On Finding Primal- and Dual-Optimal Bases

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Abstract. We show that if there exists a strongly polynomial time algorithm that finds a basis which is optimal for both the primal and the dual problems, given an optimal solution for one of the problems, then there exists a strongly polynomial algorithm for the general linear programming problem. On other hand, we give a strongly polynomial time algorithm that finds such a basis, given any pair of optimal solutions (not necessarily basic) for the primal and the dual problems. Such an algorithm is needed when one is using an interior point method and is interested in finding a basis which is both primal- and dual-optimal.

Subject classification: Programming, Linear, Algorithms and Theory

Introduction

The reader is referred, for example, to [2] for information about standard results in linear programming which are used in this work. The simplex method for linear programming-has the nice property that if the problem has an optimal solution then a basis is found which is both primal-optimal and dual-optimal, i.e., both of the basic solutions which are defined by such a basis in the primal and the dual problems are optimal in their respective problems. For brevity we call such a basis an *optimal basis*. An optimal basis

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is a useful object for post-optimality analysis. The known polynomial-time algorithms for linear programminggenerate optimal solutions which are not necessarily basic. Given any primal-optimal solution (not necessarily basic), it is easy to find a primal-optimal basis. Analogously, given any dual-optimal solution, it is easy to find a dual-optimal basis. However, none of the two bases found in this way is guaranteed to be an optimal basis. In fact, the dual solution associated with a primal-optimal basis and the primal solution associated with a dual-optimal basis may both be infeasible in their respective problems. Furthermore, if the problem is put into the combined primal-dual form, a primal-optimal basis of the combined form yields a primal-optimal basis and a dual-optimal one for the original problem, but these two bases may be distinct.

Since no polynomial-time variant of the simplex method is known, this raises the question whether an optimal basis can be found in polynomial time in terms of the input size of a problem with rational data. We answer this question in the affirmative. Actually, we prove a stronger result using the concept of strongly polynomial time complexity. For simplicity, we say that an algorithm for linear programmingruns in strongly polynomial time if it performs no more than $p(m, n)$ arithmetic operations and comparisons. (When applied to problems with rational data, the strongly polynomial algorithms of this paper involve only numbers of polynomial size.) It is not known whether there exists a strongly polynomial algorithm for the general linear programming problem. Since this question is open and seems difficult, we consider here a related problem concerning basic solutions. We prove the following two complementary theorems which shed some light on the complexity of finding optimal bases:

Theorem 0.1. *If there exists a strongly polynomial time algorithm that finds an optimal basis, given an optimal solution for* either *the primal* or *the dual, then there exists a strongly polynomial algorithm for the general linear programming problem.*

Theorem 0.2. *There exists a strongly polynomial time algorithm that finds an optimal basis, given optimal solutions for* both *the primal and the dual.*

We give the necessary definitions and the proofs in Section 1.

1. The results

Consider the linear programming problemin standard form

$$(P) \qquad \begin{aligned} \text{Maximize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where the rows of $A \in R^{m \times n}$ are assumed to be linearly independent. The dual problem is:

$$(D) \qquad \begin{aligned} \text{Minimize} \quad & b^T y \\ \text{subject to} \quad & A^T y \geq c \end{aligned}$$

A *basis* is a nonsingular submatrix $B \in R^{m \times m}$ of A. A basis B is called *primal-optimal* if $B^{-1}b$ is an optimal solution for (P) (variables not corresponding to columns of B are set to zero). Denote, as usual, by c_B the restriction of c to the components corresponding to the columns of B. A basis B is called *dual-optimal* if $(B^T)^{-1}c_B$ is an optimal solution for (D). We call B *optimal* if it is both primal- and dual-optimal. It is well-known, in view of the duality theorem, that an optimal basis is characterized by the inequalities

$$B^{-1}b \geq 0$$

and

$$c_B^T B^{-1} A \geq c^T \ .$$

We are interested here in the complexity of finding an optimal basis. It follows from the theory of the simplex method that such a basis exists if the problem has an optimal solution.

The proof of Theorem 0.1 uses Theorem 0.2, and hence we first prove the latter.

Proof of Theorem 0.2: Suppose x and y are given optimal solutions for (P) and (D), respectively. Let X denote the submatrix of A consisting of the columns A_j such that $x_j > 0$. By the complementary slackness condition,

$$y^T X = c_X^T,$$

where c_X denotes the restriction of c to the coordinates corresponding to X. If the columns of X are linearly dependent, we can find a vector $z \neq 0$ such that $Xz = 0$, and hence $c_X^T z = 0$. It follows that for some scalar t, the vector x', defined by $x'_j = x_j + tz_j$ for j in X and $x'_j = 0$ otherwise, is an optimal solution with a smaller set of positive coordinates. Such a vector x' can obviously be found in strongly polynomial time . Successive applications of this principle finally yield an optimal solution x' where the set X' of columns A_j such that $x'_j > 0$ is linearly independent. If X' has m columns then we have found an optimal basis. Otherwise, we expand the set X as follows.

First, consider the submatrix Y of A consisting of the columns A_j such that $y^T A_j = c_j$. If Y contains any column which is independent of the columns in X', then add this column to X'. Repeat this step until all the columns of Y are linearly dependent on the columns of the resulting set X'. (This procedure can be easily implemented by Gaussian eliminations.) If at this point the matrix X' still consists of less than m columns then we expand further as follows.

We now wish to move to a point y' with one additional independent column in the set Y. By assumption, the rows of A are linearly independent and hence there exists a column A_j which is linearly independent of the columns in X'. Thus, this column is not in Y and hence $y^T A_j > c_j$. We now solve the set of equations:

$$z^T X' = 0$$

$$z^T A_j = 1 \ .$$

A solution exists since the A_j together with the columns of X' constitute a linearly independent set. Now, consider vectors of the form $y - tz$, where t is any scalar. First, since every column of Y is now a linear combination of the columns of X', we have

$$(y - tz)^T Y = c_Y^T \ .$$

Moreover, consider the number

$$t_0 = \min\left\{ \frac{y^T A_k - c_k}{z^T A_k} \quad : \quad z^T A_k > 0 \right\} \ .$$

Notice that t_0 is well-defined since $z^T A_j = 1$. The vector $y' = y - t_0 z$ is also a dual-optimal solution as can be verified from the above. The set Y' of columns A_k such that $y'^T A_k = c_k$ now contains at least one column which is linearly independent of the columns of X'. We now add such columns to X', one at a time, making sure that X' remains linearly independent. We then find another vector z, and this process is repeated until X' contains m linearly independent columns. Note that throughout this process we maintain optimal solutions x' and y' such that for every column A_k not in X', we have $x'_k = 0$, and the set Y' contains X'. Thus, at the end X' is an optimal basis. The whole process does not take more than n "pivot" steps since columns which leave the set X prior to the generation of the first set X' never come back, and the set X' only increases. Thus, the whole algorithm runs in strongly polynomial time . ∎

We note that if the given optimal solutions are vertices of the respective polyhedra then the optimal basis found by the above procedure yields the same pair of solutions.

For the proof of Theorem 0.1 we first state the subject problems:

Problem 1.1. Given A, b, c, find an optimal basis or conclude that no such basis exists.

Problem 1.2. Given A, b, c, such that (P) and (D) are known to have optimal solutions, find an optimal basis.

Problem 1.3. Given A, b, c and an optimal solution x of (P), find an optimal basis.

The proof of Theorem 0.1 follows from the following reductions:

Proposition 1.4. *If there exists a strongly polynomial time algorithm for Problem 1.3 then there exists one for Problem 1.2.*

Proof: We reduce Problem 1.2 to Problem 1.3. Given A, b, c, such that (P) and (D) have optimal solutions, consider the following problem where (P) and (D) are combined, and y is replaced by $z - w$:

$$(PD) \qquad \begin{array}{llllllll} \text{Maximize} & c^T x & -b^T z & +b^T w & & & & \\ \text{subject to} & Ax & & & & & = & b \\ & & -A^T z & +A^T w & & +v & = & -c \\ & & & & & x, z, w, v & \geq & 0\ . \end{array}$$

By the duality theorem, (PD) has an optimal solution and the optimal value is 0. Consider the following problem with an additional variable ξ:

$$(PD') \qquad \begin{array}{lllllll} \text{Maximize} & c^T x & -b^T z & +b^T w & & & \\ \text{subject to} & Ax & & & & -b\xi & = 0 \\ & & -A^T z & +A^T w & +v & +c\xi & = 0 \\ & & & & & x, z, w, v, \xi & \geq 0 \quad . \end{array}$$

The dual of (PD') amounts to the following system of inequalities:

$$(DP') \qquad \begin{array}{rrcl} c^T u & -b^T y & \geq & 0 \\ & A^T y & \geq & c \\ -Au & & \geq & -b \\ Au & & \geq & b \\ & u & \geq & 0. \end{array}$$

Since (DP') is feasible, it follows that the optimal value of (PD') is also 0. Thus, we have a trivial optimal solution for (PD'), namely, set x, z, w, v and ξ to zero. Now, by assumption, an optimal basis for (PD') can be found in strongly polynomial time . Given an optimal basis, we can compute optimal solutions for both (PD') and (DP'). We are interested in the latter. Let (y, u) be such an optimal solution for (DP'). Obviously, u is an optimal solution for (P) and y is an optimal solution for (D). By Theorem 0.2, we can now find an optimal basis for (P) in strongly polynomial time .

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Proposition 1.5. *If there exists a strongly polynomial time algorithm for Problem 1.2 then there exists one for Problem 1.1.*

Proof: Suppose there exists a strongly polynomial time algorithm for Problem 1.2. Given A, b, c, consider the following problem:

$$(S) \qquad \begin{array}{lccccl} \text{Maximize} & c^T x & -b^T y & -\xi & \\ \text{subject to} & Ax & & +b\xi & = b \\ & & A^T y & +c\xi & \geq c \\ & c^T x & -b^T y & & \leq 0 \\ & & & x, \xi & \geq 0 \end{array}$$

The problem (S) can obviously be put into the standard form of (P). Now, (S) is feasible (set x and y to zero and $\xi = 1$). Moreover, the value of the objective function on the feasible domain of (S) is bounded by 0. Thus, (S) has an optimal solution which by assumption can be found in strongly polynomial time . The x part and the y part of such a solution are optimal solutions for (P) and (D), respectively, if and only if

$\xi = 0$. If $\xi = 0$, then by Theorem 0.2 an optimal basis for (P) can be found in strongly polynomial time . ∎

2. Conclusion

An algorithm for finding an optimal basis has to work on the problem from two "sides": the primal and the dual. If an algorithm concentrates on the primal side or simply discards all the information obtained throughout its execution and reports only a primal-optimal solution, then finding a dual-optimal solution given a primal-optimal one may be as hard as solving the problem from the beginning. This observation is important for the implementation of interior point algorithms. If an algorithm does not generate values for the dual variables then in the worst case it may be hard to find a dual-optimal solution. If the algorithm generates both primal and dual values then it is relatively easy to find an optimal basis.

Finally, we note that the efficiency of the algorithm given in the proof of Theorem 0.2 is very closely related to the amount of degeneracy in the problem. The more degenerate the problem is, more steps might be needed to construct an optimal basis from a pair of optimal solutions for the primal and the dual problems. We also note that the work of our algorithms can be carried out in a tableau form just like the simplex method.

References

[1] N. Megiddo, "A note on degeneracy in linear programming," *Mathematical Programming* **35** (1986), 365–367.

[2] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York, 1986.