Notre Dame Journal of Formal Logic
Volume 58, Number 3, 2017

Randomness and Semimeasures

Laurent Bienvenu, Rupert Hölzl,
Christopher P. Porter, and Paul Shafer

Abstract A *semimeasure* is a generalization of a probability measure obtained by relaxing the additivity requirement to superadditivity. We introduce and study several randomness notions for left-c.e. semimeasures, a natural class of effectively approximable semimeasures induced by Turing functionals. Among the randomness notions we consider, the generalization of weak 2-randomness to left-c.e. semimeasures is the most compelling, as it best reflects Martin-Löf randomness with respect to a computable measure. Additionally, we analyze a question of Shen, a positive answer to which would also have yielded a reasonable randomness notion for left-c.e. semimeasures. Unfortunately, though, we find a negative answer, except for some special cases.

1 Introduction

Suppose we have an algorithmic procedure P that, upon receiving an infinite binary sequence as an input, yields either an infinite binary sequence or a finite binary string as the output. The question we investigate here is:

($\mathcal{Q}$) What is the typical infinite output of P?

In the case in which P always produces an infinite output or produces an infinite output with probability 1, there is already a complete answer to ($\mathcal{Q}$), insofar as we understand typicality in terms of Martin-Löf randomness. In this case, the typical outputs of P are determined precisely by the behavior of P on all random inputs: the procedure P and the Lebesgue measure λ together induce a measure λ_P (in a sense to be made precise below) so that the typical infinite outputs of P are exactly the sequences that are random with respect to the measure λ_P.

In this article, we attempt to answer the question ($\mathcal{Q}$) in the case where the algorithmic procedure P does not produce an infinite output with probability 1. Whereas

Received October 14, 2013; accepted September 23, 2014
First published online March 15, 2017
2010 Mathematics Subject Classification: Primary 03D32
Keywords: semimeasures, measures, algorithmic randomness, randomness
 10.1215/00294527-3839446

an algorithmic procedure that yields an infinite output with probability 1 induces a computable measure, a procedure that yields an infinite output with probability less than 1 induces what is known as a *left-c.e. semimeasure*, where a semimeasure is a function $\rho : 2^{<\omega} \to [0, 1]$ which satisfies

(i) $\rho(\varepsilon) = 1$ and
(ii) $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$,

where ε denotes the empty string, and a semimeasure is left-c.e. if it is effectively approximable from below.

As will be discussed in the next section, we can reformulate the question ($\mathcal{Q}$) as follows:

($\mathcal{Q}'$) Which infinite sequences are random with respect to the left-c.e. semimeasure induced by the Turing functional Φ?

Clearly, answering this question requires a definition of randomness with respect to a left-c.e. semimeasure, but there is currently no such definition available.

One attempt at answering ($\mathcal{Q}'$) was suggested by Shen at a recent meeting in Dagstuhl (see Becher, Bienvenu, Downey, and Mayordomo [1]). There he asked the following question, which was already raised in Shen, Bienvenu, and Romashchenko [15].

Question 1.1 If Φ and Ψ are Turing functionals that induce the same left-c.e. semimeasure, does it follow that $\Phi(\mathsf{MLR}) = \Psi(\mathsf{MLR})$?

The relevance of Shen's question to the task of defining randomness with respect to a left-c.e. semimeasure is the following. Suppose Question 1.1 has a positive answer. Then we can define $Y \in 2^\omega$ to be ρ-random if and only if there is some $X \in 2^\omega$ such that $\Phi(X) = Y$ for any Φ that induces ρ. We call this the *pushforward* definition of randomness with respect to a semimeasure.

In this article, we show that Shen's question has a positive answer in the restricted case in which Φ and Ψ induce a *computable* semimeasure. Moreover, we show that if the definition of Martin-Löf randomness for computable measures is extended to computable semimeasures, the resulting definition is equivalent to the pushforward definition of randomness with respect to a computable semimeasure.

The situation is much less straightforward when we consider left-c.e. semimeasures. First, we show that Shen's question has a negative answer in this more general setting, and thus we need a different strategy for answering ($\mathcal{Q}'$). Toward this end, we consider two general approaches to defining randomness with respect to a left-c.e. semimeasure:[1]

(1) defining randomness with respect to a semimeasure by a direct adaptation of standard definitions of randomness with respect to computable measures, and
(2) defining randomness with respect to a semimeasure in terms of a specific measure derived from trimming back a given semimeasure to a measure.

Although we prove a number of results about these candidate definitions, no definition has yet emerged as the most well behaved. However, some of the results we present indicate that weak 2-randomness is a promising notion in this context.

The remainder of the article is organized as follows. In Section 2, we provide the necessary background on randomness with respect to computable and noncomputable measures. We also discuss some basic results about the relationship between semimeasures and Turing functionals. In Section 3, we answer Shen's question when restricted to the collection of computable semimeasures by formulating a definition of randomness with respect to a computable semimeasure. In Section 4, we answer the general version of Shen's question in the negative, but we do show that a related question involving a notion of randomness that is stronger than Martin-Löf randomness has a positive answer. In Section 5, we pursue the first strategy for answering ($\mathcal{Q}'$) discussed above, directly modifying a number of different definitions of randomness with respect to a measure. Lastly, in Section 6, we discuss the measure obtained by trimming back a semimeasure and explore the notions of randomness with respect to such measures.

We assume that the reader is familiar with the basic notions from computability theory: computable functions, partial computable functions, computably enumerable sets, Turing functionals, Turing degrees, and the Turing jump. (For details on algorithmic randomness, see Downey and Hirschfeldt [3] or Nies [11].)

Let us fix some notation and terminology. We denote by 2^ω the set of infinite binary sequences, also known as *Cantor space*. We denote the set of finite strings by $2^{<\omega}$ and the empty string by ε. The set of nonnegative dyadic rationals, that is, rationals of the form $m/2^n$ for $m, n \in \omega$ is written Q_2^+. Given $X \in 2^\omega$ and an integer n, $X{\upharpoonright}n$ is the string that consists of the first n bits of X, and $X(n)$ is the $(n+1)$st bit of X (so that $X(0)$ is the first bit of X). If σ and τ are strings, then $\sigma \preceq \tau$ means that σ is an initial segment of τ. Similarly for $X \in 2^\omega$, $\sigma \preceq X$ means that σ is an initial segment of X. Given a string σ, the *cylinder* $[\![\sigma]\!]$ is the set of elements of 2^ω having σ as an initial segment. Similarly, given $S \subseteq 2^{<\omega}$, $[\![S]\!]$ is defined to be the set $\bigcup_{\sigma\in S}[\![\sigma]\!]$. The cylinders form a basis for the usual topology on the Cantor space (the product topology), and thus the open sets for this topology are those of the form $[\![S]\!]$ for some S. An open set $\mathcal{U}$ is said to be *effectively open* (or Σ^0_1) if $\mathcal{U} = [\![S]\!]$ for some c.e. set of strings S. An *effectively closed set* (or Π^0_1) is the complement of an effectively open set. A sequence of open sets $(\mathcal{U}_n)_{n\in\omega}$ is said to be *uniformly effectively open* if there exists a sequence $(S_n)_{n\in\omega}$ of uniformly c.e. sets of strings such that $\mathcal{U}_n = [\![S_n]\!]$.

2 Preliminaries

In this section, we will review the basic results of randomness with respect to computable and noncomputable measures.

2.1 Randomness with respect to computable measures The standard definition of algorithmic randomness is Martin-Löf randomness, first introduced by Martin-Löf [10]. Martin-Löf's original definition was given in terms of the Lebesgue measure, but he also recognized that it held for a larger class of probability measures. We first define Martin-Löf randomness with respect to a computable measure μ.

A measure μ on 2^ω is *computable* if $\sigma \mapsto \mu([\![\sigma]\!])$ is computable as a real-valued function, that is, if there is a computable function $\tilde{\mu} : 2^{<\omega} \times \omega \to \mathbb{Q}_2^+$ such that

$$\left|\mu\big([\![\sigma]\!]\big) - \tilde{\mu}(\sigma, i)\right| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$. From now on, we will write $\mu([\![\sigma]\!])$ as $\mu(\sigma)$, and similarly, for $V \subseteq 2^{<\omega}$, we will write $\mu([\![V]\!])$ as $\mu(V)$.

By Carathéodory's theorem, a Borel measure μ on 2^ω is uniquely determined by the values $\mu(\sigma)$ for $\sigma \in 2^{<\omega}$, and conversely, given a function $f : 2^{<\omega} \in [0,1]$ such that $f(\varepsilon) = 1$ and $f(\sigma) = f(\sigma 0) + f(\sigma 1)$ for all σ, there exists a (unique) Borel measure μ such that $\mu([\![\sigma]\!]) = f(\sigma)$ for all σ.

The *uniform (or Lebesgue) measure* λ is the measure for which each bit of the sequence has value 0 with probability 1/2, independent of the values of the other bits. It can be defined as the unique Borel measure such that $\lambda(\sigma) = 2^{-|\sigma|}$ for all strings σ.

Definition 2.1 Let μ be a computable measure on 2^ω.

(i) A *μ-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly effectively open subsets of 2^ω such that, for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

(ii) $X \in 2^\omega$ passes the μ-Martin-Löf test $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} \mathcal{U}_i$.

(iii) $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathsf{MLR}_\mu$, if X passes every μ-Martin-Löf test. When μ is the uniform measure λ, we often abbreviate MLR_μ by MLR.

An important feature of Martin-Löf randomness is the existence of a universal test. For every computable measure μ, there is a universal μ-Martin-Löf test $(\widehat{\mathcal{U}}_i)_{i\in\omega}$ having the property that $X \in \mathsf{MLR}_\mu$ if and only if $X \notin \bigcap_{i\in\omega} \widehat{\mathcal{U}}_i$.

Definition 2.2 For any computable measure μ, we tacitly assume that a universal μ-Martin-Löf test $(\widehat{\mathcal{U}}_i)_{i\in\omega}$ has been fixed, and we denote by MLR^d_μ the Π^0_1-set $(\widehat{\mathcal{U}}_d)^{\mathsf{c}}$, so that MLR_μ is the nondecreasing union of the sets MLR^d_μ.

Given a measure μ, we say that $X \in 2^\omega$ is an *atom of* μ if $\mu(\{X\}) > 0$. Kautz [6, Lemma IV.3.7] proved the following useful fact about the atoms of a computable measure.

Lemma 2.3 *$X \in 2^\omega$ is computable if and only if X is an atom of some computable measure.*

There is an intimate connection between Martin-Löf random sequences and a class of effective functionals that induce computable measures, a connection that we would like to preserve when we formulate a definition of randomness with respect to a semimeasure. Recall that a *Turing functional* $\Phi :\subseteq 2^\omega \to 2^\omega$ may be defined as a c.e. set of pairs of strings (σ, τ) such that if $(\sigma,\tau), (\sigma',\tau') \in \Phi$ and $\sigma \preceq \sigma'$, then $\tau \preceq \tau'$ or $\tau' \preceq \tau$. For each $\sigma \in 2^{<\omega}$, we define Φ^σ to be the maximal string (for the prefix order) in $\{\tau : (\exists \sigma' \preceq \sigma)((\sigma',\tau) \in \Phi)\}$. To obtain a map defined on 2^ω from this c.e. set of pairs, for each $X \in 2^\omega$, we let Φ^X be the maximal (for the prefix order) sequence Y such that $\Phi^{X\upharpoonright n}$ is a prefix of Y for all n. Note that Y can be finite or infinite. We will thus set $\mathsf{dom}(\Phi) = \{X \in 2^\omega : \Phi^X \in 2^\omega\}$. When $\Phi^X \in 2^\omega$, we will often write Φ^X as $\Phi(X)$ to emphasize the functional Φ as a map from 2^ω to 2^ω. For $\tau \in 2^{<\omega}$, let $\Phi^{-1}(\tau)$ be the set $\{\sigma \in 2^{<\omega} : \exists \tau' \succeq \tau \colon (\sigma,\tau') \in \Phi\}$. Similarly, for $S \subseteq 2^{<\omega}$ we define $\Phi^{-1}(S) = \bigcup_{\tau \in S} \Phi^{-1}(\tau)$. When $\mathcal{A}$ is a subset of 2^ω, we denote by $\Phi^{-1}(\mathcal{A})$ the set $\{X \in \mathsf{dom}(\Phi) : \Phi(X) \in \mathcal{A}\}$. Note in particular that $\Phi^{-1}([\![\tau]\!]) = [\![\Phi^{-1}(\tau)]\!] \cap \mathsf{dom}(\Phi)$.

The Turing functionals that induce computable measures are precisely the *almost total* Turing functionals, where a Turing functional Φ is almost total if

$$\lambda\big(\mathsf{dom}(\Phi)\big) = 1.$$

Given an almost total Turing functional Φ, the measure induced by Φ, denoted λ_Φ, is defined by

$$\lambda_\Phi(\sigma) = \lambda\big(\llbracket \Phi^{-1}(\sigma) \rrbracket\big) = \lambda\big(\{X : \Phi^X \succeq \sigma\}\big).$$

It is not difficult to verify that λ_Φ is a computable measure. Moreover, given a computable measure μ, one can show that there is some almost total functional Φ such that $\mu = \lambda_\Phi$.

The following two results are very useful. The first one, due to Zvonkin and Levin [17, Theorem 4.2.6] and known as the preservation of randomness theorem, says that randomness is preserved under almost total Turing functionals. The second one, due to Shen (unpublished, but see [15, Theorem 5.1]), is a partial converse of the preservation of randomness theorem. It says that sequences that are random with respect to some computable measure must have some unbiased random source. We thus refer to this result as the *no randomness ex nihilo principle*, to reflect that one cannot produce randomness solely out of nonrandom sources.

Theorem 2.4 *Let Φ be an almost total Turing functional.*

(i) Preservation of randomness: If $X \in \mathsf{MLR}$, then $\Phi(X) \in \mathsf{MLR}_{\lambda_\Phi}$.
(ii) No randomness ex nihilo principle: If $Y \in \mathsf{MLR}_{\lambda_\Phi}$, then there is some $X \in \mathsf{MLR}$ such that $\Phi(X) = Y$.

It will be helpful to introduce several other notions of algorithmic randomness for computable measures. First, the definition of Martin-Löf randomness can be straightforwardly relativized to an oracle (for details, see [3] or [11]). In particular, for each n, if we relativize Martin-Löf randomness to $\emptyset^{(n)}$, the nth jump of the empty set, this yields a notion known as $(n+1)$*-randomness*. Another definition of randomness we consider is weak 2-randomness.

Definition 2.5 Let μ be a computable measure.

(i) A *generalized μ-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly Σ^0_1 subsets of 2^ω such that

$$\lim_{i\to\infty} \mu(\mathcal{U}_i) = 0.$$

(ii) $X \in 2^\omega$ passes the generalized μ-Martin-Löf test $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} \mathcal{U}_i$.
(iii) $X \in 2^\omega$ is *μ-weakly 2-random*, denoted $X \in \mathsf{W2R}_\mu$, if X passes every generalized μ-Martin-Löf test. When μ is the uniform measure λ, we often abbreviate $\mathsf{W2R}_\mu$ by $\mathsf{W2R}$.

As every μ-Martin-Löf test is a generalized μ-Martin-Löf test, it follows that $\mathsf{W2R}_\mu \subseteq \mathsf{MLR}_\mu$. In general, the converse does not hold, as shown by the following result. Recall that $A, B \in 2^\omega$ form a Turing minimal pair if, for any $C \in 2^\omega$, $C \leq_T A$ and $C \leq_T B$ imply that $C \equiv_T \emptyset$.

Theorem 2.6 *Let μ be a computable measure on 2^ω. If $X \in 2^\omega$ is not computable, then X is μ-weakly 2-random if and only if X is μ-Martin-Löf random and forms a Turing minimal pair with $\emptyset'$.*

The proof of this theorem is a generalization of the proof of the result in the case in which μ is the Lebesgue measure (see Porter [12, proof of Theorem 2.69] for details). One direction of the original theorem was proved by Downey, Nies, Weber, and Yu [4], while the other direction was proved by Hirschfeldt and Miller (unpublished; see [3, Theorem 7.2.11]).

2.2 Randomness with respect to noncomputable measures Let $\mathscr{P}(2^\omega)$ be the collection of probability measures on 2^ω. It can be equipped with a natural topology, the so-called *weak topology*. The set $\mathscr{B}$ of subsets of $\mathscr{P}(2^\omega)$ of type

$$\Big\{\mu : \bigwedge_{i=1}^{n}\big[\ell_i < \mu(\sigma_i) < r_i\big]\Big\},$$

where the σ_i's are strings and the ℓ_i's and r_i's are rational numbers, form a base for this topology. Note that such sets can be encoded by an integer, and we call B_i the *set of code* i.

We will consider two general approaches to defining randomness for a noncomputable measure $\mu \in \mathscr{P}(2^\omega)$, depending on whether our test has access to the measure as an oracle. If we allow our test to have access to the measure as an oracle, we first need to code it as an infinite binary sequence. For this we fix a surjective partial map $\Theta : 2^\omega \to \mathscr{P}(2^\omega)$, defined on a Π^0_1-subset of 2^ω, which must have the following property: from every enumeration of $X \in \mathrm{dom}(\Theta)$ (seen as a subset of ω), one can uniformly enumerate the B_i's containing μ, and from any enumeration of the B_i's containing μ, one can uniformly enumerate some preimage of μ by Θ. We say that an enumeration of X is a *representation* of $\Theta(X)$. There are a number of equivalent ways to carry this out (see Reimann [13] or Day and Miller [2]), the easiest one being to define $\Theta(X)$ to be the measure (if it exists and is unique) contained in B_i for each $i \in X$.

The important caveat is that there are measures such that, among all their representations, there is none of smallest Turing degree (this follows, for example, from the existence of a neutral measure, as shown by Levin [8]), a phenomenon which occurs no matter what particular representation is chosen. Therefore, there is no canonical way—in terms of Turing degree—to represent a measure by a unique member of 2^ω, and subsequently, in any definition where one wants to treat μ as an oracle, one needs to quantify over representations of μ.

Definition 2.7 Let μ be a measure on 2^ω, and let R be a representation of μ.

(i) An *R-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly $\Sigma^0_1(R)$ subsets of 2^ω such that, for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

(ii) $X \in 2^\omega$ passes the R-Martin-Löf test $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega}\mathcal{U}_i$.

(iii) $X \in 2^\omega$ is *R-Martin-Löf random*, denoted $X \in \mathrm{MLR}^R_\mu$, if X passes every R-Martin-Löf test.

(iv) $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if there is some representation R of μ such that X is R-Martin-Löf random.

An alternative approach to defining randomness with respect to a noncomputable measure dispenses with the representations, resulting in what is known as *blind randomness* (or *Hippocratic randomness*, as it was called by Kjos-Hanssen [7], where the definition first appeared).

Definition 2.8 Let μ be a measure on 2^ω.

(i) A *blind μ-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly Σ^0_1 (i.e., effectively open) subsets of 2^ω such that, for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

(ii) $X \in 2^\omega$ passes the blind μ-Martin-Löf test $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} \mathcal{U}_i$.

(iii) $X \in 2^\omega$ is *blind μ-Martin-Löf random*, denoted $X \in \mathsf{bMLR}_\mu$, if X passes every blind μ-Martin-Löf test.

2.3 Some basic facts about left-c.e. semimeasures Recall from the Introduction that a semimeasure $\rho : 2^{<\omega} \to [0,1]$ satisfies

(i) $\rho(\varepsilon) = 1$ and

(ii) $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$.

Henceforth, we will restrict our attention to the class of left-c.e. semimeasures, where a semimeasure ρ is *left-c.e.* if, uniformly in σ, there is a computable function $\tilde{\rho} : 2^{<\omega} \times \omega \to \mathbb{Q}_2^+$, nondecreasing in its first argument, and such that for all σ,

$$\lim_{i\to+\infty} \tilde{\rho}(\sigma, i) = \rho(\sigma).$$

That is, the values of ρ on basic open sets are uniformly approximable from below.

Just as computable measures are precisely the measures that are induced by almost-total Turing functionals, left-c.e. semimeasures are precisely the semimeasures that are induced by Turing functionals.

Theorem 2.9 (Zvonkin and Levin [17, Remark 3.7])

(i) For every Turing functional Φ, the function

$$\lambda_\Phi(\sigma) = \lambda\big(\llbracket \Phi^{-1}(\sigma) \rrbracket\big) = \lambda\big(\{X : \Phi^X \succeq \sigma\}\big)$$

is a left-c.e. semimeasure.

(ii) For every left-c.e. semimeasure ρ, there is a Turing functional Φ such that $\rho = \lambda_\Phi$.

Another significant fact about left-c.e. semimeasures is the existence of a *universal* left-c.e. semimeasure: there exists a left-c.e. semimeasure M such that, for every left-c.e. semimeasure ρ, there exists a $c \in \omega$ such that $\rho \leq c \cdot M$. One way to obtain a universal left-c.e. semimeasure is to effectively list all left-c.e. semimeasures $(\rho_e)_{e\in\omega}$ (which can be obtained from an effective list of all Turing functionals by appealing to Theorem 2.9) and set $M = \sum_{e\in\omega} 2^{-e-1}\rho_e$. Alternatively, one can induce it by means of a universal Turing functional. Let $(\Phi_i)_{i\in\omega}$ be an effective enumeration of all Turing functionals. Then the functional $\widehat{\Phi}$ such that

$$\widehat{\Phi}(1^e 0X) = \Phi_e(X)$$

for every $e \in \omega$ and $X \in 2^\omega$ is a universal Turing functional, and we can set $M = \lambda_{\widehat{\Phi}}$. Then M is an universal left-c.e. semimeasure, since for any left-c.e.

semimeasure ρ, there is some Φ_e such that $\rho = \lambda_{\Phi_e}$, and thus by the definition of $\widehat{\Phi}$, we have $\lambda_{\Phi_e} \leq 2^{e+1} \cdot \lambda_{\widehat{\Phi}}$.

3 Shen's Question for Computable Semimeasures

In this section, we provide a positive answer to Shen's question for the case of computable measures. That is, we prove the following.

Theorem 3.1 *If Φ and Ψ are Turing functionals such that $\lambda_\Phi = \lambda_\Psi$ and λ_Φ is computable, then* $\Phi(\mathsf{MLR}) = \Psi(\mathsf{MLR})$.

To prove Theorem 3.1, we extend the definition of Martin-Löf randomness with respect to computable measures to a definition of Martin-Löf randomness with respect to computable semimeasures.

The definition of a computable semimeasure is just a slight modification of the definition of a computable measure: a semimeasure ρ is *computable* if there is a computable function $\tilde{\rho} : 2^{<\omega} \times \omega \to \mathbb{Q}_2^+$ such that

$$\left|\rho(\sigma) - \tilde{\rho}(\sigma, i)\right| \leq 2^{-i}$$

for every $\sigma \in 2^{<\omega}$ and $i \in \omega$.

To define Martin-Löf randomness with respect to a computable semimeasure, we have to exercise some caution. In general, for a given Σ_1^0 class $\mathcal{U}$, $\rho(\mathcal{U})$ is not well defined, as there may exist prefix-free sets $E_0, E_1 \subseteq 2^{<\omega}$ such that $[\![E_0]\!] = [\![E_1]\!] = \mathcal{U}$ but $\rho(E_0) \neq \rho(E_1)$, if one defines $\rho(E) = \sum_{\sigma \in E} \rho(\sigma)$ for $E \subseteq 2^{<\omega}$.

To remedy this problem, we will only apply semimeasures to c.e. subsets of $2^{<\omega}$ rather than to effectively open subsets of 2^ω. Moreover, since any c.e. set $E \subseteq 2^{<\omega}$ may be replaced with a prefix-free c.e. set $F \subseteq 2^{<\omega}$ such that $[\![F]\!] = [\![E]\!]$ and $\rho(F) \leq \rho(E)$ for any semimeasure ρ, we can always assume that a given c.e. subset of $2^{<\omega}$ is prefix-free. This replacement can be done uniformly, so whenever we need to consider a uniformly c.e. sequence $(E_i)_{i\in\omega}$ of subsets of $2^{<\omega}$, we may assume that the sets E_i are all prefix-free.

Definition 3.2 Let ρ be a computable semimeasure.

(i) A *ρ-Martin-Löf test* is a uniformly c.e. sequence $(U_i)_{i\in\omega}$ of subsets of $2^{<\omega}$ such that

$$\rho(U_i) \leq 2^{-i}$$

for each $i \in \omega$.

(ii) $X \in 2^\omega$ passes the ρ-Martin-Löf test $(U_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} [\![U_i]\!]$.

(iii) $X \in 2^\omega$ is *ρ-Martin-Löf random*, denoted $X \in \mathsf{MLR}_\rho$, if X passes every ρ-Martin-Löf test.

We now verify that randomness with respect to a *computable* semimeasure satisfies both randomness preservation and the No Randomness Ex Nihilo principle. Whereas the proof of randomness preservation for computable measures is essentially the same as the standard proof of randomness preservation for computable measures, the proof of the no randomness ex nihilo principle for computable semimeasures is considerably more delicate than the original proof.

Theorem 3.3 (Randomness preservation for computable semimeasures) *If Φ is a Turing functional that induces a computable semimeasure ρ, then $X \in$* $\mathsf{MLR} \cap \mathsf{dom}(\Phi)$ *implies that* $\Phi(X) \in \mathsf{MLR}_\rho$.

Proof Suppose that we have $X \in \mathsf{dom}(\Phi)$ such that $\Phi(X) \notin \mathsf{MLR}_\rho$. Then there is a ρ-Martin-Löf test $(U_i)_{i\in\omega}$ such that $\Phi(X) \in \bigcap_{i\in\omega} [\![U_i]\!]$. We define

$$\mathcal{V}_i = \bigcup_{\tau \in U_i} [\![\Phi^{-1}(\tau)]\!].$$

Clearly, the collection $(\mathcal{V}_i)_{i\in\omega}$ is uniformly Σ^0_1. Then

$$\lambda(\mathcal{V}_i) \le \sum_{\tau\in U_i} \lambda\big([\![\Phi^{-1}(\tau)]\!]\big) = \sum_{\tau \in U_i} \rho(\tau) = \rho(U_i) \le 2^{-i},$$

so $(\mathcal{V}_i)_{i\in\omega}$ is a Martin-Löf test. Lastly, $\Phi(X) \in [\![U_i]\!]$ for each $i \in \omega$; thus for each $i \in \omega$ there is some $\tau \preceq \Phi(X)$ such that $\tau \in U_i$. This implies that $X \in [\![\Phi^{-1}(\tau)]\!]$, and so we have $X \in \mathcal{V}_i$. Thus, $X \notin \mathsf{MLR}$. □

Theorem 3.4 (No randomness ex nihilo principle for computable semimeasures) *Let Φ be a Turing functional that induces a computable semimeasure ρ. If $Y \in \mathsf{MLR}_\rho$, then there is some $X \in \mathsf{MLR}$ such that $\Phi(X) = Y$.*

Proof First, we define a collection of Turing functionals $(\widehat{\Phi}_e)_{e\in\omega}$ that will serve as approximations for the functional Φ. Note that $\mathsf{dom}(\Phi) = \bigcap_{\ell\in\omega} \mathcal{S}_\ell$, where for each ℓ,

$$\mathcal{S}_\ell = \big\{X \in 2^\omega : (\exists k)\ |\Phi^{X\restriction k}| \ge \ell\big\}$$

(which is uniformly effectively open).

For each e, we define a sequence of finite sets of strings $(C^e_\ell)_{\ell\in\omega}$ such that for every ℓ,

(i) $[\![C^e_\ell]\!] \subseteq \mathcal{S}_\ell$, and
(ii) $\lambda(\mathcal{S}_\ell \setminus [\![C^e_\ell]\!]) \le 2^{-\ell-e-1}$.

The sequence $(C^e_\ell)_{\ell\in\omega}$ can be effectively obtained, since ρ is a computable semimeasure that is induced by Φ, which implies that $\lambda(\mathcal{S}_\ell)$ is computable uniformly in ℓ. Each $[\![C^e_\ell]\!]$ is clopen, and therefore so are the sets $\bigcap_{k\le\ell}[\![C^e_k]\!]$ for each $\ell \in \omega$. Then let $(D^e_\ell)_{e,\ell\in\omega}$ be a computable bisequence of sets of finite strings such that

$$[\![D^e_\ell]\!] = \bigcap_{k\le\ell} [\![C^e_k]\!].$$

Next we set

$$\widehat{\Phi}_e := \big\{(\sigma,\tau) \in \Phi : \sigma \in D^e_{|\tau|}\big\},$$

so that

$$\mathsf{dom}(\widehat{\Phi}_e) = \bigcap_{\ell\in\omega} [\![D^e_\ell]\!] = \bigcap_{\ell\in\omega} [\![C^e_\ell]\!].$$

Since each $[\![D^e_\ell]\!]$ is clopen, it follows that $\mathsf{dom}(\widehat{\Phi}_e)$ is a Π^0_1 class uniformly in e. Moreover, $\widehat{\Phi}_e$ is a restriction of Φ such that

$$\lambda\big(\mathsf{dom}(\Phi) \setminus \mathsf{dom}(\widehat{\Phi}_e)\big) \le \sum_{\ell\in\omega} \lambda\big(\mathcal{S}_\ell \setminus [\![C^e_\ell]\!]\big) \le \sum_{\ell\in\omega} 2^{-e-\ell-1} \le 2^{-e}.$$

Note also that, for each e, ℓ, we have

$$\lambda\big(\mathcal{S}_\ell \setminus [\![D^e_\ell]\!]\big) = \lambda\Big(\bigcap_{k\le\ell} \mathcal{S}_k \setminus \bigcap_{k\le\ell} [\![C^e_k]\!]\Big) \le \sum_{k\le\ell} 2^{-e-k-1} \le 2^{-e}$$

(where we use the definition of the D^e_ℓ's and the fact that the $\mathcal{S}_k$ are nonincreasing), an inequality we will need at the end of the proof.

Now, for each $e \in \omega$, let Θ_e be the predicate on $2^{<\omega}$ defined by

$$\Theta_e(\tau) \text{ if and only if } \forall X\big[X \notin \mathsf{MLR}^e \ \vee\ X \notin [\![D^e_{|\tau|}]\!] \ \vee\ \Phi^X \perp \tau\big],$$

where $\Phi^X \perp \tau$ means that Φ^X has length at least $|\tau|$ and is incomparable with τ, and MLR^e is the complement of the eth level of the universal Martin-Löf test (with respect to the Lebesgue measure). The predicate $[X \notin \mathsf{MLR}^e \vee X \notin [\![D^e_{|\tau|}]\!] \vee \Phi^X \perp \tau]$ is Σ^0_1 over X; therefore, by effective compactness, Θ_e is also Σ^0_1 uniformly in e. For each e, let V_e be a maximal prefix-free set of strings among those satisfying Θ_e. Note that $[\![V_e]\!]$ is effectively open uniformly in e. Let us evaluate $\lambda_\Phi(V_e)$:

$$\begin{aligned}\lambda_\Phi(V_e) &= \lambda\big(\{X : (\exists\tau \in V_e)\ \Phi^X \succeq \tau\}\big)\\ &\le \lambda\Big(\{X : (\exists\tau)\ \Phi^X \succeq \tau \wedge \big(X \notin [\![D^e_{|\tau|}]\!] \vee X \in (\mathsf{MLR}^e)^c\big)\}\Big)\\ &\le \lambda\Big(\bigcup_l \mathcal{S}_l \setminus [\![D^e_l]\!]\Big) + \lambda\big((\mathsf{MLR}^e)^c\big)\\ &\le 2^{-e} + 2^{-e}.\end{aligned}$$

Thus, $(V_e)_{e\in\omega}$ is a λ_Φ-Martin-Löf test. This means that for every λ_Φ-Martin-Löf random Y, there must be an e such that $Y \notin [\![V_e]\!]$, or in other words (by the definition of V_e): for every prefix $Y \restriction \ell$, there is some $X_\ell \in \mathsf{MLR}^e \cap [\![D^e_\ell]\!]$ such that $\widehat{\Phi}_e^{X_\ell} \succeq Y \restriction \ell$. By compactness, one can assume, up to extraction of a subsequence, that the sequence $(X_\ell)_{\ell\in\omega}$ converges to some X^*. Since $X_\ell \in [\![D^e_\ell]\!]$ for all ℓ, and since the sets $[\![D^e_\ell]\!]$ are closed and nonincreasing, it follows that X^* belongs to all $[\![D^e_\ell]\!]$, that is, X^* is in the domain of $\widehat{\Phi}_e$. By the continuity of Turing functionals, $\widehat{\Phi}_e^{X^*} = \lim_\ell \widehat{\Phi}_e^{X_\ell} = \lim_\ell Y \restriction \ell = Y$. Moreover, each X_ℓ belongs to MLR^e and MLR^e is closed, so X^* belongs to MLR^e as well. Therefore Y has a Martin-Löf random preimage by $\widehat{\Phi}_e$, namely, X^*. Since $\widehat{\Phi}_e$ is a restriction of Φ, the result follows. $\square$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1 Given $Y \in \Phi(\mathsf{MLR})$, it follows from Theorem 3.3 that $Y \in \mathsf{MLR}_\rho$. Since Ψ induces ρ, by Theorem 3.4 there is some $X \in \mathsf{MLR}$ such that $\Psi(X) = Y$. This shows that $\Phi(\mathsf{MLR}) \subseteq \Psi(\mathsf{MLR})$, and a symmetric argument shows that $\Psi(\mathsf{MLR}) \subseteq \Phi(\mathsf{MLR})$. $\square$

Note that the positive answer to Shen's question is an immediate consequence of randomness preservation and the no randomness ex nihilo principle. We see this again in Corollary 4.4 below in the context of left-c.e. semimeasures and 2-randomness.

As our definition of randomness with respect to a computable semimeasure behaves much like Martin-Löf randomness with respect to a computable measure, it is reasonable to ask if there are any sequences that are random with respect to some computable semimeasure but no computable measure. We answer this question in the negative.

Proposition 3.5 *We have that $X \in 2^\omega$ is random with respect to a computable measure if and only if X is random with respect to a computable semimeasure.*

Proof As every computable measure is a computable semimeasure, one direction is immediate. Suppose now that X is not random with respect to any computable measure. Let ρ be a computable semimeasure. We define the function $g : 2^{<\omega} \to [0, 1]$ to be

$$g(\sigma) = \rho(\sigma) - \big(\rho(\sigma 0) + \rho(\sigma 1)\big)$$

for every $\sigma \in 2^{<\omega}$. Clearly, g is computable since ρ is. Next we define $\mu : 2^{<\omega} \to [0, 1]$ so that $\mu(\varepsilon) = 1$ and for $|\sigma| \geq 1$,

$$\mu(\sigma) = \rho(\sigma) + \sum_{\tau \prec \sigma} 2^{|\tau|-|\sigma|} g(\tau).$$

Clearly, μ is computable and $\rho(\sigma) \leq \mu(\sigma)$ for every $\sigma \in 2^{<\omega}$. We just need to verify that μ is a measure, which we prove by induction. For any $\sigma \in 2^{<\omega}$

$$\begin{aligned}
\mu(\sigma 0) + \mu(\sigma 1) &= \rho(\sigma 0) + \sum_{\tau \prec \sigma 0} 2^{|\tau|-|\sigma 0|} g(\tau) + \rho(\sigma 1) + \sum_{\tau \prec \sigma 1} 2^{|\tau|-|\sigma 1|} g(\tau) \\
&= \rho(\sigma 0) + \frac{1}{2} \sum_{\tau \prec \sigma 0} 2^{|\tau|-|\sigma|} g(\tau) + \rho(\sigma 1) + \frac{1}{2} \sum_{\tau \prec \sigma 1} 2^{|\tau|-|\sigma|} g(\tau) \\
&= \rho(\sigma 0) + \rho(\sigma 1) + \sum_{\tau \preceq \sigma} 2^{|\tau|-|\sigma|} g(\tau) \\
&= \rho(\sigma 0) + \rho(\sigma 1) + g(\sigma) + \sum_{\tau \prec \sigma} 2^{|\tau|-|\sigma|} g(\tau) \\
&= \rho(\sigma) + \sum_{\tau \prec \sigma} 2^{|\tau|-|\sigma|} g(\tau) \\
&= \mu(\sigma).
\end{aligned}$$

Now since $X \notin \mathsf{MLR}_\mu$ by hypothesis, there is some μ-Martin-Löf test $(\mathcal{U}_i)_{i \in \omega}$ such that $X \in \bigcap_{i \in \omega} \mathcal{U}_i$. By letting U_i be such that $[\![U_i]\!] = \mathcal{U}_i$ for each $i \in \omega$, $\rho(U_i) \leq \mu(\mathcal{U}_i)$ for every i, which implies that $(U_i)_{i \in \omega}$ is a ρ-Martin-Löf test. Thus, $X \notin \mathsf{MLR}_\rho$. □

4 Shen's Question for Left-c.e. Semimeasures

In this section, we prove that Question 1.1, Shen's original question for left-c.e. semimeasures, has a negative answer.

Theorem 4.1 *There exist Turing functionals Φ and Ψ such that $\lambda_\Phi = \lambda_\Psi$ and yet $\Phi(\mathsf{MLR}) \neq \Psi(\mathsf{MLR})$.*

Proof We define Φ and Ψ as c.e. sets of pairs $(\sigma, \tau) \in 2^{<\omega} \times 2^{<\omega}$. Recall that Chaitin's Ω is defined to be

$$\Omega := \sum_{U(\sigma)\downarrow} 2^{-|\sigma|},$$

where U is a universal prefix-free Turing machine (see [3] or [11] for more details). Furthermore, it is well known that Ω is Martin-Löf random and left-c.e. Let $(\Omega_s)_{s \in \omega}$ be a computable nondecreasing sequence of dyadic rationals converging to Ω. We can think of each Ω_s as a finite string, so that for $n < |\Omega_s|$, $\Omega_s(n)$ is the nth bit of the string Ω_s.

We define the functional

$$\Phi = \bigcup_n \{(\Omega_s \upharpoonright n, 0^n) : s \geq n\}.$$

It is easy to see that $\mathsf{dom}(\Phi) = \{\Omega\}$ and $\Phi(\Omega) = 0^\omega$. Indeed, if $X \neq \Omega$, X and Ω disagree on some bit, say, the kth bit, and then for some t we have, for all $s \geq t$, $\Omega_s \upharpoonright k = \Omega \upharpoonright k \neq X \upharpoonright k$ and thus by construction $|\Phi^X| < t$, that is, $X \notin \mathsf{dom}(\Phi)$.

Next, we define Ψ. For each $(\sigma, 0^{|\sigma|})$ that we enumerate into Φ at stage s, let τ be the leftmost string of length $|\sigma|$ such that $(\tau, 0^{|\sigma|})$ has not yet been enumerated into Ψ and enumerate this pair into Ψ. Observe that this construction ensures that (1) for all n, $\lambda_\Phi(0^n) = \lambda_\Psi(0^n)$, and thus $\lambda_\Phi = \lambda_\Psi$ as both are equal to 0 on strings that are not of type 0^n, and (2) the domain of Ψ contains 0^ω and is closed downward under the lexicographic order. A set which is closed downward under the lexicographic order is either the empty set, the singleton 0^ω, or a set of positive measure. It is not the empty set and it cannot have positive measure, because otherwise there would exist a positive r such that $\lambda_\Psi(0^n) > r$ for all n. This is impossible since $\lambda_\Psi = \lambda_\Phi$ and $\lambda_\Phi(0^n)$ tends to 0. Thus, $\mathsf{dom}(\Psi) = \{0^\omega\}$ and $\Psi(0^\omega) = 0^\omega$, which in particular implies that $\Psi(\mathsf{MLR}) = \emptyset \neq \Phi(\mathsf{MLR})$. $\square$

Remark 4.2 The above proof actually works for any Δ^0_2 Martin-Löf random sequence. Further, it is not necessary that $\lambda(\mathsf{dom}(\Phi)) = 0$. If we define $\widehat{\Phi}$ and $\widehat{\Psi}$ by

$$\begin{cases} \widehat{\Phi}(0X) = \Phi(X) \\ \widehat{\Phi}(1X) = X \end{cases} \quad \text{and} \quad \begin{cases} \widehat{\Psi}(0X) = \Psi(X) \\ \widehat{\Psi}(1X) = X \end{cases}$$

(where Φ and Ψ are defined in the previous proof), we then have

$$\lambda\big(\mathsf{dom}(\widehat{\Phi})\big) = \lambda\big(\mathsf{dom}(\widehat{\Psi})\big) = 1/2,$$

while $\lambda_{\widehat{\Phi}} = (\lambda_\Phi + \lambda)/2 = (\lambda_\Psi + \lambda)/2 = \lambda_{\widehat{\Psi}}$, and $\widehat{\Phi}(0\Omega) = \Phi(\Omega)$ has no Martin-Löf random preimage via $\widehat{\Psi}$.

Although Question 1.1 has a negative answer, if we rephrase the question in terms of a stronger notion of randomness, then we can answer the question in the affirmative. To do so, we have to extend our definition of randomness with respect to a computable semimeasure to a definition of 2-randomness with respect to a $\emptyset'$-computable semimeasure.

First, we extend several definitions from the previous section.

- A semimeasure ρ is $\emptyset'$-computable if the values $(\rho(\sigma))_{\sigma \in 2^{<\omega}}$ are uniformly $\emptyset'$-computable.
- For a $\emptyset'$-computable semimeasure ρ, a ρ-$\emptyset'$-Martin-Löf test is a uniformly $\emptyset'$-c.e. sequence $(U_i)_{i\in\omega}$ of subsets of $2^{<\omega}$ such that $\rho(U_i) \leq 2^{-i}$.
- A sequence $X \in 2^\omega$ passes the ρ-$\emptyset'$-Martin-Löf test $(U_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} [\![U_i]\!]$.
- For a $\emptyset'$-computable semimeasure ρ, $X \in 2^\omega$ is ρ-2-random, denoted $X \in \mathsf{2MLR}_\rho$, if X passes every ρ-$\emptyset'$-Martin-Löf test.

Using these definitions, the following result is obtained from relativizing the proof of Theorems 3.3 and 3.4.

Corollary 4.3 *Let ρ be a left-c.e. semimeasure, and let Φ be a Turing functional such that $\rho = \lambda_\Phi$.*

(i) *For every* $X \in \mathsf{2MLR} \cap \mathsf{dom}(\Phi)$, $\Phi(X) \in \mathsf{2MLR}_\rho$.
(ii) *If* $Y \in \mathsf{2MLR}_\rho$, *then there is some* $X \in \mathsf{2MLR}$ *such that* $\Phi(X) = Y$.

Corollary 4.3, together with an argument similar to the one in the proof of Theorem 3.1, yields the following.

Corollary 4.4 *If* Φ *and* Ψ *are Turing functionals such that* $\lambda_\Phi = \lambda_\Psi$, *then* $\Phi(\mathsf{2MLR}) = \Psi(\mathsf{2MLR})$.

5 The Direct Adaptation Approach

A positive answer to Shen's question would have yielded a definition of randomness with respect to a left-c.e. semimeasure: for a left-c.e. semimeasure ρ, the sequences that are random with respect to ρ would simply be the images of the Martin-Löf random sequences under any functional that induces ρ. But as we have answered Shen's question in the negative, we need a different strategy to define randomness with respect to a left-c.e. semimeasure.

In this section, we discuss certain desiderata for our definition and then we consider several definitions of randomness with respect to a left-c.e. semimeasure that are obtained by directly modifying standard definitions of randomness with respect to a computable measure.

5.1 Desiderata for a definition of randomness with respect to a left-c.e. semimeasure Given that the collection of left-c.e. semimeasures extends the collection of computable measures, we would like our theory of randomness with respect to a left-c.e. semimeasure to extend the standard theory of randomness with respect to a computable measure. To this end, it would be ideal to find a definition of randomness with respect to a semimeasure that satisfies a number of conditions, which we describe below.

First, as every computable measure is a left-c.e. semimeasure, it seems natural to require the following.

(i) *Coherence*: X is random with respect to a computable measure μ if and only if X is random with respect to μ considered as a left-c.e. semimeasure.

Second, as the relationship between almost-total Turing functionals and computable measures is analogous to the relationship between Turing functionals and left-c.e. semimeasures, we would like to extend the analogy by requiring the following two conditions.

(ii) *Randomness preservation*: If X is random and Φ is a Turing functional, then $\Phi(X)$ is random with respect to the semimeasure λ_Φ.
(iii) *No randomness ex nihilo principle*: If Y is random with respect to the semimeasure λ_Φ for some Turing functional Φ, then there is some random X such that $\Phi(X) = Y$.

Lastly, in the theory of randomness with respect to a measure (computable or noncomputable), a computable sequence is random with respect to some measure μ only if it is an atom of μ, as shown by Reimann and Slaman [14]. We extend this to the case of left-c.e. semimeasures.

(iv) *Computable sequence condition*: If X is computable and random with respect to a left-c.e. semimeasure ρ, then $\inf_n \rho(X{\upharpoonright}n) > 0$.

With these conditions in mind, we now turn to a first candidate definition for randomness with respect to a semimeasure.

5.2 Martin-Löf randomness with respect to a left-c.e. semimeasure First, we consider the same modification of Martin-Löf randomness that we made in Section 3 when defining randomness for a computable semimeasure.

Definition 5.1 Let ρ be a left-c.e. semimeasure.

(i) A *ρ-Martin-Löf test* is a sequence $(U_i)_{i\in\omega}$ of uniformly c.e. subsets of $2^{<\omega}$ such that, for each i,

$$\rho(U_i) \leq 2^{-i}.$$

(ii) $X \in 2^\omega$ passes the ρ-Martin-Löf test $(U_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega}[\![U_i]\!]$.

(iii) $X \in 2^\omega$ is *ρ-Martin-Löf random*, denoted $X \in \mathsf{MLR}_\rho$, if X passes every ρ-Martin-Löf test.

One interesting consequence of this definition is that the universal left-c.e. semimeasure M is universal for Martin-Löf randomness with respect to a left-c.e. semimeasure.

Proposition 5.2 *Let $\mathcal{S}$ be the collection of left-c.e. semimeasures. Then* $\mathsf{MLR}_M = \bigcup_{\rho\in\mathcal{S}} \mathsf{MLR}_\rho$.

Proof Clearly, $\mathsf{MLR}_M \subseteq \bigcup_{\rho\in\mathcal{S}} \mathsf{MLR}_\rho$. For the other direction, note that for any left-c.e. semimeasure ρ, every M-Martin-Löf test can be transformed into a ρ-Martin-Löf test since there is some c such that $\rho(\sigma) \leq c \cdot M(\sigma)$ for every $\sigma \in 2^{<\omega}$. Thus, if $X \notin \mathsf{MLR}_M$, it follows that $X \notin \mathsf{MLR}_\rho$ for any left-c.e. semimeasure ρ. □

Even though every universal left-c.e. semimeasure is universal in the sense of Proposition 5.2, the converse does not hold.

Proposition 5.3 *There is a nonuniversal left-c.e. semimeasure $\widetilde{M}$ such that*

$$\mathsf{MLR}_{\widetilde{M}} = \mathsf{MLR}_M = \bigcup_{\rho\in\mathcal{S}} \mathsf{MLR}_\rho.$$

Proof First, define a semimeasure ρ by $\rho(\sigma) = 2^{-j} M(\sigma)$, where j is the largest integer such that $1^j \preceq \sigma$. The semimeasure ρ is left-c.e., but it cannot be universal as there is no c such that $c \cdot \rho(\sigma) \geq M(\sigma)$ for every $\sigma \in 2^{<\omega}$.

Consider a ρ-Martin-Löf test $(T_i)_{i\in\omega}$. For each $j \in \omega$, define an M-Martin-Löf test $(S_i^j)_{i\in\omega}$ by $S_i^j = \{\sigma \in T_{i+j} : \sigma \succeq 1^j 0\}$. For each $\sigma \in S_i^j$, $\sigma = 1^j 0\tau$ for some $\tau \in 2^{<\omega}$. It follows that $M(\sigma) = M(1^j 0\tau) = 2^j \rho(1^j 0\tau) = 2^j \rho(\sigma)$. Thus we have

$$\sum_{\sigma\in S_i^j} M(\sigma) = 2^j \sum_{\sigma\in S_i^j} \rho(\sigma) \leq 2^j \sum_{\sigma\in T_{i+j}} \rho(\sigma) \leq 2^j 2^{-(i+j)} = 2^{-i}.$$

Clearly, every sequence containing a 0 that is covered by $(T_i)_{i\in\omega}$ is covered by $(S_i^j)_{i\in\omega}$ for some j. Thus ρ is almost the desired measure: we have $\mathsf{MLR}_M \subseteq \mathsf{MLR}_\rho \cup \{1^\omega\}$. Consider then the measure δ_{1^ω}, where $\delta_{1^\omega}(\sigma) = 1$ if $\sigma = 1^n$ for some $n \in \omega$ and $\delta_{1^\omega}(\sigma) = 0$ otherwise. Let $\widetilde{M} = (1/2)\rho + (1/2)\delta_{1^\omega}$. Then $\widetilde{M}$ is not universal, and $\mathsf{MLR}_{\widetilde{M}} \subseteq \mathsf{MLR}_M$ by Proposition 5.2. Finally, one easily checks that $\mathsf{MLR}_{\widetilde{M}} = \mathsf{MLR}_\rho \cup \mathsf{MLR}_{\delta_{1^\omega}} = \mathsf{MLR}_\rho \cup \{1^\omega\}$. Hence $\mathsf{MLR}_M \subseteq \mathsf{MLR}_{\widetilde{M}}$ as well. □

Now we evaluate the adequacy of our definition in terms of the desiderata laid out in Section 5.1. Clearly, this definition satisfies the condition of coherence. Moreover, we can show that it also satisfies randomness preservation.

Theorem 5.4 *If $X \in \mathsf{MLR}$ and Φ is a Turing functional such that $X \in \mathsf{dom}(\Phi)$, then $\Phi(X) \in \mathsf{MLR}_{\lambda_\Phi}$.*

Proof Suppose that there is a λ_Φ-Martin-Löf test $(U_i)_{i\in\omega}$ such that $\Phi(X) \in \bigcap_{i\in\omega}[\![U_i]\!]$. Then $([\![\Phi^{-1}(U_i)]\!])_{i\in\omega}$ is a uniform sequence of Σ^0_1 subsets of 2^ω, and

$$\lambda\big([\![\Phi^{-1}(U_i)]\!]\big) = \lambda_\Phi(U_i) \leq 2^{-i}$$

for every i, so $([\![\Phi^{-1}(U_i)]\!])_{i\in\omega}$ is a Martin-Löf test containing X. □

Remark 5.5 Despite satisfying these two conditions, in general ρ-Martin-Löf randomness fails to satisfy the no randomness ex nihilo principle and the computable sequence condition. First, for the counterexample to the no randomness ex nihilo principle, let ρ be the semimeasure constructed in the proof of Theorem 4.1. There we constructed functionals Φ and Ψ inducing ρ such that $\mathsf{dom}(\Phi) = \{\Omega\}$, $\mathsf{dom}(\Psi) = \{0^\omega\}$, and $\Phi(\Omega) = 0^\omega = \Psi(0^\omega)$. By Theorem 5.4, $0^\omega \in \mathsf{MLR}_\rho$. However, Ψ induces ρ and yet maps no Martin-Löf random sequence to 0^ω. The same example provides a counterexample to the computable sequence condition: 0^ω is ρ-Martin-Löf random, but $\inf_n \rho(0^n) = 0$.

We can also construct a left-c.e. semimeasure ρ that fails to satisfy the computable sequence condition in the strongest possible way: ρ has no atoms and yet $\mathsf{MLR}_\rho = 2^\omega$.

Theorem 5.6 *There is a nonatomic left-c.e. semimeasure ρ such that every $X \in 2^\omega$ is Martin-Löf random for ρ.*

Proof Let $(E^e_n)_{\langle e,n\rangle\in\omega}$ be an effective list of all uniformly c.e. sequences of subsets of $2^{<\omega}$. We satisfy the requirements

$$\mathcal{R}_e\colon \bigcap_{n\in\omega}[\![E^e_n]\!] \neq \emptyset \to (\exists n \in \omega)\big(\rho(E^e_n) > 2^{-n}\big).$$

Satisfying all of these requirements ensures that if $(E_n)_{n\in\omega}$ defines a ρ-Martin-Löf test, then $\bigcap_{n\in\omega}[\![E^e_n]\!] = \emptyset$. Therefore, every $X \in 2^\omega$ is ρ-Martin-Löf random.

For each e, we build a left-c.e. semimeasure ρ_e (where we relax the requirement $\rho_e(\varepsilon) = 1$ to $\rho_e(\varepsilon) \leq 1$) as follows.

- Start with $\rho_e(\sigma) = 0$ for all σ.
- If at some stage some τ enters E^e_{e+2}, set $\rho_e(\tau') = 2^{-e-1}$ for all prefixes of τ (including τ itself) and finish the construction.

Clearly, ρ_e is a left-c.e. semimeasure such that $\rho_e(E^e_{e+2}) > 2^{-e-2}$ if $E^e_{e+2} \neq \emptyset$, and $\rho_e(\varepsilon) \leq 2^{-e-1}$. Thus, define ρ by $\rho(\varepsilon) = 1$ and $\rho(\sigma) = \sum_{e\in\omega}\rho_e(\sigma)$ for all σ with $|\sigma| > 0$. Then ρ is a left-c.e. semimeasure such that $\rho(E^e_{e+2}) > 2^{-e-2}$ if $E^e_{e+2} \neq \emptyset$. □

Note that Proposition 5.2 and Theorem 5.6 together imply the following.

Corollary 5.7 *We have $\mathsf{MLR}_M = 2^\omega$.*

In light of the fact that ρ-Martin-Löf randomness does not always satisfy the desiderata from Section 5.1, we consider other definitions of randomness for a left-c.e. semimeasure.

5.3 Weak 2-randomness with respect to a left-c.e. semimeasure We can obtain the definition of weak 2-randomness for a left-c.e. semimeasure by modifying the notion of a generalized Martin-Löf test.

Definition 5.8 Let ρ be a left-c.e. semimeasure.

(i) A *generalized ρ-Martin-Löf test* is a sequence $(U_i)_{i\in\omega}$ of uniformly c.e. subsets of $2^{<\omega}$ such that

$$\lim_{i\to\infty} \rho(U_i) = 0.$$

(ii) $X \in 2^\omega$ passes the generalized ρ-Martin-Löf test $(U_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} [\![U_i]\!]$.

(iii) $X \in 2^\omega$ is *ρ-weakly 2-random*, denoted $X \in \mathsf{W2R}_\rho$, if X passes every generalized ρ-Martin-Löf test.

Weak 2-randomness for a left-c.e. semimeasure is more well behaved than the previous definition considered in this section, as it satisfies both randomness preservation and the computable sequence condition.

Theorem 5.9 *Let ρ be a left-c.e. semimeasure, and let Φ be a Turing functional that induces ρ. Then for every $X \in \mathsf{W2R} \cap \mathsf{dom}(\Phi)$, $\Phi(X) \in \mathsf{W2R}_\rho$.*

Proof The proof is nearly identical to the proof of Theorem 5.4. □

Proposition 5.10 *Let ρ be a left-c.e. semimeasure. Suppose that X is computable and that $X \in \mathsf{W2R}_\rho$. Then $\inf_n \rho(X\restriction n) > 0$.*

Proof Suppose that X is computable and that $\inf_n \rho(X\restriction n) = 0$. Then setting $U_i = \{X\restriction i\}$ for each $i \in \omega$ yields a generalized ρ-Martin-Löf test capturing X. □

Clearly, $\mathsf{W2R}_\rho \subseteq \mathsf{MLR}_\rho$, but for some semimeasures ρ (such as any ρ such that ρ-Martin-Löf randomness violates the computable sequence condition), the inclusion is strict. We should note further that the universal left-c.e. semimeasure M is universal for weak 2-randomness, as is the nonuniversal $\widetilde{M}$ from Proposition 5.3:

$$\mathsf{W2R}_{\widetilde{M}} = \mathsf{W2R}_M = \bigcup_{\rho\in\mathcal{S}} \mathsf{W2R}_\rho.$$

We have seen that weak 2-randomness for a semimeasure satisfies coherence, randomness preservation, and the computable sequence condition, but we currently do not know whether it satisfies the no randomness ex nihilo principle. We will return to this question at the end of Section 6.

We now turn to another general approach to defining randomness with respect to a semimeasure, an approach that is found implicitly in Levin and V'yugin [9] and V'yugin [16].

6 Trimming a Semimeasure Back to a Measure

We can also define randomness with respect to a semimeasure by trimming back our semimeasure to a measure and then considering the sequences that are random with respect to the resulting measure.

6.1 Definition of a derived measure and examples To better understand this approach, it is helpful to think of a semimeasure as a network flow through the full binary tree $2^{<\omega}$ seen as a directed graph. We initially assign 1 to be the value of the flow at the root of the tree, which implies that $\rho(\varepsilon) = 1$. Some amount of this flow at each node σ is passed along to the node corresponding to $\sigma 0$, some is passed along to the node corresponding to $\sigma 1$, and, potentially, some of the flow is lost, yielding the condition that $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$.

We obtain a measure $\overline{\rho}$ from ρ if we ignore all of the flow that is lost and just consider the behavior of the flow that never leaves the network. We will refer to $\overline{\rho}$ as the measure derived from ρ. This can be formalized as follows.

Definition 6.1 ([9, p. 360]) Let ρ be a semimeasure. We have

$$\begin{aligned}\overline{\rho}(\sigma) &:= \inf_{n \geq |\sigma|} \sum_{\tau \succeq \sigma \ \& \ |\tau| = n} \rho(\tau) \\ &= \lim_{n \to \infty} \sum_{\tau \succeq \sigma \ \& \ |\tau| = n} \rho(\tau).\end{aligned}$$

The fact that one can use either inf or lim in the expression is due to the fact that the term $\sum_{\tau \succeq \sigma \ \& \ |\tau| = n} \rho(\tau)$ is nonincreasing in n by the semimeasure inequality $\rho(\tau) \geq \rho(\tau 0) + \rho(\tau 1)$.

The following are two simple examples illustrating the different behaviors of ρ and $\overline{\rho}$.

Example 6.2 Let $\rho(\sigma) = 4^{-|\sigma|}$ for every $\sigma \in 2^{<\omega}$. Then for each $\sigma \in 2^{<\omega}$ and each $n \in \omega$,

$$\sum_{\tau \succeq \sigma \ \& \ |\tau| = n} \rho(\tau) = 2^{n-|\sigma|} 4^{-n} = 2^{-n} 2^{-|\sigma|}.$$

Thus, $\overline{\rho}(\sigma) = 0$ for every $\sigma \in 2^{<\omega}$.

Example 6.3 Let ρ be a semimeasure such that $\rho(\sigma) = \frac{1}{2}\lambda(\sigma) + \frac{1}{2^{2|\sigma|+1}}$. Then for each $\sigma \in 2^{<\omega}$ and each $n \in \omega$,

$$\begin{aligned}\sum_{\tau \succeq \sigma \ \& \ |\tau| = n} \rho(\tau) &= 2^{n-|\sigma|}\left(\frac{1}{2}\lambda(\tau) + \frac{1}{2^{2|\tau|+1}}\right) \\ &= 2^{n-|\sigma|}\left(\frac{1}{2}2^{-n} + \frac{1}{2^{2n+1}}\right) \\ &= \frac{1}{2}2^{-|\sigma|} + \frac{2^{n-|\sigma|}}{2^{2n+1}} \\ &= \frac{1}{2}\lambda(\sigma) + 2^{-(n+1)}\lambda(\sigma).\end{aligned}$$

Thus, $\overline{\rho}(\sigma) = \frac{1}{2}\lambda(\sigma)$ for every $\sigma \in 2^{<\omega}$.

This latter example yields what we will refer to as a *Lebesgue-like* semimeasure.

Definition 6.4 A semimeasure ρ is *Lebesgue-like* if there is some $\alpha \in (0, 1]$ such that

$$\overline{\rho} = \alpha \cdot \lambda.$$

Let us now show that $\overline{\rho}$ is indeed a measure which enjoys some nice properties, both from the analytic viewpoint and in connection with Turing functionals. The following proposition is probably folklore; an explicit reference is hard to find in the literature.

Proposition 6.5 *Let ρ be a semimeasure, and let $\overline{\rho}$ be defined as above. Then $\overline{\rho}$ is the largest measure μ such that $\mu \le \rho$. In particular, if ρ is a measure, then $\overline{\rho} = \rho$. Moreover, if*

$$\rho(\sigma) = \lambda\big(\{X : \Phi^X \succeq \sigma\}\big),$$

then

$$\overline{\rho}(\sigma) = \lambda\big(\big\{X \in \mathsf{dom}(\Phi) : \Phi^X \succeq \sigma\big\}\big).$$

(Thus, trimming ρ back to $\overline{\rho}$ amounts to restricting the Turing functional Φ that induces ρ to those inputs on which Φ is total.)

Proof The fact that $\overline{\rho}$ is a measure is clear from the definition

$$\overline{\rho}(\sigma) = \lim_{n\to\infty} \sum_{\tau\succeq\sigma \ \& \ |\tau|=n} \rho(\tau),$$

since we then have

$$\begin{aligned}\overline{\rho}(\sigma 0) + \overline{\rho}(\sigma 1) &= \lim_{n\to\infty} \sum_{\tau\succeq\sigma 0 \ \& \ |\tau|=n} \rho(\tau) + \lim_{n\to\infty} \sum_{\tau\succeq\sigma 1 \ \& \ |\tau|=n} \rho(\tau)\\ &= \lim_{n\to\infty} \sum_{\tau\succeq\sigma \ \& \ |\tau|=n} \rho(\tau)\\ &= \overline{\rho}(\sigma).\end{aligned}$$

Now, if μ is a measure such that $\mu(\sigma) \le \rho(\sigma)$ for all σ, then for any given σ:

$$\begin{aligned}\mu(\sigma) &= \inf_{n\ge|\sigma|} \sum_{\tau\succeq\sigma \ \& \ |\tau|=n} \mu(\tau)\\ &\le \inf_{n\ge|\sigma|} \sum_{\tau\succeq\sigma \ \& \ |\tau|=n} \rho(\tau)\\ &= \overline{\rho}(\sigma).\end{aligned}$$

(For the first equality, we used the measure property $\mu(\tau) = \mu(\tau 0) + \mu(\tau 1)$.)

Suppose now that ρ is induced by some Turing functional Φ, that is,

$$\rho(\sigma) = \lambda\big(\{X : \Phi^X \succeq \sigma\}\big)$$

for all σ. Set $\mu(\sigma) = \lambda(\{X \in \mathsf{dom}(\Phi) : \Phi^X \succeq \sigma\})$.

Let $\mathcal{D}_n$ be the set of X such that Φ^X is of length at least n. The sets $\mathcal{D}_n$ are nonincreasing in n. Moreover, $\mathsf{dom}(\Phi) = \bigcap_{n\in\omega} \mathcal{D}_n$. Therefore, for all σ:

$$\mu(\sigma) = \lim_{n\to\infty} \lambda\big(\{X \in \mathcal{D}_n : \Phi^X \succeq \sigma\}\big).$$

By definition, for $n \ge |\sigma|$,

$$\lambda\big(\{X \in \mathcal{D}_n : \Phi^X \succeq \sigma\}\big) = \sum_{\tau\succeq\sigma \ \& \ |\tau|=n} \rho(\tau).$$

Putting the two together, we have

$$\mu(\sigma) = \lim_{n\to\infty} \sum_{\tau\succeq\sigma \ \& \ |\tau|=n} \rho(\tau) = \overline{\rho}(\sigma),$$

as wanted. □

6.2 The complexity of $\overline{\rho}$ We now show that for a given left-c.e. semimeasure ρ, $\overline{\rho}$ can encode a lot of information. More precisely, for any $\emptyset'$-right-c.e. real α (i.e., α is the limit of a $\emptyset'$-computable nonincreasing sequence of rationals), we code α into the values of $\overline{\rho}$ for some left-c.e. semimeasure ρ. Further, we can even make $\overline{\rho}$ Lebesgue-like, as shown by the next theorem. The equivalence of (1), (2), and (3) is well known but it is hard to find a reference for this result, so we include the proof for completeness.

Theorem 6.6 *The following are equivalent for $\alpha \in [0,1]$:*

(1) α is $\emptyset'$-right c.e.;
(2) $\alpha = \limsup_n q_n$ for a computable sequence of rationals $(q_n)_{n\in\omega}$;
(3) $\alpha = \inf r_n$ where $(r_n)_{n\in\omega}$ is a uniform sequence of left-c.e. reals;
(4) there is a left-c.e. semimeasure ρ such that $\overline{\rho} = \alpha \cdot \lambda$.

Proof (1) $\Rightarrow$ (2): Let $\alpha \in [0,1]$ be $\emptyset'$-right c.e, and assume that α is irrational because the implication is clear for rational α. Thus there is a $\emptyset'$-computable function g such that $(g(i))_{i\in\omega}$ is a strictly decreasing sequence of rationals in $[0,1]$ converging to α. By the limit lemma, there is a total computable function f that outputs rationals in $[0,1]$ and is such that $(\forall i \in \omega)(g(i) = \lim_s f(i,s))$.

We define our sequence of rationals $(q_n)_{n\in\omega}$ as follows. Let $(i_s)_{s\in\omega}$ be an effective sequence of natural numbers in which every number appears infinitely often. At stage s, enumerate $f(i_s,s)$ as the next rational in the sequence if it has not yet been enumerated, and $(\forall k < i_s)(f(i_s,s) < f(k,s))$.

We show that, for every $i \in \omega$,

(i) $(\exists n_0 \in \omega)(q_{n_0} = g(i))$, and
(ii) $(\exists n_1 \in \omega)(\forall n > n_1)(q_n < g(i))$.

For (i), given i, let s be such that $(\forall k \le i)(g(k) = f(k,s))$ and $i_s = i$. Then at stage s we have $(\forall k < i_s)(f(i_s,s) = g(i_s) < g(k) = f(k,s))$, so at this stage $f(i_s,s) = g(i)$ will be enumerated if it has not been enumerated already. For (ii), given i, let s_0 be such that $(\forall k \le i)(\forall s \ge s_0)(g(k) = f(k,s))$ and such that (by (i)) every $g(k)$ for $k \le i$ has been enumerated by stage s_0. Consider an $f(i_s,s)$ that is enumerated at some stage $s > s_0$. It is impossible that $i_s \le i$ because in this case at stage s we would have $f(i_s,s) = g(i_s)$, and by assumption this number was already enumerated. Thus $i_s > i$, and to be enumerated at stage s, $f(i_s,s)$ must satisfy $f(i_s,s) < f(i,s) = g(i)$ as desired.

The conclusion $\alpha = \limsup_n q_n$ now follows from (i) and (ii). By (i), every tail of the sequence $(q_n)_{n\in\omega}$ contains an element of the form $g(i)$ for some i; hence, since $\alpha < g(i)$, we have $\alpha \le \limsup_n q_n$. By (ii), $(\forall i \in \omega)(\limsup_n q_n \le g(i))$, hence $\limsup_n q_n \le \alpha$.

(2) $\Rightarrow$ (3): Suppose that $\alpha = \limsup_n q_n$ for a computable sequence of rationals $(q_n)_{n\in\omega}$. Let $r_n := \sup(q_i)_{i\ge n}$. Clearly, each r_n is left-c.e. and $\inf_n r_n = \limsup q_n = \alpha$.

(3) $\Rightarrow$ (4): Since each r_i is left-c.e., let $r_{i,s}$ be the sth rational in the approximation of r_i. To define ρ, we let $\rho_s(\sigma) = 2^{-|\sigma|}\min_{i\le|\sigma|} r_{i,s}$. Then $\rho(\sigma) = 2^{-|\sigma|}\min_{i\le|\sigma|} r_i$.

It is routine to verify that ρ is a semimeasure. Now observe that

$$\begin{aligned}\overline{\rho}(\sigma) &= \inf_n \sum_{\tau\succeq\sigma\ \&\ |\tau|=n} \rho(\tau)\\ &= \inf_n \sum_{\tau\succeq\sigma\ \&\ |\tau|=n} 2^{-|\tau|} \min_{i\leq|\tau|} r_i\\ &= \inf_n\ 2^{n-|\sigma|}2^{-n} \min_{i\leq n} r_i\\ &= 2^{-|\sigma|} \inf_n \min_{i\leq n} r_i\\ &= \alpha\cdot 2^{-|\sigma|}.\end{aligned}$$

(4) $\Rightarrow$ (1): We have that $\emptyset'$ computes $\sum_{x:|x|=n}\rho(x)$ uniformly in n. Then $\overline{\rho}(\varepsilon)=\inf_n\sum_{x:|x|=n}\rho(x)$ is $\emptyset'$-right-c.e. □

The following corollary tells us that $\overline{\rho}$ can be as complicated as possible.

Corollary 6.7 *There is a left-c.e. semimeasure ρ such that $\overline{\rho}=\alpha\cdot\lambda$ and $\alpha\equiv_T\emptyset''$. In particular, every representation of $\overline{\rho}$ computes $\emptyset''$.*

Proof Recall that $\mathsf{Tot}=\{e:\Phi_e \text{ is total}\}$. Let $\alpha=\sum_{e\in\mathsf{Tot}}2^{-e}$, which is $\emptyset'$-right-c.e., and apply Theorem 6.6. □

Despite the fact that for a given left-c.e semimeasure ρ, $\overline{\rho}$ can encode a lot of information, we cannot obtain every $\emptyset'$-computable measure as the $\overline{\rho}$ of some left-c.e. semimeasure ρ, as the following result shows. The witnessing measure μ we construct even has a low representation in the sense described at the beginning of Section 2.2 because the (in this case rational-valued) function $\sigma\mapsto\mu(\sigma)$ is low and clearly computes a representation of μ.

Proposition 6.8 *There is a measure μ such that $\mu(\sigma)$ is a positive rational for all strings σ, the function $\sigma\mapsto\mu(\sigma)$ is low, and $\mu\neq\beta\cdot\overline{\rho}$ for every left-c.e. real β and every left-c.e. semimeasure ρ. In particular, $\mu\neq\overline{\rho}$ for any left-c.e. semimeasure ρ.*

Proof Let $\mathbb{Q}^{>0}$ denote the set of positive rationals. For each $n\in\omega$, let $2^{\leq n}$ denote the set of strings of length at most n, and let $2^{<n}$ denote the set of strings of length less than n. Define a *partial measure* to be a function of the form $m\colon 2^{\leq n}\to\mathbb{Q}^{>0}$ for some $n\in\omega$ such that $m(\varepsilon)=1$ and $(\forall\sigma\in 2^{<n})(m(\sigma)=m(\sigma 0)+m(\sigma 1))$. The partial measures form a partial order $\mathbb{P}$ when ordered by extension: $m_0\sqsubseteq m_1$ if $\mathsf{dom}(m_0)\supseteq\mathsf{dom}(m_1)$ and $(\forall\sigma\in\mathsf{dom}(m_1))(m_0(\sigma)=m_1(\sigma))$. Similarly, if m is a partial measure and μ is a measure, we write $\mu\sqsubseteq m$ if $(\forall\sigma\in\mathsf{dom}(m))(\mu(\sigma)=m(\sigma))$.

To ensure $\mu\neq\beta\cdot\overline{\rho}$, it suffices to ensure that there is a $\sigma\in 2^{<\omega}$ such that $\mu(\sigma)>\beta\cdot\rho(\sigma)$ because then $\mu(\sigma)>\beta\cdot\rho(\sigma)\geq\beta\cdot\overline{\rho}(\sigma)$. To this end, let $(\beta)_{e\in\omega}$ be an effective list of all left-c.e. reals, and let $(\rho_e)_{e\in\omega}$ be an effective list of all left-c.e. semimeasures.

We satisfy the following list of requirements for all $e,i\in\omega$:

$$\begin{aligned}\mathcal{R}_{\langle e,i\rangle} &: (\exists\sigma\in 2^{<\omega})\big(\mu(\sigma)>\beta_e\cdot\rho_i(\sigma)\big),\\ \mathcal{L}_e &: (\exists m\sqsupseteq\mu)\big(\Phi_e^m(e)\downarrow\ \vee\ (\forall m'\sqsubseteq m)\big(\Phi_e^{m'}(e)\uparrow\big)\big).\end{aligned}$$

To each requirement we associate the subset of $\mathbb{P}$ consisting of the partial measures that satisfy the requirement:

$$R_{\langle e,i\rangle} = \{m \in \mathbb{P} : (\exists\sigma \in \mathsf{dom}(m))(m(\sigma) > \beta_e \cdot \rho_i(\sigma))\},$$
$$L_e = \{m \in \mathbb{P} : \Phi_e^m(e)\downarrow \ \vee (\forall m' \sqsubseteq m)(\Phi_e^{m'}(e)\uparrow)\}.$$

Claim *For every $e, i \in \omega$, $R_{\langle e,i\rangle}$ is a dense subset of $\mathbb{P}$.*

Proof Let $m\colon 2^{\leq n} \to \mathbb{Q}^{>0}$ be a given member of $\mathbb{P}$, and let $q = m(0^n)$. The fact that ρ_i is a semimeasure implies that, for all $k \geq n$, $\sum\{\rho_i(\sigma) : \sigma \succeq 0^n \wedge |\sigma| = k\} \leq \rho_i(0^n)$. Therefore $\inf\{\rho_i(\sigma) : \sigma \succeq 0^n\} = 0$, so there is a $\sigma \succ 0^n$ such that $\beta_e \cdot \rho_i(\sigma) \leq q/2$. We may extend m to a partial measure m' that satisfies $m'(\sigma) = 3q/4$ and $m'(\tau) = q/4(2^{|\sigma|-n} - 1)$ for all $\tau \succeq 0^n$ with $|\tau| = |\sigma|$ and $\tau \neq \sigma$. Then $m' \in R_{\langle e,i\rangle}$ because $m'(\sigma) = 3q/4 > q/2 \geq \beta_e \cdot \rho_i(\sigma)$. □

Claim *For every $e \in \omega$, L_e is a dense subset of $\mathbb{P}$.*

Proof Let m be given. If there is an $m' \sqsubseteq m$ such that $\Phi_e^{m'}(e)\downarrow$, then $m' \in L_e$. If not, then $m \in L_e$. □

The sets $R_{\langle e,i\rangle}$ and L_e are dense in $\mathbb{P}$ and uniformly c.e. in $\emptyset'$ (L_e is even $\emptyset'$-computable), so $\emptyset'$ can compute a measure μ such that $\mu(\sigma)$ is a positive rational for each string σ and such that μ meets all of the requirements. That is, $(\forall e, i \in \omega)(\exists m \sqsupseteq \mu)(m \in R_{\langle e,i\rangle})$ and $(\forall e \in \omega)(\exists m \sqsupseteq \mu)(m \in L_e)$. Therefore $\sigma \mapsto \mu(\sigma)$ is low, and $\mu \neq \beta \cdot \overline{\rho}$ for every left-c.e. real β and every left-c.e. semimeasure ρ. □

It is well known that for a measure μ, the atoms of μ are computable from any representation of μ (which can be shown by generalizing the proof of Lemma 2.3). Thus, given the computational power of $\overline{\rho}$, one might expect that the atoms of $\overline{\rho}$ for some left-c.e. semimeasure ρ will include some noncomputable sequences. But this does not hold.

Proposition 6.9 *A set $X \in 2^\omega$ is computable if and only if there exists a left-c.e. semimeasure ρ such that X is an atom of $\overline{\rho}$.*

Proof The left-to-right direction is trivial: for a given computable $X \in 2^\omega$, we define a left-c.e. semimeasure ρ by setting $\rho(X{\upharpoonright}n) = 1$ for all n. For the other direction, let ρ be a left-c.e. semimeasure, and assume that X is an atom of $\overline{\rho}$. Write $\alpha = \lim_n \overline{\rho}(X{\upharpoonright}n)$, and choose $q \in \mathbb{Q}$ such that $\frac{1}{2}\alpha < q < \alpha$. Then there exists a large enough N such that $\rho(X{\upharpoonright}N)$ is strictly smaller than $2q$. To decide all further bits of X, say, $X(n)$ for $n \geq N$, we proceed inductively as follows. Wait until one of $\rho(X{\upharpoonright}n^\frown 0)$ and $\rho(X{\upharpoonright}n^\frown 1)$ attains or exceeds q, and output the according bit. This bit is the correct value of $X(n)$, since $\rho(X{\upharpoonright}n^\frown X(n))$ must eventually attain or exceed q while $\rho(X{\upharpoonright}n^\frown(1-X(n)))$ cannot attain q, as otherwise their sum would be at least $2q$ and would therefore exceed $\rho(X{\upharpoonright}N)$, contradicting our choice of N. □

Although the measure derived from a left-c.e. semimeasure cannot have a noncomputable atom, one interesting difference between these derived measures and computable measures is that whereas there is no computable measure μ such that every computable sequence is a μ-atom (because for each computable measure μ one can effectively find a sequence X such that $\lim_{n\to\infty} \mu(X{\upharpoonright}n) = 0$), *every* computable sequence is an atom of $\overline{M}$, because $\overline{M}$ dominates every computable measure up to a positive multiplicative constant.

6.3 Notions of randomness with respect to $\overline{\rho}$ We now apply the definitions of Martin-Löf randomness with respect to noncomputable measures, introduced in Section 2.2, to the measure derived from a semimeasure, and we compare the resulting definitions to the definitions studied in Section 5.

As noted in Section 2.2, there are two general approaches to defining a randomness test $(\mathcal{U}_i)_{i\in\omega}$ with respect to a noncomputable measure μ: either allow $(\mathcal{U}_i)_{i\in\omega}$ to have access to a representation of μ as an oracle and require $\mu(\mathcal{U}_i) \leq 2^{-i}$ for every i, or simply require the latter condition without using a representation of μ as an oracle.

Taking the former approach yields the following example.

Proposition 6.10 *Let ρ be the semimeasure from Corollary 6.7, so that $\overline{\rho} = \alpha \cdot \lambda$ for some $\alpha \equiv_T \emptyset''$. Then $\overline{\rho}$-Martin-Löf randomness is 3-randomness.*

Proof Let $j \in \omega$ satisfy $2^{-(j+1)} < \alpha < 2^{-j}$, which implies that $2^j < \frac{1}{\alpha} < 2^{(j+1)}$. First, we show that $\mathsf{MLR}^{\emptyset''} \subseteq \mathsf{MLR}_{\overline{\rho}}$. Since $\emptyset''$ computes a representation of $\overline{\rho}$, we have $\mathsf{MLR}^{\emptyset''}_{\overline{\rho}} \subseteq \mathsf{MLR}_{\overline{\rho}}$. Now for any $\emptyset''$-Martin-Löf test $(\mathcal{U}_i)_{i\in\omega}$ (with respect to $\overline{\rho}$), we have $\alpha \cdot \lambda(\mathcal{U}_i) \leq 2^{-i}$, which implies that $\lambda(\mathcal{U}_i) \leq 2^{j+1-i}$. Thus $(\mathcal{U}_i)_{i\geq j+1}$ is a $\emptyset''$-Martin-Löf test (with respect to λ) that covers $\bigcap_{i\in\omega} \mathcal{U}_i$. Thus $\mathsf{MLR}^{\emptyset''} \subseteq \mathsf{MLR}_{\overline{\rho}}$.

To show that $\mathsf{MLR}_{\overline{\rho}} \subseteq \mathsf{MLR}^{\emptyset''}$, let $(\mathcal{U}_i)_{i\in\omega}$ be a $\emptyset''$-Martin-Löf test with respect to λ. Then since

$$\alpha \cdot \lambda(\mathcal{U}_i) \leq 2^{-j}\lambda(\mathcal{U}_i) \leq 2^{-(i+j)},$$

it follows that $(\mathcal{U}_i)_{i\in\omega}$ is a $\emptyset''$-Martin-Löf test with respect to $\overline{\rho}$. But since every representation of $\overline{\rho}$ computes $\emptyset''$, it follows that for any such representation R, $(\mathcal{U}_i)_{i\in\omega}$ is an R-Martin-Löf test with respect to $\overline{\rho}$. Thus $\mathsf{MLR}^{R}_{\overline{\rho}} \subseteq \mathsf{MLR}^{\emptyset''}$ for all representations R of $\overline{\rho}$, and hence $\mathsf{MLR}_{\overline{\rho}} \subseteq \mathsf{MLR}^{\emptyset''}$. □

This example shows a defect of using $\overline{\rho}$ to define randomness with respect to ρ. As $\overline{\rho}$ is a multiple of the Lebesgue measure, we would expect that $\overline{\rho}$-Martin-Löf randomness is just Martin-Löf randomness with respect to the Lebesgue measure. But the α encodes information that can be used to derandomize any sequence that is not 3-random. However, the blind approach to $\overline{\rho}$-randomness avoids this problem.

Proposition 6.11 *Let ρ be the semimeasure from Corollary 6.7. Then blind $\overline{\rho}$-Martin-Löf randomness is the same as Martin-Löf randomness.*

Proof By an argument similar to the one in the proof of Proposition 6.10, every Martin-Löf test is covered by a blind $\overline{\rho}$-Martin-Löf test, and vice versa. □

As a consequence of these two examples, we have the following.

Corollary 6.12 *There is a left-c.e. semimeasure ρ such that $\mathsf{MLR}_{\overline{\rho}} \subsetneq \mathsf{bMLR}_{\overline{\rho}}$.*

We also have the following.

Proposition 6.13 *There is a left-c.e. semimeasure ρ such that $\mathsf{bMLR}_{\overline{\rho}} \subsetneq \mathsf{MLR}_{\rho}$.*

Proof Let ρ be the left-c.e. semimeasure from the proof of Theorem 4.1, so that $\mathsf{MLR}_\rho = \{0^\omega\}$. Since ρ is induced by a functional Φ such that $\lambda(\mathsf{dom}(\Phi)) = 0$, by the characterization of $\overline{\rho}$ given in Proposition 6.5,

$$\overline{\rho}(2^\omega) = \overline{\rho}\big([\![\varepsilon]\!]\big) = \lambda\big\{X : X \in \mathsf{dom}(\Phi)\big\} = 0.$$

Thus $\mathsf{MLR}_{\overline{\rho}} = \emptyset$. □

The above proof also shows that $\mathsf{bMLR}_{\overline{\rho}}$ does not satisfy randomness preservation, since Φ induces ρ (and hence $\overline{\rho}$), but $\Phi(\mathsf{MLR}) = \{0^\omega\} \neq \mathsf{bMLR}_{\overline{\rho}}$. Thus, blind Martin-Löf randomness for $\overline{\rho}$ does not provide an adequate definition of randomness for ρ according to the desiderata laid out in Section 5.1.

Blind weak 2-randomness with respect to $\overline{\rho}$ fares much better than $\overline{\rho}$-Martin-Löf randomness and blind Martin-Löf randomness with respect to $\overline{\rho}$. As we now show, blind weak 2-randomness for $\overline{\rho}$ is equivalent to weak 2-randomness for ρ and, hence, satisfies randomness preservation. First, we need a lemma generalizing the definition of $\overline{\rho}(\sigma)$.

Lemma 6.14 *Let ρ be a left-c.e. semimeasure. Let $E \subseteq 2^{<\omega}$ be prefix-free. For each $m \in \omega$, let $E^m = \{\sigma \in 2^{<\omega} : (\exists \tau \in E)(\tau \preceq \sigma \wedge |\sigma| = |\tau| + m)\}$. Then $\overline{\rho}([\![E]\!]) = \lim_{m\to\infty} \rho(E^m)$.*

Proof For all $m \in \omega$, $\rho(E^{m+1}) \leq \rho(E^m)$. Thus it suffices to show that for every $k \in \omega$ there is some $m \in \omega$ such that $\rho(E^m) \leq \overline{\rho}([\![E]\!]) + 1/k$.

Recall that, for all $\tau \in 2^{<\omega}$, $\overline{\rho}(\tau) = \inf_m \sum\{\rho(\sigma) : \sigma \succeq \tau \wedge |\sigma| = |\tau| + m\}$. Thus if E is finite, then for all $k \in \omega$ there is an $m \in \omega$ such that $\rho(E^m) \leq \overline{\rho}([\![E]\!]) + 1/k$. Suppose instead that E is infinite, and let $k \in \omega$. The fact that E is prefix-free implies that $\rho(E)$ is finite. Thus there is an $\ell \in \omega$ such that $\sum\{\rho(\sigma) : \sigma \in E \wedge |\sigma| > \ell\} < 1/2k$. Now let $E_0 = \{\tau \in E : |\tau| \leq \ell\}$, let $E_1 = \{\tau \in E : |\tau| > \ell\}$, and let m be such that $\rho(E_0^m) \leq \overline{\rho}([\![E_0]\!]) + 1/2k$. Then

$$\begin{aligned}\rho(E^m) &= \rho(E_0^m) + \rho(E_1^m)\\ &\leq \rho(E_0^m) + \rho(E_1)\\ &\leq \overline{\rho}\big([\![E_0]\!]\big) + 1/2k + 1/2k\\ &\leq \overline{\rho}\big([\![E]\!]\big) + 1/k,\end{aligned}$$

as required. □

Theorem 6.15 *Let ρ be a left-c.e. semimeasure. Then $X \in 2^\omega$ is weakly 2-random for ρ if and only if X is blindly weakly 2-random for $\overline{\rho}$.*

Proof For every $E \subseteq 2^{<\omega}$, $\overline{\rho}([\![E]\!]) \leq \rho(E)$, and therefore every generalized ρ-Martin-Löf test is also a blind generalized $\overline{\rho}$-Martin-Löf test. Thus if X is blindly weakly 2-random for $\overline{\rho}$, then X is weakly 2-random for ρ.

Conversely, suppose that X is not blindly weakly 2-random for $\overline{\rho}$. Let $(\mathcal{U}_n)_{n\in\omega}$ be a blind generalized $\overline{\rho}$-Martin-Löf test capturing X, and let $(E_n)_{n\in\omega}$ be a uniformly c.e. sequence of prefix-free subsets of $2^{<\omega}$ such that, for all $n \in \omega$, $\mathcal{U}_n = [\![E_n]\!]$. Let $(F_n)_{n\in\omega}$ be the uniformly c.e. sequence of prefix-free subsets of $2^{<\omega}$, where σ is enumerated in F_n if and only if every E_i with $i \leq n$ enumerates a $\tau_i \preceq \sigma$, and $|\sigma| = \max\{|\tau_i| : i \leq n\} + n$. Then, for all $n \in \omega$, $[\![F_n]\!] = \bigcap_{i\leq n}[\![E_i]\!]$. Therefore

$X \in \bigcap_{n\in\omega}[\![E_n]\!] = \bigcap_{n\in\omega}[\![F_n]\!]$. Furthermore, for all $n \in \omega$, $\rho(F_{n+1}) \leq \rho(F_n)$. It remains to show that $\lim_{n\to\infty} \rho(F_n) = 0$. To see this, observe that, for all $m, n \in \omega$, $\rho(F_{n+m}) \leq \rho(E_n^{n+m})$. Thus, for all $n \in \omega$,

$$\begin{aligned} \lim_{m\to\infty} \rho(F_m) &= \lim_{m\to\infty} \rho(F_{n+m}) \\ &\leq \lim_{m\to\infty} \rho(E_n^{n+m}) \\ &= \overline{\rho}\big([\![E_n]\!]\big), \end{aligned}$$

where the last equality is by Lemma 6.14. Thus, for all $n \in \omega$, $\lim_{n\to\infty} \rho(F_n) \leq \overline{\rho}([\![E_n]\!])$. Since $\lim_{n\to\infty} \overline{\rho}([\![E_n]\!]) = 0$, we must have $\lim_{n\to\infty} \rho(F_n) = 0$ as well. $\square$

The relationships between the various notions considered here are summarized by the following diagram, where a strict inequality means that there is some semimeasure ρ separating the two notions:

$$\begin{array}{ccc} \mathsf{W2R}_{\overline{\rho}} & \subsetneq & \mathsf{MLR}_{\overline{\rho}} \\ \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} \\ \mathsf{bW2R}_{\overline{\rho}} & \subsetneq & \mathsf{bMLR}_{\overline{\rho}} \\ \| & & \rotatebox{90}{$\subsetneq$} \\ \mathsf{W2R}_{\rho} & \subsetneq & \mathsf{MLR}_{\rho} \end{array}$$

6.4 The no randomness ex nihilo principle for weak 2-randomness with respect to a semimeasure As we showed in Section 5.3, for each left-c.e. semimeasure ρ, ρ-weak 2-randomness satisfies coherence, randomness preservation, and the computable sequence condition. The status of the no randomness ex nihilo principle, however, is still open.

Question 6.16 Let ρ be a left-c.e. semimeasure. If Φ is a Turing functional that induces ρ and $Y \in \mathsf{W2R}_\rho$, is there some $X \in \mathsf{W2R}$ such that $\Phi(X) = Y$?

A positive answer to Question 6.16 would also allow us to answer Shen's question for weak 2-randomness, which also remains open.

Question 6.17 If Φ and Ψ are Turing functionals such that $\lambda_\Phi(\sigma) = \lambda_\Psi(\sigma)$ for every $\sigma \in 2^{<\omega}$, does it follow that $\Phi(\mathsf{W2R}) = \Psi(\mathsf{W2R})$?

Some partial progress on answering Question 6.16 has been made. We show that the no randomness ex nihilo principle holds for weak 2-randomness with respect to any computable measure.

Theorem 6.18 *Let Φ be an almost-total Turing functional. If $Y \in \mathsf{W2R}_{\lambda_\Phi}$, there is some $X \in \mathsf{W2R}$ such that $\Phi(X) = Y$.*

Proof Let $(\mathcal{U}_i^e)_{e,i\in\omega}$ be a (noneffective) listing of all generalized λ-Martin-Löf tests. That is, every generalized Martin-Löf test is of the form $(\mathcal{U}_i^e)_{i\in\omega}$ for some e. Without loss of generality, we can assume that the first test $(\mathcal{U}_i^0)_{i\in\omega}$ is the universal Martin-Löf test. Let $Y \in \mathsf{W2R}_{\lambda_\Phi}$. Since Y is in particular λ_Φ-Martin-Löf random, by Theorem 2.4, $\Phi^{-1}(Y) \cap \mathsf{MLR} \neq \emptyset$. In other words, for some i_0, the preimage of Y under Φ meets the Π_1^0 class $\mathcal{C}_0 = (\mathcal{U}_{i_0}^0)^{\mathsf{c}}$. We further note that Φ is total on $\mathcal{C}_0$. Indeed, Φ is almost total, which means that $\mathsf{dom}(\Phi)^{\mathsf{c}}$ has measure 0. But $\mathsf{dom}(\Phi)^{\mathsf{c}}$ is a Σ^0_2 set, that is, a union of effectively closed sets, which thus must all have measure 0. Since no Martin-Löf random real can be contained in an effectively closed set of measure 0 and since $\mathcal{C}_0$ contains only Martin-Löf random elements, this shows that $\mathcal{C}_0 \cap \mathsf{dom}(\Phi)^{\mathsf{c}} = \emptyset$, that is, $\mathcal{C}_0 \subseteq \mathsf{dom}(\Phi)$.

We now build a sequence of nonempty Π^0_1 classes $\mathcal{C}_1, \mathcal{C}_2, \ldots$ in such a way that

- $\mathcal{C}_i \supseteq \mathcal{C}_{i+1}$ for every $i \geq 0$,
- for all n, all members of $\mathcal{C}_n$ pass all the tests $(\mathcal{U}^e_i)_{i\in\omega}$ for $e \leq n$, and
- for all n, $\Phi^{-1}(Y) \cap \mathcal{C}_n \neq \emptyset$.

Note that since all $\mathcal{C}_i$'s are contained in $\mathcal{C}_0$, this in particular means that Φ is total on all $\mathcal{C}_i$'s. Suppose that $\mathcal{C}_0, \ldots, \mathcal{C}_n$ with these properties have already been built. To build $\mathcal{C}_{n+1}$, we do the following. Suppose that for all i we have

$$Y \in \Phi(\mathcal{C}_n) \setminus \Phi\big(\mathcal{C}_n \cap (\mathcal{U}^{n+1}_i)^{\mathsf{c}}\big).$$

The preimage of the set $\Phi(\mathcal{C}_n) \setminus \Phi(\mathcal{C}_n \cap (\mathcal{U}^{n+1}_i)^{\mathsf{c}})$ under Φ is contained in $\mathcal{U}^{n+1}_i$ and therefore its measure tends to 0 as i tends to infinity. By the definition of the induced measure λ_Φ, this implies that

$$\lambda_\Phi\big(\Phi(\mathcal{C}_n) \setminus \Phi\big(\mathcal{C}_n \cap (\mathcal{U}^{n+1}_i)^{\mathsf{c}}\big)\big) \to 0$$

and thus the set $\bigcap_i (\Phi(\mathcal{C}_n) \setminus \Phi(\mathcal{C}_n \cap (\mathcal{U}^{n+1}_i)^{\mathsf{c}}))$ is a Π^0_2 set of λ_Φ-measure 0 containing Y, contradicting the fact that Y is λ_Φ-weakly 2-random. Thus, there exists j such that $Y \in \Phi(\mathcal{C}_n \cap (\mathcal{U}^{n+1}_j)^{\mathsf{c}})$, and we set $\mathcal{C}_{n+1} = \mathcal{C}_n \cap (\mathcal{U}^{n+1}_j)^{\mathsf{c}}$. This ensures that all elements pass the $(n+1)$st generalized Martin-Löf test. This finishes the construction of the $\mathcal{C}_n$'s.

To finish the proof, since $\Phi^{-1}(Y) \cap \mathcal{C}_i \neq \emptyset$ for every $i \in \omega$, choose $X_i \in \Phi^{-1}(Y) \cap \mathcal{C}_i$ for each i. By the compactness of 2^ω, one can assume, up to extraction of a subsequence, that the sequence $(X_i)_{i\in\omega}$ converges to a limit X^*. For any given n, almost all i are greater than n, and thus $X_i \in \mathcal{C}_i \subseteq \mathcal{C}_n$. Since $\mathcal{C}_n$ is closed, it implies that the limit X^* belongs to $\mathcal{C}_n$. This being true for all n, by the construction of the $\mathcal{C}_n$'s, X^* passes all generalized Martin-Löf tests and, therefore, is weakly 2-random. Moreover, by the continuity of Turing functionals on their domain:

$$\lim_{i\to\infty} \Phi(X_i) = \Phi(X^*),$$

but $\Phi(X_i) = Y$ for all i; therefore, $\Phi(X^*) = Y$. This establishes the existence of a weakly 2-random sequence in $\Phi^{-1}(\{Y\})$ and completes the proof. □

The above proof of Theorem 6.18 is essentially analytic. Let us mention that a completely different proof, of computability-theoretic flavor, can also be given. Suppose that Y is λ_Φ-weakly 2-random. Then by Theorem 2.6, Y does not compute any noncomputable Δ^0_2 set. Let $\mathcal{C}_0$ be the set defined in the previous proof (on which Φ is total), and let $\mathcal{P} = \mathcal{C}_0 \cap \Phi^{-1}(Y)$, so that $\mathcal{P} \subseteq \mathsf{MLR} \cap \Phi^{-1}(Y)$. It is well known that, given a Π^0_1 class and a countable collection of reals $(A_i)_{i\in\omega}$, there is a member of the Π^0_1 class which does not compute any A_i. (Jockusch and Soare [5] proved this fact for a single A, but it is easy to see that their construction, a forcing argument, can be extended to a countable collection of A_i's.) Taking the collection $(A_i)_{i\in\omega}$ to consist of the noncomputable Δ^0_2 sets, relativizing the previous theorem to Y, and using the fact that each A_i is not Y-computable, there exists a member X of $\mathcal{P}$ which does not compute any A_i. Thus, $X \in \Phi^{-1}(\{Y\})$ is Martin-Löf random and does not compute any noncomputable Δ^0_2 set. Applying Theorem 2.6 again, this shows that X is weakly 2-random.

Note

1. Another possible approach would be to adapt the Levin–Schnorr theorem. When μ is a computable measure, X is μ-Martin-Löf random if and only if

$$K(X\restriction n) > -\log \mu(X\restriction n) - O(1),$$

where K is the prefix-free Kolmogorov complexity. Hence one could also use this inequality as a definition of randomness for left-c.e. measures μ. We will not consider this approach here.

References

[1] Becher, V., L. Bienvenu, R. Downey, and E. Mayordomo, "Computability, Complexity and Randomness (Dagstuhl Seminar 12021)," *Dagstuhl Reports*, vol. 2 (2012), pp. 19–38. 302

[2] Day, A. R., and J. S. Miller, "Randomness for non-computable measures," *Transactions of the American Mathematical Society*, vol. 365 (2013), pp. 3575–91. Zbl 1307.03026. MR 3042595. DOI 10.1090/S0002-9947-2013-05682-6. 306

[3] Downey, R. G., and D. R. Hirschfeldt, *Algorithmic Randomness and Complexity: Theory and Applications of Computability*, Springer, New York, 2010. Zbl 1221.68005. MR 2732288. 303, 305, 306, 311

[4] Downey, R. G., A. Nies, R. Weber, and L. Yu, "Lowness and Π^0_2 nullsets," *Journal of Symbolic Logic*, vol. 71 (2006), pp. 1044–52. Zbl 1112.03040. MR 2251554. DOI 10.2178/jsl/1154698590. 306

[5] Jockusch, C. G., Jr., and R. I. Soare, "Π^0_1 classes and degrees of theories," *Transactions of the American Mathematical Society*, vol. 173 (1972), pp. 33–56. Zbl 0262.02041. MR 0316227. DOI 10.2307/1996261. 325

[6] Kautz, S., "Degrees of random sets," Ph.D. dissertation, Cornell University, Ithaca, N.Y., 1991. MR 2686787. 304

[7] Kjos-Hanssen, B., "The probability distribution as a computational resource for randomness testing," *Journal of Logic and Analysis*, vol. 2 (2010), no. 10. Zbl 1286.68251. MR 2737709. 307

[8] Levin, L. A., "Uniform tests for randomness," *Doklady Mathematics*, vol. 17 (1976), no. 2, pp. 337–40. Zbl 0341.94013. MR 0429327. Erratum, *Doklady Mathematics*, vol. 231 (1976), p. 264. MR 0429327. 306

[9] Levin, L. A., and V. V. V'yugin, "Invariant properties of informational bulks," pp. 359–64 in *Proceedings of the Sixth Symposium on Mathematical Foundations of Computer Science (Tatranská, Lomnica, 1977)*, vol. 53 of *Lecture Notes in Computer Science*, Springer, Berlin, 1977. Zbl 0409.94019. MR 0520740. 316, 317

[10] Martin-Löf, P., "The definition of random sequences," *Information and Control*, vol. 9 (1966), pp. 602–19. Zbl 0244.62008. MR 0223179. DOI 10.1016/S0019-9958(66)80018-9. 303

[11] Nies, A., *Computability and Randomness*, vol. 51 of *Oxford Logic Guides*, Oxford University Press, Oxford, 2009. Zbl 1169.03034. MR 2548883. 303, 305, 311

[12] Porter, C., "Mathematical and philosophical perspectives on algorithmic randomness," Ph.D. dissertation, University of Notre Dame, South Bend, Ind., 2012. MR 3217940. 306

[13] Reimann, J., "Effectively closed sets of measures and randomness," *Annals of Pure and Applied Logic*, vol. 156 (2008), pp. 170–82. Zbl 1153.03021. MR 2474448. DOI 10.1016/j.apal.2008.06.015. 306

[14] Reimann, J., and T. A. Slaman, "Measures and their random reals," *Transactions of the American Mathematical Society*, vol. 267 (2015), pp. 5081–97. Zbl 06429156. MR 3335411. DOI 10.1090/S0002-9947-2015-06184-4. 313

[15] Shen, A., L. Bienvenu, and A. Romashchenko, "Sparse sets," preprint, https://hal.inria.fr/hal-00274010 (accessed 3 February 2017). 302, 305

[16] V'yugin, V. V., "Algebra of invariant properties of binary sequences," *Problems of Information Transmission*, vol. 18 (1982), pp. 147–61. Zbl 0525.03035. MR 0689342. 316

[17] Zvonkin, A. K., and L. A. Levin, "The complexity of finite objects and the basing of the concepts of information and randomness on the theory of algorithms," *Russian Mathematical Surveys*, vol. 25 (1970), pp. 83–124. Zbl 0222.02027. MR 0307889. 305, 307

Acknowledgments

Bienvenu's work was supported by Agence Nationale de la Recherche (ANR) project "EQINOCS" (ANR-11-BS02-004). Hölzl's work was supported by a Feodor Lynen postdoctoral research fellowship through the Alexander von Humboldt Foundation, and is supported by Ministry of Education of Singapore grant R146-000-184-112 (MOE2013-T2-1-062). Porter's work was supported by National Science Foundation grant OISE-1159158 as part of the International Research Fellowship Program. Shafer is an FWO Pegasus Long Postdoctoral Fellow, and he also acknowledges the support of the Fondation Sciences Mathématiques de Paris.

Bienvenu
LIRMM
CNRS & Université de Montpellier
Montpellier
France
laurent.bienvenu@computability.fr
http://www.lirmm.fr/~bienvenu

Hölzl
Faculty of Computer Science
Bundeswehr University Munich
Neubiberg
Germany
r@hoelzl.fr
http://hoelzl.fr

Porter
Department of Mathematics and Computer Science
Drake University
Des Moines, Iowa
USA
cp@cpporter.com
http://cpporter.com

Shafer
School of Mathematics
University of Leeds
Leeds
United Kingdom
p.e.shafer@leeds.ac.uk
http://www1.maths.leeds.ac.uk/~matpsh/