again, this is an A-invariant global type.

In the definition above, all types may have infinitely many variables.

Remark 3.4. If $\{J_0, \ldots, J_k\}$ is a set of mutually indiscernible sequences over $C \supseteq A$, and $\langle a_i \mid i < \alpha \rangle$ is a Morley sequence of a global A-invariant type over $\{J_0, \ldots, J_k\} \cup C$ then $\{J_0, \ldots, J_k, \langle a_i \mid i < \alpha \rangle\}$ is mutually indiscernible over C.

(Why? On the one hand, $\{J_0, \ldots, J_k\}$ is mutually indiscernible over $C \cup \{a_i \mid i < \alpha\}$ since $\operatorname{tp}(\langle a_i \mid i < \alpha \rangle / \{J_0, \ldots, J_k\} \cup C)$ does not split over A. On the other hand, $\langle a_i \mid i < \alpha \rangle$ is a Morley sequence over $\{J_0, \ldots, J_k\} \cup C$, and as such is indiscernible over that set.)

We also need to recall the notions of heir and coheir:

Definition 3.5. A global type $p(x)$ is called a coheir over a set A, if it is finitely satisfiable in A. Note that in this case, it is invariant over A, and $p^{(\alpha)}$ is also a coheir over A.

It is called an heir over A if for every formula over A, $\varphi(x, b) \in p$, there exists some $a' \in A$ such that $\varphi(x, a') \in p$.

Claim 3.6. If p is an A-invariant global type and $p^{(\omega)}$ is both an heir and a coheir over A, then any Morley sequence of p over A, $\langle a_i \mid i < \omega \rangle$ satisfies $a_{\geq i} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{ist}}_A a_{<i}$ for any $i < \omega$.

Proof. The type $p^{(\omega)}$ is a global A-invariant (so also A-Lascar invariant) type that extends $\operatorname{tp}(a_{\geq i}/Aa_{<i})$, and if $c \models p^{(\omega)}|AC$ then $\operatorname{tp}(C/Ac)$ is finitely satisfiable over A (since $p^{(\omega)}$ is an heir over A), and it follows that $C \mathop{\smile\!\!\!\!\!\!\mid}^{f}_A c$. $\square$

Claim 3.7. Given any global type $p(x)$ and a set A, we can find a model $M \supseteq A$ such that p is an heir over M.

Proof. Construct inductively a sequence of models M_i for $i < \omega$. Let M_0 be any model containing A. Let $M_{i+1} \supseteq M_i$ be such that for every formula $\varphi(x, y)$ over M_i, if $\varphi(x, a) \in p$ then there is some such a in M_{i+1}. Finally, let $M = \bigcup_{i<\omega} M_i$. $\square$

Lemma 3.8. *Let M be a model. Suppose that p is an M-invariant global type such that $p^{(\omega)}$ is an heir-coheir over M. Suppose I is an endless Morley sequence of p over M. If I is indiscernible over Ma then* $\operatorname{tp}(a/MI)$ *does not fork over M.*

Proof. By Fact 3.2, it is enough to see that the type does not divide over M. Suppose $\varphi(x, b_0) \in \operatorname{tp}(a/MI)$ divides over M, where φ is over M and $b_0 \subseteq I$. For $i \geq 1$ choose tuples $b_i \subseteq I$ of the same length as b_0 that appear after b_0 in increasing order. By Claim 3.6, $b_i \mathop{\smile\!\!\!\!\!\!\;\mid}^{\text{ist}}_M b_{<i}$ so by "Kim's lemma" (Fact 3.2), it must witness dividing. But this is a contradiction to the fact that I is indiscernible over Ma. □

3.2. **Proof of the main theorem.** The following is the key definition in the proof.

Definition 3.9. Suppose

(1) p is a global A-invariant type such that $p|_A$ is dependent (we call such types A-invariant and A-dependent).
(2) B is some set containing A.
(3) $\varphi(x, y)$ is a formula over A.
(4) a is a tuple of length $\lg(y)$.

Then we define $\operatorname{alt}(\varphi, B, a, p)$ to be the maximal number n such that there is a realization $\langle a_i \mid i < n \rangle \models p^{(n)}|_B$, such that $\varphi(a_i, a)$ alternates for $i < n$, i.e. such that $\varphi(a_i, a) \Leftrightarrow \neg\varphi(a_{i+1}, a)$ for $i < n - 1$.

Note that $\operatorname{alt}(\varphi, B, a, p)$ exists by Claim 2.1 (4). Observe that $\operatorname{alt}(\varphi, B, a, p) \geq \operatorname{alt}(\varphi, B', a, p)$ when $B' \supseteq B \supseteq A$, but not necessarily the other way. Sometimes there is equality:

Lemma 3.10. *Suppose p is a global A-invariant and A-dependent type, a some tuple and I an indiscernible sequence over Aa.*

Then: for every infinite subset $I' \subseteq I$ and for any formula $\varphi(x, y)$ over A, $alt(\varphi, IA, a, p) = alt(\varphi, I'A, a, p)$.

Proof. Obviously, $\operatorname{alt}(\varphi, IA, a, p) \leq \operatorname{alt}(\varphi, I'A, a, p)$.

Conversely, suppose we have some n such that $\bar{a} = \langle a_i \mid i < n \rangle \models p^{(n)}|_{I'A}$ alternates as in the definition. Let $\bar{x} = (x_0, \ldots, x_{n-1})$. We want to show that the type

$$p^{(n)}(\bar{x})|_{IA} \cup \left\{ \varphi(x_i, a)^{(\text{if } \varphi(a_i, a))} \mid i < n \right\}$$

is consistent.

Take any finite subset and write it as $\psi(\bar{x}, b, c) \wedge \xi(\bar{x}, a)$ where $b \subseteq I$, $c \subseteq A$. As I' is infinite, and I is indiscernible over Aa we can find $b' \in I'$ such that $b' \equiv_{Aa} b$ so $\mathfrak{C} \models \exists \bar{x} \psi(\bar{x}, b', c) \wedge \xi(\bar{x}, a)$ iff $\mathfrak{C} \models \exists \bar{x} \psi(\bar{x}, b, c) \wedge \xi(\bar{x}, a)$. Now, $\psi(\bar{x}, b, c) \in p^{(n)}$, and $p^{(n)}$ is A invariant, hence $\psi(\bar{x}, b', c) \in p^{(n)}$, and since $\bar{a}$ satisfies $\psi(\bar{x}, b', c) \wedge \xi(\bar{x}, a)$, we are done. □

We will deduce Main Theorem B from the following theorem:

Theorem 3.11. *Suppose $p(x)$ is a dependent type over C with* $\text{rk-dp}(p) \geq \kappa$. *Assume this is witnessed by $c \models p$ and $\{I_i \mid i < \kappa\}$ where I_i has order type ω for $i < \kappa$.*

Then there are

- *$C' \supseteq C$ with $|C' \backslash C|$ finite, $c' \models p$ and J_0*

such that

- *$\{J_0\} \cup \{I_i \mid 0 < i < \kappa\}$ is mutually indiscernible over C'; $c' \equiv_{C \cup \{I_i \mid 0 < i < \kappa\}} c$; J_0 is not indiscernible over $C'c'$ and*
- *all the tuples in J_0 satisfy p.*

Proof. Denote $I_i = \langle f_{i,j} \mid j < \omega \rangle$. By compactness, we can find $f_{i,j}$ for $j \in \mathbb{Z}$ and $i < \kappa$ such that, letting $I'_i = \langle f_{i,j} \mid j \in \mathbb{Z}, j < 0 \rangle$, $\{I'_i \frown I_i \mid i < \kappa\}$ is mutually indiscernible over C. Let U be a non-principal ultrafilter on ω. For $i < \kappa$, let p_i be global coheir over I'_i defined by:

- for a formula $\psi(z, y)$ and a tuple $a \in \mathfrak{C}$, $\psi(z, a) \in p_i$ iff $\{n < \omega \mid \models \psi(f_{i,-n}, a)\} \in U$.

So each p_i is invariant over $\bigcup_{i<\kappa} I'_i$ and we can consider the type $\left(\bigotimes_{0<i<\kappa} p_i^{(\omega)}\right)^{(\omega)}$, and find a model $M \supseteq C \cup \bigcup_{i<\kappa} (I'_i \frown I_i)$ such that this type is an heir over M (using Claim 3.7).

Let $\langle K_i \mid i < \kappa \rangle \models \bigotimes_{i<\kappa} p_i^{(\omega)}|_M$, then:

- each K_i is a Morley sequence of p_i over M,
- since $\{I'_i \frown I_i \mid i < \kappa\}$ is mutually indiscernible over C, and p_i is finitely satisfiable in I'_i, $\langle K_i \mid i < \kappa \rangle \equiv_C \langle I_i \mid i < \kappa \rangle$ (this follows from the fact that the order type of I'_i is ω^* — ω in reverse), and
- by Remark 3.4, $\{K_i \mid i < \kappa\}$ is mutually indiscernible over M.

By the second bullet, there is an automorphism of $\mathfrak{C}$ that fixes C (but may move M and p_i) and maps $\langle K_i \mid i < \kappa \rangle$ to $\langle I_i \mid i < \kappa \rangle$. By applying it we we may assume that $\langle K_i \mid i < \kappa \rangle = \langle I_i \mid i < \kappa \rangle$.

Let $\mu = |T|^+$. Let $J = \langle d_i \mid i < \mu \rangle$ be a Morley sequence of $\left(\bigotimes_{0<i<\kappa} p_i^{(\omega)}\right)$ over MI_0 so that d_0 is the infinite tuple $\langle I_i \mid 0 < i < \kappa \rangle$. Note that I_0 is indiscernible over JM, J is indiscernible over $I_0 M$ and $\{I_i \mid i < \kappa\}$ is mutually indiscernible over $M \cup \{d_i \mid 0 < i < \mu\}$ (by Remark 3.4).

Now, I_0 is not indiscernible over Cc. So there are increasing tuples a_0 and a_1 from I_0 of the same length and a formula $\varphi(x, y)$ over C such that $\neg\varphi(c, a_0) \wedge \varphi(c, a_1)$ holds. By indiscernibility, there is an automorphism σ of $\mathfrak{C}$ that fixes JM and takes a_0 to a_1. Let $c_0 = c$ and $c_1 = \sigma(c_0)$. Then $\varphi(c_0, a_1) \wedge \neg\varphi(c_1, a_1)$ holds.

By Proposition 2.6, for some $\alpha < \mu$, the sequence $J' = \langle d_i \mid \alpha < i < \mu \rangle$ is indiscernible over Mc_0c_1. By Lemma 3.8, $c_0c_1 \mathop{\smile\!\!\!\!|}^{f}_{M} J'$.

Let $r(x)$ be a global non-forking (over M) extension of $\text{tp}(c_0/MJ')$ ($= \text{tp}(c_1/MJ')$). Since $\text{tp}(c_0/C)$ is dependent, $r(x)$ is M-dependent and M-invariant (by Proposition 2.5). Let $n =$

alt (φ, MJ, a_1, r), and let $\langle e_i \,|\, i < \omega \rangle$ be a Morley sequence of r over MJ that witnesses this, i.e. such that $\varphi(e_i, a_1)$ alternates for $i < n$. Let $\langle e_i \,|\, \omega \leq i < \omega + \omega \rangle$ be a Morley sequence of r over $MJc_0c_1e_{<\omega}$.

So:

- $I'_0 = \langle e_i \,|\, i < \omega + \omega \rangle$ is an MJ-indiscernible sequence, and moreover $\{I_i \,|\, 0 < i < \kappa\} \cup \{I'_0\}$ is a set of MJ'-mutually indiscernible sequences.

Now, both $\langle c_0, e_\omega, e_{\omega+1}, \ldots \rangle$, $\langle c_1, e_\omega, e_{\omega+1}, \ldots \rangle$ are Morley sequences of r over MJ'. But if in addition $\langle c_0, e_0, e_1, \ldots \rangle$ and $\langle c_1, e_0, e_1, \ldots \rangle$ are also Morley sequences of r over MJ', then since one of c_0, c_1, adds an alternation of the truth value of $\varphi(x, a_1)$, this is a contradiction to the choice of e_i and to Lemma 3.10 (which we can use because J is indiscernible over a_1, and J' is infinite). Let $c' \in \{c_0, c_1\}$ be such that $\langle c', e_0, e_1, \ldots \rangle$ is not a Morley sequence of r over MJ'. Note that $c' \equiv_{MJ} c$ and that MJ contains $C \cup \{I_i \,|\, 0 < i < \kappa\}$.

So, $\langle c', e_0, \ldots \rangle \not\equiv_{MJ'} \langle c', e_\omega, \ldots \rangle$ and hence the sequence I'_0 is not indiscernible over $c'MJ'$. Let J_0 be some infinite subset of I'_0 of order type ω that witnesses this, and let $C' \supseteq C$ be such that $|C' \backslash C|$ is finite, and $C' \subseteq MJ'$ so that J_0 is not indiscernible over $C'c'$.

It is now easy to check that all conditions are satisfied. □

Now let us conclude:

Proof. (of Main Theorem B) Suppose p is a dependent type over A with $\text{rk-dp}(p) \geq \kappa$. Consider the family $\mathcal{F}$ of triples (s, c, J, A') such that

- $c \models p$, $s \subseteq \kappa$, $J = \langle I_i \,|\, i < \kappa \rangle$; $A \subseteq A'$; J is a sequence of A'-mutually indiscernible sequences such that for each $i < \kappa$, I_i is not indiscernible over $A'c$; all tuples in I_i for $i \in s$ realize p.

By assumption, $\mathcal{F}$ is not empty. Define the following order relation on these triples:

- $(s, c, J, A') \leq (s', c', J', A'')$ iff ($s \subseteq s'$, $A' \subseteq A''$, if $i \in s \cup (\kappa \backslash s')$ then $I_i = I'_i$ and $c' \equiv_{A' \cup \{I_i \,|\, i \in s \cup (\kappa \backslash s')\}} c$).

It is easy to see that by compactness $\mathcal{F}$ satisfies the conditions of Zorn's Lemma, so it has a maximal member (s_0, c_0, J_0, A'_0). By Theorem 3.11, $s_0 = \kappa$ and we are done. □

References

[Che12] Artem Chernikov. Theories without the tree property of the second kind, 2012. `arXiv:1204.0832`.

[CK12] Artem Chernikov and Itay Kaplan. Forking and dividing in NTP$_2$ theories. *Journal of Symbolic Logic*, 77(1):1–20, 2012.

[KOU11] Itay Kaplan, Alf Onshuus, and Alexander Usvyatsov. Additivity of the dp-rank. *Transactions of the AMS*, 2011. accepted, `arXiv:1109.1601`.

[OU11] Alf Onshuus and Alexander Usvyatsov. On dp-minimality, strong dependence and weight. *Journal of symbolic logic*, 76(3):737–758, 2011.

[She12] Saharon Shelah. Strongly dependent theories. *Israel Journal of Mathematics*, 2012. accepted.

[Sim11] Pierre Simon. On dp-minimal ordered structures. *Journal of Symbolic Logic*, 76(2):448–460, 2011.

[Usv07] Alexander Usvyatsov. On generically stable types in dependent theories. *Journal of Symbolic Logic*, 2007. accepted, `arXiv:0709.0195`.