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Dynamic Pricing in Social Networks: The Word of Mouth Effect*

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Abstract

We study the problem of optimal dynamic pricing for a monopolist selling a product to consumers in a social network. In the proposed model, the only means of spread of information about the product is via Word of Mouth communication; consumers' knowledge of the product is only through friends who already know about the product's existence. Both buyers and non-buyers contribute to information diffusion while buyers are more likely to get engaged. By analyzing the structure of the underlying endogenous process, we show that the optimal dynamic pricing policy for durable products with zero or negligible marginal cost, drops the price to zero infinitely often. By attracting low-valuation agents with free-offers and getting them more engaged in the spread, the firm can reach out to potential high-valuation consumers in parts of the network that would otherwise remain untouched without the price drops. We provide evidence for this behavior from smartphone app market, where price histories indicate frequent free-offerings. Moreover, we show that despite infinitely often drops of the price to zero, the optimal price trajectory does not get trapped near zero. We demonstrate the validity of our results in face of strategic forward-looking agents, homophily-based engagement in word of mouth, network externalities, and consumer inattention to price changes. We further unravel the key role of the product type in the drops by showing that the price fluctuations disappear after a finite time for a nondurable product.

Keywords: Information diffusion, word of mouth, price fluctuations, social networks, network externality, homophily, zero pricing, monopoly pricing.

JEL Classification: D42, O33, D85.

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1 Introduction

How should a monopolist offering a product in a social network price its product over time? Does the profit-maximizing strategy always keep the prices monotone? Is there a steady state price? This paper introduces a new framework to investigate these questions by considering the mechanism by which information about a product diffuses in networks. In particular, our goal is to investigate the role of word of mouth (WOM) communication of consumers[1], in the optimal pricing policy of the monopoly firm. The multibillion-dollar growing market for smartphone applications, where word of mouth is often the only means of spread of information about the product, is a great real world example of such a scenario.

During the last decade, there has been significant growth in the market for smartphone applications. These applications (apps) are typically cheap, and often the only low-budget[2] means by which many of these apps spread is the word of mouth communication of their users. Many apps ask users for permission to send notifications about the product to contacts in their address books or to post a message on their online social media, when they purchase or start using the app. A good evidence for the effectiveness of word of mouth is the smartphone application *WhatsApp* which was sold to *facebook* in early 2014 for $19 billion. WOM was the key to *WhatsApp* popularity. As noted by Bloomberg (Satariano (2014)), "They [*WhatsApp* management] eschewed marketing and did not employ a public relations person, relying on the word of mouth recommendations of its users instead".[3]

The price an app developer offers for its product is also a big driver for spreading the information. As such, posting time-varying prices is a common marketing tool for spreading information about the existence of a new app among the users. Figure 1 depicts the price history for *Tadaa SLR*, an iPhone photo and video application since its release.[4] An interesting observation from this chart is the *frequent* drops of the price to zero. The same pattern can be seen in the price trends of many other smartphone applications (e.g., *XnShape*, *The Curse*, *Equalizer PRO*^{TM}, *Color Vacuum*, *ContactFlow*, *Coyn*, and *IBSnap*, only to name a few[5]).

[1] "Word of mouth communication involves the passing of information between a non-commercial communicator (i.e. someone who is not rewarded) and a receiver concerning a brand, a product, or a service", Dichter (1966).

[2] A recent survey by *AppFlood* (McCloughan (2013)) over 1000 independent small, medium and large app developers shows that the majority (78%) of developers surveyed had a per app marketing budget of $5000 or less.

[3] This example is solely meant to show the effectiveness of WOM in app marketing.

[4] The price data is gathered from "www.appshopper.com".

[5] These examples are chosen from various app categories of Photos & Videos, Games, Music, Education,

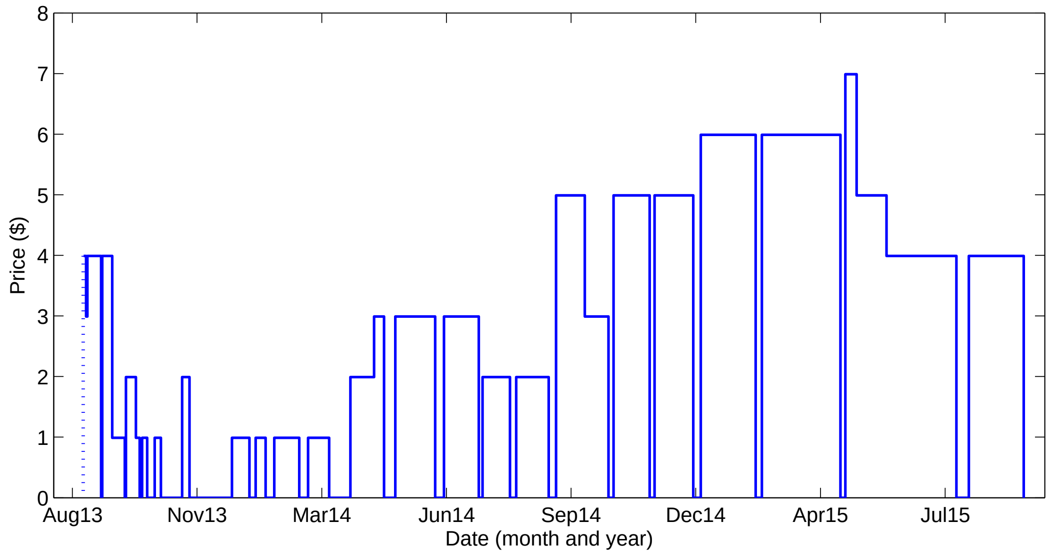

Figure 1: Price history for the iPhone application Tadaa SLR since its debut on Aug. 16, 2013.

Motivated by the above observations, we study the problem of optimal dynamic pricing of a profit-maximizing firm selling a product in a large social network where agents can only get informed about the product via Word of Mouth of previously informed friends. A key feature of this work is the explicit modeling of the effect of the price on the information diffusion via WOM. The (dynamic) price is a control variable by which the firm directly affects the information diffusion of its new product through the underlying social network. Firm's problem is then to decide, at each time step, between optimally exploiting the existing informed network or charging a lower price in favor of a higher spread of information.

The main contribution of this paper is to study pricing in social networks through the channel of information diffusion. We show that when the spread of a durable product is only via word of mouth, the optimal pricing policy is neither monotone nor reaches a steady state. Rather, the optimal policy fluctuates, dropping the price to zero infinitely often, essentially giving away the immediate profit in full to expand the informed network in order to exploit it in future. This is consistent with the real world evidence from smartphone applications[6] described above.

The key intuition behind this result is that frequent zero-price sales allow the firm to attract consumers who would not buy the product unless offered for free. Giving the product

Finance, and Utilities. See Appendix C for the price plots of some of these apps.

[6]We deal with smartphone applications as durable products. This is because when a user buys an app, she usually does not need to buy the same product any more.

for free to these low-valuation agents and getting them more engaged in the spread, the firm is able to reach sizeable parts of the network that would remain otherwise uninformed. By properly timing the drop in price, the firm can ensure that the marginal gain in future profit by selling the product in this previously unexplored part of the network prevails the loss in the immediate profit caused by offering the product for free, making the drop of the price to zero a profitable course of action. We also show that, although the optimal policy drops the price to zero infinitely often, price will not get trapped near zero.

More importantly, we show that the results remain valid in face of forward-looking agents, homophily-based[7] engagement in word-of mouth, network externalities, and consumer inattention to price changes, under surprisingly mild assumptions. Beside the WOM nature of information diffusion, we further show that the durability of the product is also a key driver for these frequent price drops. For a nondurable product, although the firm may initially make some free offers to expand its network, after a finite time it will fix the price at a level that extracts the maximum profit from the already informed population.

1.1 Literature

This paper makes contributions to four bodies of literature: WOM marketing, dynamic pricing, zero pricing, and strategic information diffusion in networks.

Since the landmark paper of Katz and Lazarsfeld (1955) in which the authors show that people rarely act on mass-media information unless it is also transmitted through personal ties, WOM has been a major focus of research in the marketing literature (Bass (1969), Godes and Mayzlin (2004), Chevalier and Mayzlin (2006), Besbes and Scarsini (2013)). WOM communication strategies are appealing because they combine the prospect of overcoming consumer resistance with significantly lower costs and fast delivery – especially through the Internet social networking (Trusov et al. (2008)). Although much of the work studying WOM communications focused on behavioral factors affecting information transfer (Herr et al. (1991), Chevalier and Mayzlin (2006), Berger and Schwartz (2011)), the rapid development of Internet social networking and communication technologies, e.g. smartphones, have boosted WOM research that is primarily concerned with network effects of social influence (see e.g., Watts and Dodds (2007), Goldenberg et al. (2009), Stephen and Toubiaz (2010), Katona et al. (2011), Campbell et al. (2013), Hervas-Drane (2015)). Our work also belongs

[7]Homophily refers to a tendency of various types of individuals to associate with others who are similar to themselves (Golub and Jackson (2012)).

to this latter category of WOM marketing literature. Recently, Campbell et al. (2013) study how far information about a product eventually diffuses through the population when consumers desire to signal their type to others. By focusing on a signaling equilibrium, they show that increasing asymmetry among consumers by restricting information of low-typed agents can boost the spread of WOM.[8] As a consequence, they find that advertising may crowd out the incentives for consumers to engage in WOM. In contrast to all of these works, we provide an analytical tractable model to analyze the impact of dynamic pricing as a marketing tool to control information diffusion via WOM.

Dynamic pricing has a rich history in economics and operation research.[9] In general, varying prices over time may have different causes. It might be because of the inability of the firms to commit to future actions (e.g. Conlisk et al. (1984), Sobel (1991)), or due to learning new experience goods (e.g. Bergemann and Välimäki (1997, 2000), Ifrach et al. (2011, 2013))[10] or the result of the inability of boundedly rational buyers to pay immediate attention to price changes (e.g. Radner et al. (2013)). Scarcity of the products with regard to the number of buyers (e.g. Gallego and van Ryzin (1994), Gershkov and Moldovanu (2009)), network externalities (e.g. Cabral et al. (1999)), stochastic incoming demand (e.g. Board (2008)), and time-varying values of buyers (e.g. Garrett (2013)) are among other causes suggested in the literature for varying prices over time. In particular, Garrett (2013) studies profit-maximizing prices in an environment where buyers arrive over time and have values for the good which evolve stochastically. The author shows that for a range of parameter values, optimal prices fluctuate over time. Prices gradually fall up to sales dates and jump thereafter, mainly due to the inter-temporal price discrimination effect introduced by Stokey (1979). In contrast to the present paper, none of these works relate pricing to the extent of the information diffusion.

How can firms profitably give away free products? Several branches of literature yield insight for this phenomenon. For example, multi-product pricing in two-sided markets (Rochet and Tirole (2003), Parker and Alstyne (2005)), forward-looking consumers and durable-goods monopolies with zero marginal cost (Coase (1972), Stokey (1981) and Gul et al.

[8]In this vein, Hervas-Drane (2015) presents a model of customer search to explain the impact of product recommendations on customer product discovery and the concentration of sales. He shows while recommendations benefit mainstream customers, when recommender systems are based on WOM and social filtering, there is a positive effect in the tail of the sales distribution on customers interested in niche products.

[9]Talluri and van Ryzin (2004) and Phillips (2005) provide an extensive review of this topic.

[10]These models are typically either two sided or one sided. Bergemann and Välimäki (1997, 2000), Ifrach et al. (2013), and Yu et al. (2013) consider two-sided learning models where buyers and sellers both learn the true value of a new product through consumer experiences. Papanastasiou et al. (2013) and Ifrach et al. (2011) analyze one-sided learning models when firm knows the product quality, buyers report their experiences and subsequent customers learn from these reports.

(1981)), and to a lesser extent bundle pricing (Hanson and Martin (1990)) are potential causes for this phenomenon. In our model the optimal price scheme sets the price to zero infinitely many times. However, the principal probing factor, in contrast to all the aforementioned works, is to control the extent of information diffusion.

The main focus here is durable products with zero or negligible marginal cost. Markets for digital goods and, in particular, smartphone applications are real world examples of this scenario. Market for smartphone applications (apps) is quite large and still rapidly growing. Before 2013 mobile apps had an economy with a market size of $25 billion. It is estimated that one billion smartphones will be sold by the end of 2015. Given that %46 of app users report having paid for their apps, the app market is expected to have 268 billion downloads that generate $77 billion worth of revenue by 2017.[11] Frequent zero-pricing of apps is already noticed by the app industry. In fact there are websites that provide lists of the paid apps that become free on a daily basis.[12] Our work is the first analytical approach to this phenomenon in the app market which suggests controlling the extent of information diffusion via dynamic prices as a potential cause for the zero price drops in this market.

This paper is also related to the growing literature on strategic interactions in social networks (e.g. Ballester et al. (2006), Bramoullé and Kranton (2007), Galeotti et al. (2010)).[13] For the most part, in these models prices are static. Su (2007), Nocke and Peitz (2007), and Hörner and Samuelson (2011) consider dynamic pricing with strategic customers, however, in contrast to our work, the price paths are found to be monotone.

The closest result in literature to ours is the work in Campbell (2012), where the author studies pricing for a nondurable product under word of mouth communications. Campbell shows price fluctuations for a nondurable product during the introductory stages. This is in line with our result for the nondurable product where we show that the firm may use price drops to zero during the early stages in order to expand the informed network. However, we show that fluctuations disappear as the size of the spread gets sufficiently large. Campbell models the word of mouth as a branching process, in which each new buyer informs a fixed number of new agents in average. The model neither considers the overlap among the friends of new buyers nor the fact that some of their friends may have already heard about the product. As Campbell emphasizes in his paper, such a model is valid only at the early

[11]See "entrepreneur.com/article/236832" for more statistics on the app market.

[12]For instance, see "www.appsliced.co", "www.appaddict.net" and "www.appspy.com".

[13]Other relevant studies include: strategic information exchange in social networks (e.g. Acemoglu et al. (2014)), optimal static pricing under presence of local network effect (Sundararajan (2008), Hartline et al. (2008), Candogan et al. (2012)), and optimal advertising strategies in social networks (Galeotti and Goyal (2009), Galeotti and Mattozzi (2011)).

stages of introducing a product to the network, when the size of the informed population is still very small. Our model, on the other hand, captures both the diminishing marginal contribution of new buyers to diffusion caused by the overlap among their friends, and the slowdown of diffusion caused by the growing size of the agents who have already heard about the product.

This work is also related to the literature on diffusion dynamics in social networks. One of the main challenges in information diffusion in social networks is developing tractable models. The combinatorial nature of networks with heterogeneity often makes analysis prohibitively difficult. Several modeling approaches have been developed to reduce the inherent complexity. An early model of diffusion is the Bass (1969) model. Although the proposed model does not capture any explicit social network structure, it still incorporates imitation from others. Some recent models use the concept of mean field theory to model diffusion over the network. The main idea of this approach is to replace all interactions to an agent with an average or effective interaction. These tractable models have been used by Jackson and Rogers (2007b) to relate stochastic dominance properties of the degree distribution of the network to the depth of diffusion, by Young (2009) to provide methodologies for characterizing different models of social influence by the time path of adoption, by Jackson and Rogers (2007a) to infer how the formation process affects average utility in the network, and by Jackson and Yariv (2007) and López-Pintado (2008) to evaluate strategic adoption decisions of individuals.[14]

From the methodological point of view, our work is also related to the literature on random graph theory. The theory of random graph has been used as a convenient modeling abstraction, which can facilitate modeling and analysis of the information diffusion in networks. Random networks find their origin in studies of random graph by Rapoport (1957) and Erdős and Rényi (1959, 1960, 1961). Random graph theory is also widely used by network scientists. For instance, it has been used by Watts and Strogatz (1998) to present their seminal small-world idea by creating highly clustered networks with small diameters, by Newman et al. (2001) to model the world-wide web and collaboration networks of company directors and scientists, and by Watts (2002) to model collective actions and the diffusion of norms and innovations. The current paper develops an endogenous network model by using selling prices as dynamic controls for the information diffusion in a social network whose structure is captured by a Poisson random graph (Bollobás (2001)). The proposed model allows a firm to strategically affect the information diffusion about the existence of a new

[14] Mean field theory is also used in revenue management, in particular, to study and model complex dynamic demand systems with the objective of maximizing performance, e.g. Gallego and van Ryzin (1994).

product within the social network by means of dynamic prices.

The rest of the paper is organized as follows. Section 2 presents a tractable model for strategic information diffusion via WOM in a large social network whose structure is represented by a Poisson random graph. Section 3 discusses the main challenge of the firm as deciding between spreading and exploiting, and presents the main results of the paper. In Section 4, we examine the robustness of the frequent zero-price drops in the face of forward-looking agents, homophily-based engagement in WOM, and network externalities. We unravel the key role of the type of the products in the price drops by studying the problem for a nondurable product in Section 5. Finally, our conclusions are presented in Section 6.

2 Model

2.1 General Description

The economy consists of a unit measure continuum of agents indexed by $i \in I = [0, 1]$. Agents form a social network, the structure of which is captured by an undirected random graph G with Poisson degree distribution with mean λ. More precisely, each agent $i \in I$ has a total of $d_i \sim \text{Poiss}(\lambda)$ friends uniformly distributed in I.[15] We denote the set of the friends of i in G with N_i. For every $i \in I$, the set of her friends N_i forms a Poisson process in I.

At each time step $t = 0, 1, 2, \cdots$, a firm offers a product to the continuum of agents in the network at price $u(t) \in \mathcal{U}$, where $\mathcal{U}$ is a finite set of admissible prices. The set of admissible prices $\mathcal{U}$ can represent any set of quantized price levels in $[0, 1]$. In particular, we assume $0 \in \mathcal{U}$ to allow for the *free offering* of the product. We denote the set of admissible prices as $\mathcal{U} = \{p_0 = 0 < p_1 < \ldots < p_m \leq 1\}$, where $m \geq 1$ is the number of nonzero price levels.

Each agent has a private valuation $\theta \in [0, 1]$ of the product, distributed according to a cumulative distribution function $F(\theta)$. We assume that F corresponds to a non-atomic PDF and is strictly increasing on $[0, 1]$. The valuations of the agents are time-invariant and for

[15] This can be thought of as a limit case of the well-known Erdős-Rényi graph (Erdős and Rényi (1959)), keeping the mean degree equal to λ and with $I = [0, 1]$ as the limit vertex set. This network model inherits independence of the edges from the Erdős-Rényi graph which proves very convenient in analyzing network behavior. A similar network model is used by Oberfield (2012), Larson (2013), and Galeotti and Goyal (2009).

now we assume they are independent of their degrees and the valuations of their friends.[16] Moreover, agents' valuations and their positions in the network are their private information, and hence, not known to the firm.

In order for an agent to buy the product, she should first be informed about its existence. At $t = 0$ to initiate the spread of information, a uniformly randomly subset of the population becomes informed about the product directly by the firm. Later on, at any time $t \geq 1$, other agents can only get informed via word of mouth from a friend who already knows about the product. We make a distinction between the rate of engagement of buyers and non-buyers in the spread of the information about the product: When an agent buys the product, she engages in word of mouth with each friend with some probability $0 < p_B \leq 1$, informing her about the product. If an agent learns about the product but does not make a purchase, she informs each friend with a lower probability $0 \leq p_{\bar{B}} < p_B$. Assigning a nonzero probability to the engagement of non-buyers in word of mouth is motivated by some recent work which provide evidence for the significant role of non-adopters in the spread of information (Banerjee et al. (2013)). It is to be noted that an informed agent buys the product if the offered price does not exceed her valuation, i.e., $u(t) \leq \theta$ where $u(t)$ is the price offered by the firm at time t.

In this framework, firm's objective is to devise an optimal dynamic pricing policy maximizing its accumulated discounted profit over an infinite time horizon. We first study this problem for the case of a *durable* product, such as many smartphone applications, in order to justify the behavior pointed out in the previous section. We then verify the robustness of the price drops to several key model assumptions. We discuss the validity of the results in face of strategic forward-looking agents, homophily-based engagement in word of mouth, network externalities, and consumer inattention to price changes. Finally, we investigate the role of the type of the product in the price drops to zero by studying the problem for a *nondurable* product. It is to be noted that an informed agent may buy a nondurable product at each time step, given that its price is lower than her valuation. However, if she buys a durable product at some time, she will not buy it thereafter.

2.2 WOM Diffusion Dynamics

In this subsection, we first present a few notations, definitions, and observations that will be used later to derive the dynamics of the information diffusion in the network. We denote the

[16] Later on, we will relax this assumption.

set of informed agents at time t by $X(t)$ and its size by $x(t)$. $X(0)$ is therefore the set of those agents directly informed by the firm, with $x(0) = x_0$ denoting the size of this set. Considering that we are dealing with a unit measure continuum of agents, an informal use of the strong law of large numbers implies that $x(t) = \text{Prob}(i \in X(t))$. As we will see in the sequel, this will prove very convenient in deriving the dynamics of the information diffusion.[17] The set of informed agents $X(t)$ is increasing, that is $X(t-1) \subseteq X(t)$. $Y(t) = X(t) - X(t-1)$ represents the set of freshly informed agents at time t whose size is denoted by $y(t)$.

We partition the set of freshly informed agents in $Y(t)$ into two subsets: those who buy the product, denoted by $B_Y(t)$, and those who do not buy, denoted by $\bar{B}_Y(t)$. Agents in both subsets contribute to information diffusion by informing a fraction of their friends about the product, which constitutes part of $Y(t+1)$. Noting that θ has the same distribution in $Y(t)$ as it has in I, the fraction of agents from $Y(t)$ that buy the product when offered the price $u(t) = p_r$ is $(1 - F(p_r))y(t)$. This yields

$$\begin{aligned} b_Y(t) &= (1 - F(p_r))y(t), \\ \bar{b}_Y(t) &= F(p_r)y(t), \end{aligned} \tag{1}$$

with p_r being the price offered at time t ($u(t) = p_r$) and lowercases denoting the size of the corresponding sets.

Another contribution to diffusion comes from the set of agents previously informed about the product who have not yet purchased. For such agents, the price has not fallen below their valuation since the time they were informed about the product. We denote the set of such agents at time t by $Z(t)$. An agent in this set may buy the product at time t and thus inform some of her friends, if the offered price at time t is below her valuation. Unlike $Y(t)$, the distribution of θ is not given by $F(\cdot)$ for the agents in $Z(t)$ and depends on the price history. However, we can use a stack of m variables (recall that m is the number of nonzero price levels) to fully describe the distribution of θ in $Z(t)$. We can partition $Z(t)$ as $\bigcup_{j=1}^{m} Z_j(t)$, where $Z_j(t)$ is the set of those agents in $Z(t)$ whose valuations lie between price levels p_{j-1} and p_j, that is, $Z_j(t) = \{i \in Z(t) | p_{j-1} \leq \theta_i < p_j\}$. Then, the distribution of θ in $Z(t)$ is fully determined by the sizes of these sets, denoted by $z(t) = [z_1(t) \dots z_m(t)]^T$. If the firm chooses $u(t) = p_r$ as the price at time t, then all the agents in $B_Z(t) = \bigcup_{j=r+1}^{m} Z_j(t)$

[17]Following the same logic, we may interchangeably use the words size, fraction, and probability in the paper.

which has a size of

$$b_Z(t) = \sum_{j=r+1}^{m} z_j(t), \tag{2}$$

will buy the product and will subsequently engage in word of mouth with their friends, while the rest of the agents in $Z(t)$ will be carried over to $Z(t+1)$. Those freshly informed agents in $Y(t)$ with valuations below p_r, which we denoted earlier with $\bar{B}_Y(t)$, constitute another part of $Z(t+1)$. Summarizing the above, we arrive at the following update rule for the size of the set of agents whose valuations are between p_{j-1} and p_j and have not yet bought the product:

$$z_j(t+1) = \begin{cases} z_j(t) + (F(p_j) - F(p_{j-1}))y(t), & 1 \leq j \leq r \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

assuming that the offered price at time t is p_r ($u(t) = p_r$). The size of the fresh buyers $B(t) = B_Y(t) \cup B_Z(t)$ is given by

$$b(t) = (1 - F(p_r))y(t) + \sum_{j=r+1}^{m} z_j(t). \tag{4}$$

In order to find the size of the informed agents at time $t+1$, we take a closer look at the three subsets involved in the information diffusion: $B_Y(t)$, $\bar{B}_Y(t)$, and $B_Z(t)$. Agents in $B_Y(t)$ are those who were just informed about the product at time t and bought it. Upon buying the product, they may inform each of their friends about the product with some probability p_B. Using the *stationary increments property*[18] of Poisson processes, the number of friends an uninformed agent $i \notin X(t)$ has in $B_Y(t)$ is a Poisson random variable with mean $\lambda b_Y(t)$. Since each such friend may inform i with probability p_B, thus the number of friends in $B_Y(t)$ from which i may hear about the product has a Poisson distribution with mean $\lambda p_B b_Y(t)$.

Agents in $\bar{B}_Y(t)$ are those just informed about the product at time t but found the price too high to buy. These agents still may inform friends about the product with some probability $p_{\bar{B}} < p_B$ (we may also have $p_{\bar{B}} = 0$). The number of friends an uninformed agent $i \notin X(t)$ has in $\bar{B}_Y(t)$ is a Poisson random variable with mean $\lambda \bar{b}_Y(t)$. Therefore, the number of friends in $\bar{B}_Y(t)$ from which i may hear about the product has a Poisson distribution with mean $\lambda p_{\bar{B}} \bar{b}_Y(t)$.

[18] According to the stationary increments property of Poisson processes, the probability distribution of the number of occurrences (herein friends) in any subset only depends on the size of the subset (Billingsley (1995)).

The last contribution to diffusion comes from agents in $B_Z(t)$, who have previously heard about the product but have not made a purchase as of time t. Any such agent has already informed some of her friends when hearing about the product for the first time. Upon buying the product, they once again get engaged in word of mouth with more friends, informing them about the product. Since these agents have already informed each friend with probability $p_{\bar{B}}$, thus the number of friends an uninformed agent $i \notin X(t)$ has in $B_Z(t)$ is a Poisson random variable with mean $\lambda(1-p_{\bar{B}})b_Z(t)$. Therefore, the number of friends in $B_Z(t)$ from which i may hear about the product has a Poisson distribution with mean $\lambda p_B(1-p_{\bar{B}})b_Z(t)$.

Putting the above three cases together, it is easy to see that the number of friends an uniformed agent $i \notin X(t)$ may hear from about the product at time t has a Poisson distribution with mean

$$\lambda p_B b_Y(t) + \lambda p_{\bar{B}} \bar{b}_Y(t) + \lambda p_B(1-p_{\bar{B}})b_Z(t). \tag{5}$$

An agent $i \in I$ will be uninformed at time $t+1$ if and only if she is neither informed nor hears from a friend at time t. We can now write the dynamics of the informed population $x(t)$ as

$$1 - x(t+1) = (1-x(t))e^{-\lambda(p_B b_Y(t) + p_{\bar{B}} \bar{b}_Y(t) + p_B(1-p_{\bar{B}})b_Z(t))}, \tag{6}$$

where $b_Y(t)$, $\bar{b}_Y(t)$, and $b_Z(t)$ are given by (1) and (2), $z(t)$ is updated using (51), and $y(t+1) = x(t+1) - x(t)$. Moreover, $y(0) = x(0) = x_0$ and $z_j(0) = 0$ for $1 \le j \le m$. The overall structure of the information diffusion via WOM is depicted in Figure 2.

Remark 1 (Diminishing Information Diffusion and the Slowdown of Spread)**.** *We can use the dynamics derived above to verify our claim in Section 1 about the diminishing marginal contribution of new buyers (non-buyers) to diffusion, and the slowdown of spread due to the growing size of the informed population. Using (6), the growth in the spread can be written as*

$$y(t+1) = (1-x(t))(1 - e^{-\lambda(p_B b_Y(t) + p_{\bar{B}} \bar{b}_Y(t) + p_B(1-p_{\bar{B}})b_Z(t))}). \tag{7}$$

From this, the marginal contribution of newly informed buyers $B_Y(t)$ to diffusion is

$$\frac{\partial y(t+1)}{\partial b_Y(t)} = \lambda p_B (1-x(t))e^{-\lambda(p_B b_Y(t) + p_{\bar{B}} \bar{b}_Y(t) + p_B(1-p_{\bar{B}})b_Z(t))}. \tag{8}$$

When only a few contribute to spread (that is $p_B b_Y(t) + p_{\bar{B}} \bar{b}_Y(t) + p_B(1-p_{\bar{B}})b_Z(t)$ is small), information diffuses at a rate of λp_B from $B_Y(t)$ to the set of uninformed agents $I - X(t)$. However, the marginal contribution to diffusion decays exponentially with the size of those

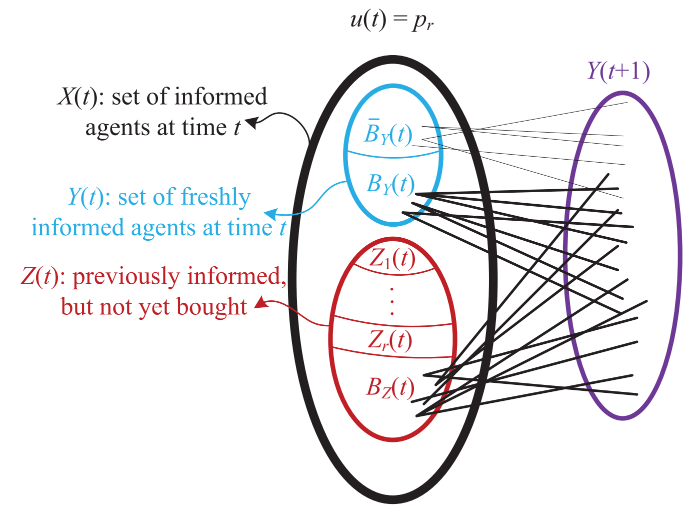

Figure 2: An overall view of the information diffusion via word of mouth. Agents in $B_Y(t) \cup B_Z(t)$ buy the product and inform their friends with probability p_B. Fresh non-buyers $\bar{B}_Y(t)$ also inform some friends but with a lower probability (thinner edges reflect the lower likelihood of getting informed via a non-buyer friend).

participating in the spread, thus lowering the average rate of diffusion to uninformed agents; the larger the set of spreaders, the higher the chance of having friends in common, hence lowering the average rate of diffusion. The model also clearly captures the slowdown effect of the agents who have already heard about the product on diffusion as $y(t+1)$ is proportional to $1 - x(t)$. Similar arguments hold for the contribution to the spread by buyers $B_Z(t)$ and by new non-buyers $\bar{B}_Y(t)$.

3 Firm's Decision Problem: To Spread or to Exploit?

The profit of the firm for a durable product is given by

$$\Pi^D(u(\cdot)) = \sum_{t=0}^{\infty} \beta^t u(t) b(t), \tag{9}$$

where $0 < \beta < 1$ is the discount factor, $b(t)$ is the size of the buyers at time t given by (52), and the marginal cost of the product is assumed to be zero. Firm's objective is to find a pricing policy that maximizes the above profit, which we denote as $u^D(\cdot)$.

Given the dynamics of the information diffusion for the WOM model developed in previous section and the profit of the firm given by (9), the firm's problem is to decide at each time step, between optimally exploiting the network it already has by offering a price that results in the maximum immediate profit, or offering a lower price in favor of a higher spread.

A related problem is to find the maximum achievable size of the informed network via WOM. For any price function $u(\cdot)$, $x(t)$ is bounded and increasing and therefore has a limit as $t \to \infty$.

Define $q(x_0, p_B, p_{\bar{B}}; u(\cdot)) = \lim_{t\to\infty} x(t)$ as the asymptotic size of the population that can be informed about the product via WOM, starting from a uniformly randomly chosen informed population of size x_0 and following a given pricing policy $u(\cdot)$, for given values of p_B and $p_{\bar{B}}$. It is easy to see that for $x_0 < 1$ this asymptotic size is always less than 1, implying that the product cannot take over the entire population I via only WOM. This is simply due to the fact that there are $e^{-\lambda}$ isolated agents (with no friend) in I, out of which $(1 - x_0)e^{-\lambda}$ of them are not in $X(0)$ and therefore will never hear about the product via WOM.

To gain more insights on the endogenous dynamics of diffusion, let us start with the case of zero price, i.e., when the product is given for free, that is $u \equiv 0$. Every agent that is informed about the product will in turn inform her friends with probability p_B. Note that since there are no non-buyers, $p_{\bar{B}}$ does not matter, so we simply choose $p_{\bar{B}} = 0$. In this case, $Z(t) = \emptyset$ and $B(t) = Y(t)$, thus the dynamics of diffusion governed by (1), (2), (51), and (6) simplifies to

$$1 - x(t+1) = (1 - x(t))e^{-\lambda p_B y(t)}, \tag{10}$$

$$y(t+1) = x(t+1) - x(t), \tag{11}$$

where $y(0) = x(0) = x_0$. Using this recursively for $t, t-1, \ldots, 0$, we obtain

$$1 - x(t+1) = (1 - x_0)e^{-\lambda p_B x(t)}. \tag{12}$$

The asymptotic size of the informed network for the case of zero price when informed agents engage in WOM with friends with probability p_B can now be obtained, noting that $q(x_0, p_B, 0; 0)$ should satisfy the above relation as well:

$$1 - q(x_0, p_B, 0; 0) = (1 - x_0)e^{-\lambda p_B q(x_0, p_B, 0; 0)}. \tag{13}$$

Based on this equation, we present several properties for $q(x_0, p_B, 0; 0)$ in the following proposition.

Proposition 1. *For every $0 < x_0 \leq 1$, the asymptotic size of the informed population for a free product where informed agents engage in the spread with probability p_B is given by the unique solution of $1 - q(x_0, p_B, 0; 0) = (1 - x_0)e^{-\lambda p_B q(x_0, p_B, 0; 0)}$ in $[0, 1]$. The solution is concave and monotonically increasing in x_0. Moreover, $q(x_0, p_B, 0; 0) > 1 - \frac{1}{\lambda p_B}$.*[19]

Proof. See the appendix. ∎

One interesting consequence of Proposition 1 is the discontinuity in $q(x_0, p_B, 0; 0)$ at $x_0 = 0$ for $\lambda p_B > 1$. Although $q(0, p_B, 0; 0) = 0$, for any nonzero x_0 and $\lambda p_B > 1$, $q(x_0, p_B, 0; 0)$ is lowerbounded by a positive constant independent of x_0. This implies that no matter how small the size of the initially informed population is, a free product with strong engagement of agents in the spread can take over a large portion of the network via WOM given the typically large average number of friends in the networks.

The zero-price case with full engagement of agents in spread ($p_B = 1$) gives an upperbound on the achievable asymptotic size of the informed population, that is $q(x_0, p_B, p_{\bar{B}}; u(\cdot)) \leq q(x_0, 1, 0; 0)$. In this case, every agent that is informed about the product will in turn inform all of her friends. The information will then spread throughout the network and all the agents that are reachable from an agent $i \in X(0)$ will eventually learn about the product. This upperbound can be obtained by solving $1 - q^0 = (1 - x_0)e^{-\lambda q^0}$ according to Proposition (1), where q^0 is the short-note for $q(x_0, 1, 0; 0)$.

As the main objective of this paper, we next show that under the optimal policy for a durable product price should drop to zero infinitely often. This matches the real world evidence from smartphone applications discussed in Section 1, where price histories witness frequent drops of the price to zero for many apps. We also present tight bounds for the asymptotic size of the spread under the optimal policy.

Theorem 1. *Under the optimal pricing policy $u^D(\cdot)$ for a durable product with zero marginal cost, where buyers and non-buyers engage in word of mouth with probabilities $0 \leq p_{\bar{B}} < p_B \leq 1$, the price drops to zero infinitely often. That is, there exists an infinite sequence of time instants $0 \leq t_0 < t_1 < \ldots$ such that $u^D(t_j) = 0$ for $j \in \mathbb{N}_0$. Moreover, the asymptotic size of*

[19] $q(x_0, p_B, 0; 0)$ can also be represented in terms of the Lambert W function, which is defined as the solution to the equation $W(z)e^{W(z)} = z$ (Corless et al. (1996)). Using this notation, we can easily show that $\lambda p_B(1 - q(x_0, p_B, 0; 0)) = -W(-\lambda p_B e^{-\lambda p_B}(1 - x_0))$. W is known to have two branches. It follows from Proposition 1 that $\lambda p_B(1 - q(x_0, p_B, 0; 0)) < 1$, requiring $W > -1$. This identifies the principal branch of the Lambert W function, denoted by W_0. Therefore, we can write $\lambda p_B(1 - q(x_0, p_B, 0; 0)) = -W_0(-\lambda p_B e^{-\lambda p_B}(1 - x_0))$. This representation enables us to use the properties of the Lambert W function, if ever needed.

the spread satisfies

$$q(x_0, p_B, 0; 0) \leq q(x_0, p_B, p_{\bar{B}}; u^D(\cdot)) \leq q(x_0, p_B + p_{\bar{B}} - p_B p_{\bar{B}}, 0; 0). \tag{14}$$

Proof. First we note that for any pricing policy, there exists at least one price level that holds infinitely often. This follows from the finiteness of the set of admissible prices $\mathcal{U}$. Let $p_r \in \mathcal{U}$ be the smallest price level which holds infinitely often for $u^D(\cdot)$. Then, any price level below p_r is used finitely in $u^D(\cdot)$. Therefore, there exists $T \geq 0$ such that $u^D(t) \geq p_r$ for all $t \geq T$.

Having infinitely many drops to zero under the optimal policy $u^D(\cdot)$ is clearly equivalent to $p_r = 0$ (that is $r = 0$). Therefore, to prove the theorem, we assume $r \geq 1$ and try to reach contradiction by constructing a new policy with a profit higher than that of $u^D(\cdot)$. For this purpose, we show that by zeroing the price to sell the product to a subset of informed agents that would not buy it otherwise, and by getting them (more) engaged in the spread of information, the monopolist can reach out to a part of the network that would remain unexplored under $u^D(\cdot)$. Dropping the price to zero to access this part of the network at a proper time, we then introduce a new policy yielding a profit higher than that of $u^D(\cdot)$ by exploiting this untouched component of the network.

Let $Y_r^D(T) \subset Y^D(T)$ denote those freshly informed agents at time T whose valuations are below p_r, i.e. $Y_r^D(T) = \{i \in Y^D(T) | 0 \leq \theta_i < p_r\}$, with a size of $y_r^D(T) = F(p_r) y^D(T)$.[20] None of the agents in $[\cup_{j=1}^r Z_j^D(T)] \cup Y_r^D(T)$ will ever buy the product under the pricing policy $u^D(\cdot)$, where $\cup_{j=1}^r Z_j^D(T)$ is the set of those previously informed agents at time T whose valuations are below p_r. Now, consider the set of agents that will remain uninformed under $u^D(\cdot)$. The size of this set is clearly $1 - q^D$, where q^D is the asymptotic size of the informed population under $u^D(\cdot)$, i.e., $q^D = q(x_0, p_B, p_{\bar{B}}; u^D(\cdot))$. Define Δ_r as the subset of these agents who have at least one friend in $\cup_{j=1}^r Z_j^D(T) \cup Y_r^D(T)$, that is

$$\Delta_r = \{i \in I | (\nexists t \in \mathbb{N}_0 : i \in x^D(t)) \wedge d_i(\cup_{j=1}^r Z_j^D(T) \cup Y_r^D(T)) \neq 0\}.\text{[21]} \tag{15}$$

The number of friends of an uninformed agent among $\cup_{j=1}^r Z_j^D(T) \cup Y_r^D(T)$ has a Poisson distribution with mean $\lambda(1 - p_{\bar{B}})(F(p_r) y^D(T) + \sum_{j=1}^r z_j^D(T))$, since each of these agents has already informed friends with probability $p_{\bar{B}}$. Therefore, by zeroing the price at any time

[20] We use superscript D to indicate that the variables correspond to the pricing policy $u^D(\cdot)$.

[21] For any $S \subseteq I$ and $i \in I$, we denote the number of friends of agent i in S with $d_i(S)$.

$t > T$ we can reach out a subset of Δ_r with the size of

$$\delta_r = (1 - q^D)(1 - e^{-\lambda p_B (1-p_{\bar{B}})(F(p_r) y^D(T) + \sum_{j=1}^{r} z_j^D(T))}), \tag{16}$$

that could not be reached under $u^D(\cdot)$. The idea now is to show that after a while there is so little profit left to be made in the future under $u^D(\cdot)$ that it is profitable to zero the price to reach out these agents in Δ_r, as will be elaborated below.

Let t_k, $k = 1, 2, \ldots$, denote the k-th price drop to p_r after time T under the optimal policy $u^D(\cdot)$. If an agent $i \in X^D(t_k)$ does not buy the product at this time, neither will she buy it in future. This means that agents in $X^D(t_k)$ do not contribute to the set of buyers $B^D(t)$ for $t > t_k$. Therefore, the size of the buyers from time $t_k + 1$ to $t_k + \tau$ for any $\tau \geq 1$ can be upperbounded by $x^D(t_k + \tau) - x^D(t_k)$, that is

$$\sum_{t=t_k+1}^{t_k+\tau} b^D(t) \leq x^D(t_k + \tau) - x^D(t_k). \tag{17}$$

Letting $\tau \to \infty$, yields

$$\sum_{t=t_k+1}^{\infty} b^D(t) \leq q^D - x^D(t_k). \tag{18}$$

Thus, the contribution of the buyers to the firm's profit after t_k can be upperbounded by

$$\begin{aligned} \Pi^D_{>t_k}(u^D(\cdot)) &= \sum_{t=t_k+1}^{\infty} \beta^t u^D(t) b^D(t) \\ &\leq \beta^{t_k+1} \sum_{t=t_k+1}^{\infty} u^D(t) b^D(t) \\ &\leq \beta^{t_k+1} p_m (q^D - x^D(t_k)). \end{aligned} \tag{19}$$

Next, consider a new policy $\tilde{u}(\cdot)$ having the same value as $u^D(\cdot)$ at all times except t_k+1 and t_k+2. Let $\tilde{u}(t_k+1) = 0$ and $\tilde{u}(t_k+2) = u^*$, where $u^*(1-F(u^*)) = \max_{u \in \mathcal{U}} u(1-F(u))$. Note that a subset of agents in Δ_r with size δ_r as in (16) are among the freshly informed agents $\tilde{Y}(t_k+2)$ since the agents in $\cup_{j=1}^{r} Z_j^D(T) \cup Y_r^D(T)$ buy the product at time t_k+1. The distribution of θ in Δ_r is given by $F(\cdot)$, hence the discounted profit made from these newly informed agents in Δ_r at time t_k+2 is $\beta^{t_k+2} u^*(1 - F(u^*))\delta_r$. Considering that $x^D(t) \to q^D$

as $t \to \infty$, we can choose k large enough such that

$$q^D - x^D(t_k) < \beta\delta_r \frac{u^*(1 - F(u^*))}{p_m}, \tag{20}$$

in which case the profit resulting from $\tilde{u}(\cdot)$ will be clearly higher than that coming from $u^D(\cdot)$. This contradicts the optimality of $u^D(\cdot)$, hence rejecting the initial assumption of $p_r \neq 0$, which completes the proof.

To bound the size of the spread under the optimal policy, we recursively use (6) and let $t \to \infty$ to obtain

$$1 - q^D = (1 - x_0)e^{-\lambda(p_B b_Y + p_{\bar{B}}\bar{b}_Y + p_B(1-p_{\bar{B}})b_Z)}, \tag{21}$$

where $b_Y = \sum_{t=0}^{\infty} b_Y(t)$, $\bar{b}_Y = \sum_{t=0}^{\infty} \bar{b}_Y(t)$, $b_Z = \sum_{t=0}^{\infty} b_Z(t)$, and q^D is the asymptotic size of the informed population under the optimal policy $u^D(\cdot)$. Since the optimal policy drops the price to zero infinitely often, every informed agent that is not a fresh buyer (i.e., $i \in \bar{B}_Y$), will buy sometime later on and hence is in B_Z. This implies $\bar{b}_Y = b_Z$. Also, every agent that will eventually get informed (a set with size q^D) will be either a fresh buyer ($i \in B_Y$) or not ($i \in \bar{B}_Y$). Thus, we have shown that $\bar{b}_Y = b_Z = q^D - b_Y$. Substituting this in (21), we get

$$1 - q^D = (1 - x_0)e^{-\lambda(p_B b_Y + (p_B + p_{\bar{B}} - p_B p_{\bar{B}})(q^D - b_Y))}, \tag{22}$$

which using Proposition 1 and noting that $p_B q^D \leq p_B b_Y + (p_B + p_{\bar{B}} - p_B p_{\bar{B}})(q^D - b_Y) \leq (p_B + p_{\bar{B}} - p_B p_{\bar{B}})q^D$ yields the bound $q(x_0, p_B, 0; 0) \leq q^D \leq q(x_0, p_B + p_{\bar{B}} - p_B p_{\bar{B}}, 0; 0)$. ■

Remark 2 (Asymptotic Size of the Informed Population for Buyers-Only Spread)**.** *For the case where non-buyers do not contribute to the spread of information (i.e., $p_{\bar{B}} = 0$), (21) gives the exact size of the spread as $q^D = q(x_0, p_B, 0; 0)$, which is the same as that of a free product. This means that for the special case where only buyers engage in WOM with their friends, the policy maximizing the profit of the firm also maximizes the asymptotic size of the spread.*

We can also easily extend the theorem to the case where the marginal cost is nonzero but sufficiently small. However, a significant marginal cost may shift the drops to a price level away from zero. This level is still below the marginal cost given that the gap between the marginal cost and the closest price level to it from below in $\mathcal{U}$ is sufficiently small.[22]

[22] See the appendix for the extension of the theorem to the case of nonzero marginal cost and the proof.

Theorem 1 shows infinitely many price drops to zero for a durable product under the optimal policy. A question that arises here is that whether it is possible for the optimal price trajectory to get trapped in a vicinity of zero. Noting that the price cannot stay at zero forever (due to the zero profit from such choice), this question translates to the possibility of getting stuck between price levels 0 and p_1 for small values of p_1. If such thing happens, one may even falsely attribute the price drops to a continuous-valued optimal price trajectory asymptotically converging to zero which manifests itself as a quantized price path bouncing between 0 and p_1. The following proposition rejects the possibility of such a price lockdown.

Proposition 2. *Under the optimal pricing policy $u^D(\cdot)$, the price jumps to a level above p_1 infinitely often, when $p_1(1-F(p_1)) < c$, where*

$$c = \max_{u \in \mathcal{U} \setminus \{p_1\}} \frac{u(1-F(u))(1-\beta\lambda(p_B + p_{\bar{B}} - p_B p_{\bar{B}})(1-q_B^0))}{1-\beta\lambda(p_B(1-F(u)) + p_{\bar{B}}F(u))(1-q_B^0)}, \tag{23}$$

and $q_B^0 = q(x_0, p_B, 0; 0)$ is the asymptotic size of the informed population under the zero price policy.

Proof. See the appendix. ∎

Based on this result, a questions still remains as to how small should p_1 be to guarantee frequent price jumps to levels above p_1. In fact, as we will see in the following example, p_1 does not need to be very small to satisfy the above condition for a wide range of parameters.

Example 1 (Buyers-Only Spread and Uniform Valuations)**.** *Assuming $\theta \sim Unif[0,1]$ and $p_{\bar{B}} = 0$, the condition on p_1 simplifies to $p_1 < \frac{1-\sqrt{1-4c}}{2}$, where*

$$c = \max_{u \in \mathcal{U} \setminus \{p_1\}} \frac{u(1-u)(1-\beta\lambda p_B(1-q_B^0))}{1-\beta\lambda p_B(1-u)(1-q_B^0)}. \tag{24}$$

It is easy to verify that c is decreasing with both β and $\lambda p_B(1-q_B^0)$. As a result, the less the value of each, the looser the bound on p_1. Although it may look counterintuitive at first, the term $\lambda p_B(1-q_B^0)$ is very helpful in loosening the bound on p_1. This term is less than 1 (from Proposition 1), and indeed is decreasing with λp_B for $\lambda p_B \geq 1$.[23] We now use this background to study a few cases in order to get an insight on the values of p_1 satisfying the above condition. We assume $0.5 \in \mathcal{U}$ to use it as a (sub)maximizer in (24). If $\lambda p_B \geq 2$,

[23] Recall that $\lambda p_B(1-q_B^0) = -W_0(-\lambda p_B e^{-\lambda p_B}(1-x_0))$, where W_0 is the principal branch of the Lambert W function (see Footnote 19). W_0 is an increasing function, and $-\lambda p_B e^{-\lambda p_B}$ is also increasing with λp_B for $\lambda p_B \geq 1$. This implies that $\lambda p_B(1-q_B^0)$ is decreasing with λp_B for $\lambda p_B \geq 1$.

then $\lambda p_B(1-q_B^0) < 0.41$,[24] which along with $\beta < 1$ yields $c > 0.18$, for which the condition in the above proposition reduces to $p_1 < 0.24$. Some information on β can loosen up this bound even further. For example, if we also know that $\beta \leq 0.5$, then the condition on p_1 becomes $p_1 < 0.33$. This bound gets closer to 0.5 for larger λp_B (or smaller β), assuring infinitely many price jumps to levels above p_1 even for values of p_1 that are not very small (see Appendix B for an illustrative plot).

4 Generalizations

In this section, we examine the robustness of the frequent zero-price drops behavior to several key assumptions that were made while developing the results of the previous section.

4.1 Forward-Looking Agents

The results of the previous section are derived based on the assumption that agents are myopic; an informed agent buys the product as soon as the firm offers a price below her valuation. At first, it may seem that the frequent free-offering policy may not be a good idea in face of forward-looking agents; a forward-looking agent may wait for a free offer even if the current offered price is below her valuation. The aim of this subsection is to show that in fact the frequent zeroing of the price still persists even if the agents are forward-looking.

We begin by making the necessary changes to the model. Assume that agents share a common discount factor $0 < \beta_c < 1$. An informed agent with valuation θ who pays price u at time t obtains utility $\beta_c^t(\theta - u)$. Therefore, given the optimal price path $u^D(\cdot)$, an informed agent has to choose a purchasing time τ, maximizing the utility

$$\sup_{\tau} \beta_c^{\tau}(\theta - u^D(\tau)). \tag{25}$$

An informed agent has the option of not buying, in which case $\tau = \infty$ and the payoff is zero. It is clear that the update rules for the size of previously informed non-buyers in (51) and freshly informed buyers and non-buyers in (1) are not valid anymore. However, they can be used to obtain bounds for the case of forward-looking agents:

$$z_j(t+1) \geq z_j(t) + (F(p_j) - F(p_{j-1}))y(t) \quad \text{for } 1 \leq j \leq r, \tag{26}$$

[24] $\lambda p_B(1-q_B^0) \leq -W_0(-2e^{-2}) = 0.4064$, for $\lambda p_B \geq 2$.

and

$$
\begin{aligned}
b_Y(t) &\leq (1 - F(p_r))y(t), \\
\bar{b}_Y(t) &\geq F(p_r)y(t),
\end{aligned}
\tag{27}
$$

where p_r is the price level chosen by firm at time t ($u(t) = p_r$). The size of informed agents still obeys the same update rule as in (6).

With this model at hand, we are now ready to unravel the rationale behind frequent zeroing of the price even when the consumers are forward-looking. We sketch the main probing factors here and refer the reader to the appendix for a comprehensive proof.

The first observation is that not everyone waits for a free offer; high-valuation agents pay to buy even knowing that the price can be zero later on. In fact, offering a price $u(t)$, all informed agents with valuations $\theta > \frac{u(t)}{1-\beta_c}$ will buy the product immediately. For such agents, $\theta - u(t) > \beta_c\theta$, thus they are willing to pay price $u(t)$ to buy at time t even if they know that it would be offered for free at time $t+1$. In other words, for consumers with high enough valuations, the myopic and forward-looking behaviors coincide. The second observation is that similar to the myopic case, there will still be a growing pile of low-valuation agents that do not buy the product unless offered for free, including those with valuations below the smallest price level p_1.

From the above discussion, we see that there are two subsets of agents whose actions are the same no matter whether they are myopic or forward-looking: these are agents with very high or very low valuations. As a result, the same mechanism as in the case of myopic agents triggers the frequent dropping of the price to zero; firm drops the price to zero to reach out uninformed high-valuation, willing-to-pay agents via low-valuation free-riders by engaging them in the spread at a higher rate; a profitable component which would remain untouched otherwise. We therefore have the following result.

Proposition 3. *Consider the same setup as in Theorem 1, with forward-looking agents sharing a common discount factor $0 < \beta_c < 1 - p_1$. Then, the optimal pricing policy $u^D(\cdot)$ will drop the price to zero infinitely often*

Proof. See the appendix. ∎

4.2 Homophily-Based WOM

In our base model in Section 2, consumers are indiscriminate in passing the information about the product to their friends. Buyers engage in WOM with each friend with the same probability p_B and non-buyers pass on the information with the same probability $p_{\bar{B}}$ to friends when they hear about the product. In this section we aim to extend our model and results to the case where agents' engagement in WOM is based on homophily: Agents tend to get engaged in WOM about the product with friends they believe to have similar valuations for the product. The rationale of zeroing the price to reach out high-valuation willing-to-pay uninformed agents via WOM of low-valuation agents may at first seem to fail here because low-valuation agents tend to engage in WOM with other low-valuation agents based on homophily. However, as we will see, under some mild assumptions on the homophily functions, the same rationale for the price drops still holds except that here we may need a chain of agents (and price drops) to reach the high-valuation uninformed agents.

The social network structure is same as before. We embed homophily in the model by considering valuation-dependent probabilities for the engagement of the informed agents in the spread. A buyer with valuation θ informs a friend whose valuation is θ' with probability $p_B(\theta, \theta')$. A non-buyer with valuation θ passes the information to a friend with valuation θ' with probability $p_{\bar{B}}(\theta, \theta')$, when she hears about the product. [25]

Assuming the independence of the engagement in WOM from the valuations in our base model of Section 2 had the advantage of keeping the distribution of θ invariant among the newly informed agents, and we only needed to keep track of the distribution of θ among non-buyers. Here, we will need to keep track of the distribution of θ in all sets in play. Using $\mu_Y(\theta, t)$ to denote the PDF of θ in $Y(t)$ and $M_Y(\theta, t)$ for the corresponding CDF[26] and assuming an offered price $u(t) = p_r$, we can find the size of the sets $B_Y(t)$ and $\bar{B}_Y(t)$ as

$$\begin{aligned} b_Y(t) &= (1 - M_Y(p_r, t))y(t), \\ \bar{b}_Y(t) &= M_Y(p_r, t)y(t), \end{aligned} \tag{28}$$

[25] This is similar to the multi-type random networks model of Golub and Jackson (2012), where types affect the formation of the links. Here, types (valuations) affect the likelihood of engaging two friends in WOM about the product.

[26] We use the same convention of using $\mu_S(\theta, t)$ and $M_S(\theta, t)$ to denote the PDF and CDF of θ in any set $S(t) \subseteq I$.

and the distribution of θ in each as

$$\mu_{B_Y}(\theta,t) = \mathbb{1}(\theta \geq p_r)\frac{\mu_Y(\theta,t)}{1-M_Y(p_r,t)},$$
$$\mu_{\bar{B}_Y}(\theta,t) = \mathbb{1}(\theta < p_r)\frac{\mu_Y(\theta,t)}{M_Y(p_r,t)}. \tag{29}$$

Similar relations hold for the size of buyers and non-buyers among previously informed agents $Z(t)$:

$$b_Z(t) = (1-M_Z(p_r,t))z(t),$$
$$\bar{b}_Z(t) = M_Z(p_r,t)z(t), \tag{30}$$

with corresponding distributions

$$\mu_{B_Z}(\theta,t) = \mathbb{1}(\theta \geq p_r)\frac{\mu_Z(\theta,t)}{1-M_Z(p_r,t)},$$
$$\mu_{\bar{B}_Z}(\theta,t) = \mathbb{1}(\theta < p_r)\frac{\mu_Z(\theta,t)}{M_Z(p_r,t)}. \tag{31}$$

In order to find the dynamics of the diffusion, we pick an uninformed agent $i \in \bar{X}(t)$ with valuation θ' (where $\bar{X}(t) = I \setminus X(t)$) and study the likelihood of her getting informed about the product via a contributor to the spread (that is, agents in $B_Y(t) \cup B_Z(t) \cup \bar{B}_Y(t)$). Following the same steps as in our base model, we can easily see that the number of informed agents that may engage in WOM with this agent has a Poisson distribution in each of these sets, except that the mean values here are time-varying and valuation-dependent:

$$\lambda_{B_Y}(\theta',t) = \lambda b_Y(t)\mathbb{E}_{B_Y}[p_B(\theta,\theta')|\theta'] = \lambda b_Y(t)\int p_B(\theta,\theta')\mu_{B_Y}(\theta,t)d\theta,$$
$$\lambda_{\bar{B}_Y}(\theta',t) = \lambda \bar{b}_Y(t)\mathbb{E}_{\bar{B}_Y}[p_{\bar{B}}(\theta,\theta')|\theta'] = \lambda \bar{b}_Y(t)\int p_{\bar{B}}(\theta,\theta')\mu_{\bar{B}_Y}(\theta,t)d\theta, \tag{32}$$
$$\lambda_{B_Z}(\theta',t) = \lambda b_Z(t)\mathbb{E}_{B_Z}[p_B(\theta,\theta')(1-p_{\bar{B}}(\theta,\theta'))|\theta'] = \lambda b_Z(t)\int p_B(\theta,\theta')(1-p_{\bar{B}}(\theta,\theta'))\mu_{B_Z}(\theta,t)d\theta,$$

and the update rule for the size of the informed population $x(t)$ becomes

$$1-x(t+1) = (1-x(t))\mathbb{E}_{\bar{X}}[e^{-\lambda(\theta',t)}] = (1-x(t))\int e^{-\lambda(\theta',t)}\mu_{\bar{X}}(\theta',t)d\theta', \tag{33}$$

where $\lambda(\theta',t) = \lambda_{B_Y}(\theta',t) + \lambda_{\bar{B}_Y}(\theta',t) + \lambda_{B_Z}(\theta',t)$ and $\mu_{\bar{X}}(\theta',t)$ is the PDF of θ' in $\bar{X}(t)$.

Finally, we need the update rules for $\mu_{\bar{X}}$, μ_Y, and μ_Z:

$$\begin{aligned}
\mu_{\bar{X}}(\theta, t+1) &= \frac{1-x(t)}{1-x(t+1)}\mu_{\bar{X}}(\theta,t)e^{-\lambda(\theta,t)}, \\
\mu_Y(\theta, t+1) &= \frac{1-x(t)}{y(t+1)}\mu_{\bar{X}}(\theta,t)(1-e^{-\lambda(\theta,t)}), \\
\mu_Z(\theta, t+1) &= \frac{\bar{b}_Y(t)\mu_{\bar{B}_Y}(\theta,t)+\bar{b}_Z(t)\mu_{\bar{B}_Z}(\theta,t)}{\bar{b}_Y(t)+\bar{b}_Z(t)}.
\end{aligned} \tag{34}$$

To sum up, the model dynamics for homophily-based WOM diffusion is governed by (28)-(34). Clearly, this is not as compact as the base model in Section 2 where $p_B(\theta,\theta') \equiv p_B$ and $p_{\bar{B}}(\theta,\theta') \equiv p_{\bar{B}}$ but is remarkably still tractable. Extracting a few observations from these equations, we will be able to use a similar proof argument for the profitability of the infinitely often dropping of the price to zero. Using the update rule for $\mu_{\bar{X}}(\theta,t)$ in (34) recursively, we can obtain

$$\mu_{\bar{X}}(\theta, t+1) = \frac{1-x_0}{1-x(t+1)}e^{-\sum_{\tau=0}^{t}\lambda(\theta,\tau)}f(\theta), \tag{35}$$

where $f(\cdot)$ is the PDF of θ in I. The maximum contribution of an informed agent to the summation $\sum_{\tau=0}^{t}\lambda(\theta,\tau)$ is upperbounded by λ if she ever buys the product and zero otherwise. This gives a lowerbound for the PDF of θ in uninformed regions as

$$\mu_{\bar{X}}(\theta, t+1) \geq \frac{1-x_0}{1-x(t+1)}e^{-\lambda q}f(\theta), \tag{36}$$

where q is the asymptotic size of the informed population. Integrating the above relation over any range of valuations $[\underline{\theta}, \bar{\theta}]$, we get

$$(M_{\bar{X}}(\bar{\theta}, t+1) - M_{\bar{X}}(\underline{\theta}, t+1))(1-x(t+1)) \geq e^{-\lambda q}(F(\bar{\theta}) - F(\underline{\theta}))(1-x_0). \tag{37}$$

We refer to the above property as *valuation diversity preservation of WOM among uninformed agents*: if a range of valuations has an initial nonzero measure among uninformed agents $I - X_0$, it will have a nonzero measure in the unexplored part of the network at all times, at least as large as its initial size in $I - X_0$ scaled by $e^{-\lambda q}$. This property holds for any general functions $p_B(\theta,\theta')$ and $p_{\bar{B}}(\theta,\theta')$, and is due to the nature of WOM and not homophily. This will prove very useful.

To proceed further, we impose two conditions on $p_B(\theta,\theta')$ and $p_{\bar{B}}(\theta,\theta')$. The first condition is a *local homophily condition*, requiring a buyer to engage in WOM with those friends having very similar valuations with a probability bounded away from zero. That is, there exist $\delta, \underline{p} > 0$ such that if $|\theta - \theta'| < \delta$, then $p_B(\theta,\theta') > \underline{p}$. The second condition is that the contribution of a non-buyer to spread should be upperbounded by a positive number less than 1. That is, there exists $\bar{p} < 1$ such that $p_{\bar{B}}(\theta, \theta') < \bar{p}$. We refer to this condition as the *limited engagement of non-buyers in spread*. These two conditions are indeed very general and hold even for many non-homophilous functions. We can use the local homophily condition to show that, no matter the pricing policy, there will always be a nonzero mass of low-valuation yet informed non-buyers around.[27] Moreover, we can show that having a nonzero measure set of agents with valuations in the range $[\theta_0, \theta_0 + \frac{\delta}{2}]$ among non-buyers, the firm can reach out a nonzero mass of higher valuations in range of $[\theta_0 + \frac{\delta}{2}, \theta_0 + \delta]$. It should be now clear that by dropping the price to zero (possibly several times in a row depending on the radius of homophily δ) the firm can reach out high-valuation willing-to-pay agents from low-valuation agents. We therefore have the following result.

Proposition 4. *Suppose that the probabilities of the engagement of the buyers and non-buyers in WOM spreading of the product are given by the valuation-dependent functions* $p_B, p_{\bar{B}} : [0,1]^2 \to [0,1]$ *satisfying the following conditions:*
i) there exist $\delta, \underline{p} > 0$ *such that if* $|\theta - \theta'| < \delta$*, then* $p_B(\theta, \theta') > \underline{p}$*, and*
ii) there exists $\bar{p} < 1$ *such that* $p_{\bar{B}}(\theta, \theta') < \bar{p}$*.*
Then, the optimal pricing policy $u^D(\cdot)$ *for a durable product drops the price to zero infinitely often.*

Proof. See the appendix. ■

4.3 Network Externalities

A firm offering its product in a social network can leverage the spread of its product from network externalities. An informed agent who does not buy the product at a given price may do so later on, if many of her friends buy the product, even if the firm does not lower the price. This raises another interesting question as to whether the price drops would be still profitable in the presence of network externalities. The aim of this section is to formalize and answer this question.

When the product exhibits network externalities, an informed agent buys the product if the offered price does not exceed the sum of her valuation and the total externalities from her friends whom she knows are already using the product. Denote as $\mathfrak{B}(t) = \cup_{\tau=0}^{t-1} B(\tau)$ the set of all previous buyers at time t and let $0 < \alpha \leq 1$ represent the network externality effect.

[27] See the proof of Proposition 4 for the details.

Then, an informed agent $i \in I$ buys the product at time t if the offered price $u(t)$ does not exceed her *augmented* valuation defined as $\theta_i^a(t) = \theta_i + \alpha d_i^{WOM}(\mathfrak{B}(t))$, where $d_i^{WOM}(\mathfrak{B}(t))$ denotes the number of friends at time t who have already bought the product and have engaged in WOM with agent i about the product.

The first step in the analysis is to identify the set of buyers and non-buyers at time t. As before, new buyers are either among the freshly informed agents $Y(t)$ or among those previously informed non-buyers, denoted as $Z(t)$. Define the set of those agents in $Y(t)$ whose augmented valuations are below θ^a as $Y(\theta^a, t)$ and its size by $y(\theta^a, t)$. Note that $y(\theta^a, t)$ fully characterizes the distribution of the augmented valuation in $Y(t)$.[28] Similarly, we use $Z(\theta^a, t)$ and $z(\theta^a, t)$ to represent the set of those agents in $Z(t)$ whose augmented valuations are below θ^a and its size, respectively.[29] Having been offered a price $u(t) \in \mathcal{U}$, agents in $\bar{B}_Y(t) = Y(u(t), t)$ and $\bar{B}_Z(t) = Z(u(t), t)$ do not have a high enough augmented valuation to buy the product at this price and will form $Z(t+1)$. For agents in $B_Y(t) = Y(t) \setminus \bar{B}_Y(t)$ and $B_Z(t) = Z(t) \setminus \bar{B}_Z(t)$, augmented valuations are higher than (or equal to) $u(t)$ and therefore they will buy the product.

Upon buying the product, buyers and non-buyers engage in WOM with their friends with probabilities p_B and $p_{\bar{B}}$. We need to distinguish between the agents who only hear about the product from non-buyers and those who also hear from some of the buyers as well. Consider the partition $Y(t+1) = Y_0(t+1) \cup Y_{>0}(t+1)$. Here, $Y_0(t+1)$ are those freshly informed agents who have not heard from any friend in $B_Y(t) \cup B_Z(t)$, if they have any friend among them. $Y_{>0}(t)$, on the other hand, is the set of those freshly informed agents who have heard about the product from a friend in $B_Y(t) \cup B_Z(t)$. Recalling that an agent only receives externality from friends whom she knows are using the product, the augmented valuation for any freshly informed agent will be $\theta_i^a(t+1) = \theta_i + \alpha d_i^{WOM}(B(t))$. Using this we can find the update rule for $y_{>0}(\theta^a, t+1)$ and $y_0(\theta^a, t+1)$ as

$$y_{>0}(\theta^a, t+1) = (1 - x(t))e^{-\lambda_B(t)} \sum_{d=1}^{\infty} \frac{(\lambda_B(t))^d}{d!} F(\theta^a - \alpha d),$$
$$y_0(\theta^a, t+1) = (1 - x(t))e^{-\lambda_B(t)}(1 - e^{-\lambda p_{\bar{B}} \bar{b}_Y(t)})F(\theta^a), \tag{38}$$

where $\lambda_B(t) = \lambda p_B(b_Y(t) + (1 - p_{\bar{B}})b_Z(t))$ is the average number of friends among the buyers $B(t)$ that an uninformed agent hears from.

[28] In fact, $y(\theta^a, t)$ is the CDF of θ^a in $Y(t)$ multiplied by its size $y(t)$.

[29] In general, for any $S(t) \subseteq I$, we use the notation $S(\theta^a, t)$ to denote the set of those agents in $S(t)$ whose augmented valuations are below θ^a and $s(\theta^a, t)$ to denote its size.

Finding the update rule for $Z(\theta^a, t+1)$ is much more involved, mainly due to the engagement of non-buyers in the spread. Whether or not an agent is informed about the product via a non-buyer changes the future likelihood of them getting engaged in WOM, if the non-buyer ever buys the product. To include this into the diffusion dynamics we need to keep track of both the time an agent first hears about the product and whether she first learns about it only through non-buyers, or not. We write $Z(t) = Z^0(t) \cup \ldots \cup Z^{t-1}(t)$, where we use the superscript to time stamp the moment when a non-buyer first hears about the product. That is, $Z^\tau(t)$, where $0 \leq \tau \leq t-1$, are those agents who heard about the product at time τ and have not yet made a purchase by time t. We also use the subscript to indicate whether an agent has first learned about the product via only non-buyers or not. So, for each $0 \leq \tau \leq t-1$, we write $Z^\tau(t) = Z_0^\tau(t) \cup Z_{>0}^\tau(t)$.[30] Similarly we can partition the set of buyers according to the time they first hear about the product: $B(t) = B^0(t) \cup \ldots \cup B^{t-1}(t) \cup B^t(t)$, for which

$$B^\tau(t) = \begin{cases} Z^\tau(t) \setminus Z^\tau(u(t), t), & \text{for } 0 \leq \tau \leq t-1 \\ B_Y(t), & \text{for } \tau = t. \end{cases} \tag{39}$$

To complete the dynamics, we note that $Z(t+1) = Z^0(t+1) \cup \ldots \cup Z^t(t+1)$, and

$$Z^\tau(t+1) = \begin{cases} Z^\tau(u(t), t), & \text{for } 0 \leq \tau \leq t-1 \\ \bar{B}_Y(t), & \text{for } \tau = t. \end{cases} \tag{40}$$

The augmented valuation of an agent $i \in Z^\tau(t+1)$ may increase by time $t+1$ as some of her friends may buy the product at time t and let her know about it. More precisely, $\theta_i^a(t+1) = \theta_i^a(t) + \alpha d_i^{WOM}(B(t))$. Therefore, in order to find the update rule for $z(\theta^a, t+1)$ we need to determine how many friends an agent $i \in Z^\tau(t+1)$ interacts with among the new buyers $B(t)$. The distribution of interactions varies depending on the time an agent first gets informed and whether or not a buyer was involved in informing her. To see this heterogeneity, note that the likelihood of the engagement of an agent $i \in Z^\tau(t+1)$, is highest for agents in $B_0^{\tau+1}(t)$ and lowest for buyer at time t who were informed before $\tau - 1$. This is because agents in $B_0^{\tau+1}(t)$ were first informed via new non-buyers at time τ and agent i was one of them. This increases the posterior probability of them engaging in WOM. On the other hand, agents informed before $\tau - 1$ engaged in WOM once when they first heard about the product but did not get engaged in WOM with i since she just heard about the product at time τ, lowering the posterior probability of them engaging in WOM when they buy the product.

[30] Similarly, for any $S(\cdot) \subseteq I$ we use the notation $S_0(\cdot)$ to represent those in $S(\cdot)$ who were first informed only via non-buyers and $S_{>0}(\cdot)$ to denote the rest.

Using the Bayes update rule and some manipulation we can show that number of buyers in $B(t)$ that agent $i \in Z^{\tau}_{>0}(t+1)$ may hear from has a Poisson distribution with mean[31]

$$\lambda^{\tau}_{>0}(t) = \lambda p_B \left((1-p_{\bar{B}})b(t) + p_{\bar{B}}(b^{\tau-1}(t) + b^{\tau}(t) + b^{\tau+1}(t)) + \frac{e^{-\lambda p_{\bar{B}} \bar{b}_Y(\tau)}}{1-e^{-\lambda p_{\bar{B}} \bar{b}_Y(\tau)}} p_{\bar{B}} b_0^{\tau+1}(t) \right). \tag{41}$$

Using this, the update rule for $z^{\tau}_{>0}(\theta^a, t+1)$ is

$$z^{\tau}_{>0}(\theta^a, t+1) = \sum_{d=0}^{\infty} e^{-\lambda^{\tau}_{>0}(t)} \frac{(\lambda^{\tau}_{>0}(t))^d}{d!} z^{\tau}_{>0}(\min(\theta^a - \alpha d, u(t)), t), \tag{42}$$

for $0 \le \tau \le t-1$, and

$$z^{t}_{>0}(\theta^a, t+1) = \sum_{d=0}^{\infty} e^{-\lambda^{t}_{>0}(t)} \frac{(\lambda^{t}_{>0}(t))^d}{d!} y_{>0}(\min(\theta^a - \alpha d, u(t)), t), \tag{43}$$

for $\tau = t$. The update rule for $z^{\tau}_0(\theta^a, t+1)$ is even more involved. Agents in $Z^{\tau}_0(t+1)$ were informed via non-buyers in $\bar{B}_Y(\tau-1)$. Therefore, the number of WOM engagement they have with those in $\bar{B}_Y(\tau-1)$ that buy the product at time t also depends on the number of their engagement they have already had with others in $\bar{B}_Y(\tau-1)$ that have made a purchase before t. Therefore, we need to break down $z^{\tau}_0(\theta^a, t+1)$ further. We write $z^{\tau}_0(\theta^a, t+1) = \sum_{d=0}^{\infty} z^{\tau}_0(\theta^a, d, t+1)$, where $z^{\tau}_0(\theta^a, d, t+1)$ denotes the mass of non-buyers at time $t+1$, first informed via non-buyers in $\bar{B}_Y(\tau-1)$ at time τ, who has engaged with d friends in $\bar{B}_Y(\tau-1)$ that have bought the product sometime between τ and t, that is $d_i^{WOM}(\cup_{t'=\tau}^{t} B^{\tau-1}(t')) = d$. We can find the update rule for $z^{\tau}_0(\theta^a, d, t+1)$ as[31]

$$\begin{aligned} z^{\tau}_0(\theta^a, d, t+1) =& (1 - e^{-\lambda p_{\bar{B}}(\bar{b}_Y(\tau-1) - p_B \sum_{t'=\tau}^{t} b^{\tau-1}(t'))}(1-p_{\bar{B}})^d) \times \\ & \sum_{k=0}^{d} \frac{e^{-\lambda p_B b^{\tau-1}(t)} \frac{(\lambda p_B b^{\tau-1}(t))^{d-k}}{(d-k)!}}{1 - e^{-\lambda p_{\bar{B}}(\bar{b}_Y(\tau-1) - p_B \sum_{t'=\tau}^{t-1} b^{\tau-1}(t'))}(1-p_{\bar{B}})^k} \times \\ & \sum_{d'=0}^{\infty} e^{-\lambda^{\tau}_0(t)} \frac{(\lambda^{\tau}_0(t))^{d'}}{d'!} z^{\tau}_0(\min(\theta^a - \alpha(d+d'-k), u(t)), k, t), \end{aligned} \tag{44}$$

[31] To improve the readability, the details are moved to the appendix.

for $0 \leq \tau \leq t-1$, and

$$z_0^t(\theta^a, d, t+1) = (1 - e^{-\lambda p_{\bar{B}}(\bar{b}_Y(t-1) - p_B b^{t-1}(t))}(1-p_{\bar{B}})^d) \times \frac{e^{-\lambda p_B b^{t-1}(t)} \frac{(\lambda p_B b^{t-1}(t))^d}{d!}}{1 - e^{-\lambda p_{\bar{B}} \bar{b}_Y(t-1)}} \sum_{d'=0}^{\infty} e^{-\lambda_0^t(t)} \frac{(\lambda_0^t(t))^{d'}}{d'!} y_0(\min(\theta^a - \alpha(d+d'), u(t)), t), \tag{45}$$

where

$$\lambda_0^\tau(t) = \lambda_{>0}^\tau(t) - \lambda p_B b^{\tau-1}(t). \tag{46}$$

Despite the above complex dynamics, the rationale for persistence of the zero price drops is rather simple. The first observation is that unless the externality is very strong, the optimal price path will jump above α infinitely often. As we will see, this property is very similar to Proposition 2. We then use the above dynamics to show that raising the price above α will bring in some non-buyers which will inform a nonzero measure subset of low-valuation agents (with valuations below p_1) given $p_{\bar{B}} > 0$. Due to network externalities, these low-valuation agents may eventually buy the product if many of their friends do so, elevating their augmented valuations above p_1. The next observation is to show that this cannot vanish the set of low-valuation agents. In fact by recursively using (44), we can show that[31]

$$z_0^\tau(p_1, t) \geq (1-p_B) e^{-\lambda(p_B + p_{\bar{B}})} y_0(p_1, \tau), \tag{47}$$

for $t > \tau$, that is, a nonzero fraction of those low-valuation agents that get informed about the product at time τ will never buy the product unless we drop the price to zero. Therefore, the same intuition of dropping the price to reach out new parts of the network via these low-valuation agents still holds in the presence of network externalities.

Proposition 5. *Consider the same setup as in Theorem 1 with $0 < p_{\bar{B}} < p_B$, and assume that the network externality effect $0 < \alpha < 1$ satisfies*

$$\min_{u \in \mathcal{U}} \frac{\alpha(1 - \beta\lambda p_B(1 - F(u-\alpha))(1 - \hat{q}_B^0))}{u(1 - F(u-\alpha))} < 1 - \beta\lambda p_B(1 - q_B^0), \tag{48}$$

where $q_B^0 = q(x_0, p_B, 0; 0)$ and $\hat{q}_B^0 = q(x_0, p_B + p_{\bar{B}} - p_B p_{\bar{B}}, 0; 0)$. Then, the optimal pricing policy should drop the price to zero infinitely often.

Proof. See the appendix. ∎

How small the externality effect α needs to be to satisfy the condition in (48)? To get an idea, let us consider the case of uniform valuations $\theta \sim Unif[0,1]$ and assume $\frac{1+\alpha}{2} \in \mathcal{U}$ to use it as a sub-minimizer in (48). After some simple algebra and noting that $q_B^0 \leq \hat{q}_B^0$, the condition reduces to $\alpha < \frac{1-\beta\lambda p_B(1-\hat{q}_B^0)}{1+\beta\lambda p_B(1-\hat{q}_B^0)}$. We can now state the following corollary.

Corollary 1. *Suppose that $\theta \sim Unif[0,1]$ and that $\frac{1+\alpha}{2} \in \mathcal{U}$. If the network externality effect satisifes*

$$\alpha < \frac{1-\beta\lambda p_B(1-\hat{q}_B^0)}{1+\beta\lambda p_B(1-\hat{q}_B^0)}, \tag{49}$$

where $\hat{q}_B^0 = q(x_0, p_B + p_{\bar{B}} - p_B p_{\bar{B}}, 0; 0)$, then the optimal pricing policy should drop the price to zero infinitely often.

In fact, the network externality needs to be very strong to fail (49). This bound is decreasing with β and $\lambda p_B(1-\hat{q}_B^0)$. Also, $\lambda p_B(1-\hat{q}_B^0) \leq \lambda p_B(1-q_B^0) < 1$ according to Proposition 1 and as we explained in Example 1, $\lambda p_B(1-q_B^0)$ is decreasing with λp_B for $\lambda p_B \geq 1$. With this background, let us look at a couple of examples. If $\lambda p_B \geq 2$, that is each buyer passes the information to at least two friends in average, then for α to fail the condition we should have $\alpha > 0.42$. [32] We can improve this bound if we also have some information about β. For example, if we also know that $\beta < 0.5$, then to fail we should have $\alpha > 0.66$. These are extremely strong externalities, as $\alpha > 0.42$ ($\alpha > 0.66$) suggests that everyone with three (two) friends among the buyers will buy the product even if the price is at its maximum ($u(t) = 1$).

4.4 Consumer Inattention to Price Changes

The core model of Section 2 assumes that consumers have unlimited ability of tracking down the price changes. An agent who knows about the product, is available to make a purchase as soon as the price falls below her valuation. With limited attention to price changes, consumers may check for a better deal every now and then. This section aims to verify the persistence of the frequent zero-price drops when consumers are partially attentive to price changes.

To model this, we assume that each informed non-buyer checks for price changes with an attentiveness probability $0 < p_A \leq 1$ at each subsequent time instant. This results in a geometrical distribution for the next time this agent becomes available to make a purchase.

The extension of the model in this case is quite straightforward. Instead of the whole

[32] See Example 1 and Footnotes 23-24 on how to derive this.

set $Z(t)$, only a fraction p_A of them will be available to make a purchase. Therefore, the size of the buyers among the previously informed agents at time t in (2) changes to

$$b_Z(t) = p_A \sum_{j=r+1}^{m} z_j(t), \tag{50}$$

assuming a price $u(t) = p_r$. This consequently changes the update rule for the distribution of valuations among informed non-buyers in (51):

$$z_j(t+1) = \begin{cases} z_j(t) + (F(p_j) - F(p_{j-1}))y(t), & 1 \leq j \leq r \\ (1-p_A)z_j(t), & \text{otherwise} \end{cases} \tag{51}$$

implying that $(1 - p_A)$ fraction of agents who would make a purchase if they had noticed the new price $u(t) = p_r$ miss the opportunity to buy the product. The total size of the new buyers in (52) becomes

$$b(t) = (1 - F(p_r))y(t) + p_A \sum_{j=r+1}^{m} z_j(t), \tag{52}$$

and the dynamics of the spread is given by (6) as before. The rationale for dropping the price to zero to reach out to high-valuation agents via low-valuation agents is preserved in this case, except that the bridge to the otherwise untouchable profitable component is taken via a fraction p_A of the low-valuation agents that would notice the free-offer. The other point worth mentioning here is that the immediate loss in the profit by dropping the price to zero may have a higher margin in this case; some high-valuation agents may be still around at the time of the drop even if the firm has already used low prices, due to the consumers' limited attention to price changes. We formally present this result in the next proposition.

Proposition 6. *Consider the same setup as in Theorem 1 with the exception that consumers pay limited attention to price changes. That is, upon learning about the product, each non-buyers checks back for price changes with some probability $0 < p_A \leq 1$ at each subsequent time instant. Then, the optimal pricing policy $u^D(\cdot)$ will drop the price to zero infinitely often.*

Proof. See the appendix. ∎

5 Dynamic Pricing for a Nondurable Product

As we saw in the previous sections, the WOM nature of the information diffusion is a key deriver for dropping the price to zero. If agents (users) are not involved in spreading the information about the product, the firm will not have any incentive to drop the price to zero. In fact, it is easy to show that for the case of full information, in which everybody is directly informed by the firm, the optimal pricing policy is monotone (decreasing) exploiting those who are willing to pay more first and then gradually lowering the price. The aim of this section is to show that beside the WOM nature of the information diffusion, these drops are also caused by the durability of the product.

For a nondurable product, every agent $i \in X(t)$ can buy the product if the offered price is below her valuation. The size of the buyers at time t is $(1 - F(u(t)))x(t)$, and therefore the accumulated discounted profit of the firm over an infinite time horizon is given by

$$\Pi^{ND}(u(\cdot)) = \sum_{t=0}^{\infty} \beta^t u(t)(1 - F(u(t)))x(t). \tag{53}$$

Firm's objective is to find the optimal pricing policy $u^{ND}(\cdot)$ that maximizes the above profit. An informed agent can buy a nondurable product as many times as the offered price is below her valuation, while she may buy a durable product only once. Therefore, in order to keep the dynamics of the spread the same for both cases, we focus on the case where $p_B = 1$ and $p_{\bar{B}} = 0$. In this setting, despite the type of the product, an agent informs her friends about the product as soon as she buys it.

Denote by u^* the price level maximizing the immediate profit, that is, $u^* = \text{argmax}_{u \in \mathcal{U}}\, u(1 - F(u))$. If there are two such price levels in $\mathcal{U}$, denote the smaller one with u^*. A useful observation is that $u^{ND}(t) \leq u^*$ for all $t \geq 0$. Otherwise, lowering the price to u^* would increase both the immediate profit and the size of the informed population at future times. Next theorem presents a steady state fixed-price property for the optimal pricing policy of a nondurable product.

Theorem 2. *Given the optimal pricing policy $u^{ND}(\cdot)$ for a nondurable product, there exists a finite time T after which the price is set to the fixed level u^* maximizing the immediate profit, that is $u^{ND}(t) = u^*$ for $t \geq T$.*

Proof. Denote by q^{ND} the asymptotic size of the informed population under the optimal policy $u^{ND}(\cdot)$, i.e., $q^{ND} = q(x_0, 1, 0; u^{ND}(\cdot))$. We claim that when the size of the informed population gets large enough, then no price other than u^* can be used by the optimal policy. In particular, we show that if $x^{ND}(t) > \gamma q^{ND}$, where

$$\gamma = \max_{\{u \in \mathcal{U} | u < u^*\}} \frac{\beta u^*(1 - F(u^*))}{u^*(1 - F(u^*)) - (1 - \beta)u(1 - F(u))}, \tag{54}$$

then $u^{ND}(t) = u^*$. Clearly $\gamma < 1$ since $u(1 - F(u)) < u^*(1 - F(u^*))$ for every $u < u^*$ in $\mathcal{U}$.

In order to prove the above claim, we again use contradiction and assume there is some time t_0 at which $x^{ND}(t_0) > \gamma q^{ND}$, but $u^{ND}(t_0) \neq u^*$, and we try to reach contradiction by constructing a new policy with a higher profit. We construct the new policy $\tilde{u}(\cdot)$, by shifting $u^{ND}(\cdot)$ one step ahead for $t > t_0$, changing the price to u^* at t_0, and keeping the policy unchanged for $t < t_0$. More specifically, we have

$$\tilde{u}(t) = \begin{cases} u^{ND}(t), & t < t_0 \\ u^*, & t = t_0 \\ u^{ND}(t-1), & t > t_0. \end{cases} \tag{55}$$

The key observation here is to note that defining $\tilde{u}$ in this way, any agent who is informed about the product under the optimal policy $u^{ND}(\cdot)$ will also be informed under the new policy $\tilde{u}(\cdot)$ with at most one step delay. This assures $X^{ND}(t-1) \subseteq \tilde{X}(t)$ for $t > t_0$, implying that $x^{ND}(t-1) \leq \tilde{x}(t)$ for $t > t_0$. Using this, we can lowerbound the accumulated discounted profit under the new policy $\tilde{u}(\cdot)$ from time t_0 on by the immediate profit under this policy, plus the accumulated discounted profit under the optimal policy $u^{ND}(\cdot)$ from time t_0 on discounted by β to account for the one step delay. This can be written as

$$\begin{aligned} \Pi^{ND}_{\geq t_0}(\tilde{u}(\cdot)) &= \sum_{t=t_0}^{\infty} \beta^t \tilde{u}(t)(1 - F(\tilde{u}(t)))\tilde{x}(t) \\ &\geq \beta^{t_0} u^*(1 - F(u^*))x^{ND}(t_0) + \beta \Pi^{ND}_{\geq t_0}(u^{ND}(\cdot)), \end{aligned} \tag{56}$$

where $\Pi^{ND}_{\geq t_0}(u^{ND}(\cdot))$ is the accumulated discounted profit under the optimal policy $u^{ND}(\cdot)$ from time t_0 on.[33]

The profit of the firm for $t < t_0$ is the same under both policies. Therefore, in order to prove that the new policy $\tilde{u}(\cdot)$ results in a higher profit than $u^{ND}(\cdot)$, we need to show that $\Pi^{ND}_{\geq t_0}(\tilde{u}(\cdot)) > \Pi^{ND}_{\geq t_0}(u^{ND}(\cdot))$. Applying (56), it thus suffices to show that

$$(1 - \beta)\Pi^{ND}_{\geq t_0}(u^{ND}(\cdot)) < \beta^{t_0} u^*(1 - F(u^*))x^{ND}(t_0). \tag{57}$$

[33] See Appendix A for more details on how to obtain (56).

We can find an upperbound for $\Pi^{ND}_{\geq t_0}(u^{ND}(\cdot))$ as[34]

$$\Pi^{ND}_{\geq t_0}(u^{ND}(\cdot)) \leq \beta^{t_0} u^{ND}(t_0)(1 - F(u^{ND}(t_0)))x^{ND}(t_0) + \frac{\beta^{t_0+1}}{1-\beta} u^*(1 - F(u^*))q^{ND}. \tag{58}$$

Using the above upperbound along with the fact that $x^{ND}(t_0) > \gamma q^{ND}$, where γ is defined in (54), and after some simplifications, we can easily show that (57) holds. This completes the proof. ■

Although the above theorem assures that there will be no free-offering of the product after some finite time in the nondurable case, it is still possible that the firm drops the price to zero during the early stages in order to expand its network, as shown in the next proposition.

Proposition 7. *Consider the optimal pricing policy $u^{ND}(\cdot)$ for a nondurable product and assume that $\beta > \frac{1-F(u^*)}{u^F(\lambda^*-1)}$, where $\lambda^* = (1 - F(u^*))\lambda$ and*

$$u^F = \frac{\min_{p_1 \leq u \leq u^*} u(1 - F(u))}{\max_{p_1 \leq u \leq u^*} \frac{u(1-F(u))}{F(u)}}. \tag{59}$$

Then, there exists $x^c > 0$ such that for $x^{ND}(t) < x^c$:
i) $u^{ND}(t)u^{ND}(t + 1) = 0$,
ii) $x^{ND}(t + 1) > \lambda^* x^{ND}(t)$.

Proof. See the appendix. ■

Part i) of the above proposition implies that, as long as the size of the informed population is below a certain threshold, there are no successive nonzero price levels in the optimal policy. This means that the firm should initially offer the product for free at least half of the time in order to expand its network. Part ii) presents a lowerbound on the effectiveness of these free offers. Using this, it is easy to see that for $\lambda^* > 1$, these drops can result in an exponential growth of the informed population.

6 Conclusions

In this paper, we analyzed optimal dynamic pricing in social networks from the information diffusion point of view. We developed a tractable, yet rich, model for information diffusion

[34] See Appendix A for more details on how to obtain (58) and to use it in proving (57).

via word of mouth, where an agent can only get informed about a product through a friend who already knows about the product. Both buyers and non-buyers may contribute to the information diffusion except that buyers are more likely to engage in the spread. Firm can hence use dynamic prices as a tool to control the endogenous information diffusion process. Word of mouth is the only means by which many apps spread among smartphone users. Using this model, we showed that the optimal pricing policy for a durable product with zero or negligible marginal cost, such as many smartphone applications, should drop the price to zero infinitely often. The rationale for this behavior is that by dropping the price to zero and selling the product to agents with low valuations of the product and getting them more engaged in the spread, firm can reach out a new part of the network that would remain untouched otherwise. By timing the drop properly, firm can make sure that the marginal growth in future profit by exploiting this new part of the network prevails the loss in the immediate profit caused by dropping the price to zero. We also showed that although the optimal policy drops the price to zero infinitely often, the price trajectory cannot get trapped in a vicinity of zero meaning that it jumps away from this vicinity infinitely often.

We also examined the validity of our results in face of strategic forward-looking agents, homophily-based engagement in word of mouth, network externalities, and consumer inattention to price changes, by generalizing our base model to these cases. Finally, we showed that beside the word of mouth nature of the information diffusion, this behavior is also rooted in the durability of product being offered. For a nondurable product, although the firm may initially make some free offers to expand its network, after a while it will set the price at a fixed level which extracts the maximum profit from the already informed population. When the network gets large, the loss in the immediate profit by dropping the price in favor of a higher spread would become too large to compare with the marginal gain in future resulted from the excess expansion of the network.

A Proofs

Proof of Proposition 1. Let $\lambda_B = \lambda p_B$. We can use (13) to write x_0 as a function of q

$$x_0 = 1 - (1-q)e^{\lambda_B q}. \tag{60}$$

The existence of a solution $q \in [0,1]$ for any $0 < x_0 \leq 1$ now follows from the continuity of x_0 in q and that $x_0(q=0) = 0$ and $x_0(q=1) = 1$. Taking derivatives from the above equation, we obtain

$$\frac{dx_0}{dq} = (1 - \lambda_B(1-q))e^{\lambda_B q}, \tag{61}$$

$$\frac{d^2x_0}{dq^2} = \lambda_B(2 - \lambda_B(1-q))e^{\lambda_B q}. \tag{62}$$

It follows from (61) that x_0 attains its minimum at $q^* = 1 - \frac{1}{\lambda_B}$ and is strictly increasing (decreasing) for $q \geq q^*$ ($q \leq q^*$). It also follows from (62) that x_0 is convex for $q \geq q^*$. Next, we show that for $q \in [0,1]$ the constraint $0 < x_0 \leq 1$ implies $q > q^*$. This is automatically satisfied for the case where $\lambda_B < 1$ since $q^* < 0$. For $\lambda_B \geq 1$, we have $q^* \geq 0$. However, $x_0(q)$ is decreasing for $0 \leq q \leq q^*$ resulting in $x_0(q) \leq x_0(q=0) = 0$. This shows that also in this case we should have $q > q^*$.

Now, the uniqueness of the solution in $[0,1]$ for $0 < x_0 \leq 1$ follows from the fact that x_0 is strictly increasing for $q \geq q^*$. Also, since x_0 is strictly increasing and convex for $q \geq q^*$, thus q is strictly increasing, but is concave in x_0. ■

Extension of Theorem 1 to a durable product with nonzero marginal cost. Denote the marginal cost with $c > 0$ and let $p_s \in \mathcal{U}$ be some given price level. We claim that if the gap between c and p_s is small enough, then the optimal price trajectory will drop the price to a level below or equal to p_s infinitely often. This specially implies the validity of Theorem 1 for a nonzero but negligible marginal cost. In particular we claim that if

$$c - p_s < \frac{\beta\lambda p_B(1-p_{\bar{B}})(1-\hat{q}_B)e^{-\lambda p_B(1-p_{\bar{B}})\hat{q}_B}(u_s^* - p_s)(1-F(u_s^*))}{1 + \beta\lambda p_B(1-p_{\bar{B}})(1-\hat{q}_B)e^{-\lambda p_B(1-p_{\bar{B}})\hat{q}_B}(1-F(u_s^*))}, \tag{63}$$

where $\hat{q}_B = q(x_0, p_B + p_{\bar{B}} - p_B p_{\bar{B}}, 0; 0)$ and $(u_s^* - p_s)(1-F(u_s^*)) = \max_{u\in\mathcal{U}}(u-p_s)(1-F(u))$, then the optimal policy $u^D(\cdot)$ drops the price to a level below or equal to p_s infinitely often.

The proof follows the same line as of the proof of Theorem 1, so we here only highlight the differences. Defining p_r as the smallest price level which is used by firm infinitely often and assuming $p_r > p_s$ we try to reach contradiction using (63). Let $u(t) \geq p_r$ for $t \geq T$ and t_k denote the k-th drop of the price to p_r after T. To extend the results to a nonzero marginal cost, we need to slightly modify $\tilde{u}(\cdot)$. Let $\tilde{u}(\cdot)$ have the same value as $u^D(\cdot)$ at all times except t_k+1 and t_k+2, for some k that we specify later. We choose $\tilde{u}(t_k+1) = p_s$ and $\tilde{u}(t_k+2) = u_s^*$. Similar to (16), we can lowerbound the size of the freshly informed agents at time t_k+2 as

$$\begin{aligned}\tilde{y}(t_k+2) &\geq (1-q^D)(1-e^{-\lambda p_B(\tilde{b}_Y(t_k+1)+(1-p_{\bar{B}})\tilde{b}_Z(t_k+1))}) \\ &\geq (1-q^D)(1-e^{-\lambda p_B(1-p_{\bar{B}})(\tilde{b}_Y(t_k+1)+\tilde{b}_Z(t_k+1))}).\end{aligned} \tag{64}$$

The main difference with the proof of Theorem 1 is that a drop may acquire some cost if $p_s < c$. The cost of the drop is $(c-p_s)(\tilde{b}_Y(t_k+1)+\tilde{b}_Z(t_k+1))$. How much profit can we make from $\tilde{y}(t_k+2)$? Using the simple fact that $\frac{1-e^{-\zeta w}}{w}$ is decreasing with w for $w \geq 0$ and $\zeta > 0$ (and so is $(1-w)\frac{1-e^{-\zeta w}}{w}$ for $0 < w < 1$) and that $q^D \leq \hat{q}_B$, we can use (64) to obtain

$$\begin{aligned}\frac{\tilde{y}(t_k+2)}{\tilde{b}_Y(t_k+1)+\tilde{b}_Z(t_k+1)} &\geq (1-q^D)\frac{1-e^{-\lambda p_B(1-p_{\bar{B}})q^D}}{q^D} \\ &\geq (1-\hat{q}_B)\frac{1-e^{-\lambda p_B(1-p_{\bar{B}})\hat{q}_B}}{\hat{q}_B} \\ &\geq \lambda p_B(1-p_{\bar{B}})(1-\hat{q}_B)e^{-\lambda p_B(1-p_{\bar{B}})\hat{q}_B},\end{aligned} \tag{65}$$

yielding

$$\tilde{y}(t_k+2) \geq \lambda p_B(1-p_{\bar{B}})(1-\hat{q}_B)e^{-\lambda p_B(1-p_{\bar{B}})\hat{q}_B}(\tilde{b}_Y(t_k+1)+\tilde{b}_Z(t_k+1)). \tag{66}$$

Therefore, the profit from the choice $\tilde{u}(t_k+1) = p_s$ and $\tilde{u}(t_k+2) = u_s^*$ can be lowerbounded by

$$\Pi^D_{>t_k}(\tilde{u}(\cdot)) \geq \beta^{t_k+1}((u_s^*-c)(1-F(u_s^*))\beta\lambda p_B(1-p_{\bar{B}})(1-\hat{q}_B)e^{-\lambda p_B(1-p_{\bar{B}})\hat{q}_B}-(c-p_s))(\tilde{b}_Y(t_k+1)+\tilde{b}_Z(t_k+1)). \tag{67}$$

Using (63), we can easily see that

$$((u_s^*-c)(1-F(u_s^*))\beta\lambda p_B(1-p_{\bar{B}})(1-\hat{q}_B)e^{-\lambda p_B(1-p_{\bar{B}})\hat{q}_B}-(c-p_s)) > 0, \tag{68}$$

i.e., it is a positive constant. Also, $(\tilde{b}_Y(t_k+1)+\tilde{b}_Z(t_k+1)) > \sum_{j=s}^{r} z_j^D(T) + (F(p_r) - F(p_s))y^D(T) > 0$. To have the new strategy yield a higher profit it suffices to choose k such that

$$(p_m - c)(q^D - x^D(t_k)) < \beta((u_s^* - c)(1 - F(u_s^*))\beta\lambda p_B(1 - p_{\bar{B}})(1 - \hat{q}_B)e^{-\lambda p_B(1-p_{\bar{B}})\hat{q}_B} - (c - p_s))\times \\ (\sum_{j=s}^{r} z_j^D(T) + (F(p_r) - F(p_s))y^D(T)), \tag{69}$$

which can be satisfied noting that $x^D(t) \to q^D$ as $t \to \infty$. ■

Proof of Proposition 2. In order to prove the proposition, we show that if the optimal policy $u^D(\cdot)$ gets stuck between 0 and p_1 after a finite time, then we should have $p_1(1 - F(p_1)) \geq c$. Suppose that there is $T \geq 0$ such that $u^D(t) \in \{0, p_1\}$ for all $t \geq T$. Denote with t_k the k-th drop of the price to zero after T under $u^D(\cdot)$. Note that there are infinitely many such drops according to Theorem 1. We first try to find an upperbound for the accumulated profit of the firm under $u^D(\cdot)$ after t_k. Consider a new policy $\hat{u}(\cdot)$ which has the same values as $u^D(\cdot)$ before t_k but is zero afterwards,

$$\hat{u}(t) = \begin{cases} u^D(t), & t \leq t_k \\ 0, & t > t_k \end{cases} \tag{70}$$

that is a free product after t_k. Assume also that in this alternative scenario, informed agents spread the word with increased probability $\hat{p}_B = p_B + p_{\bar{B}} - p_B p_{\bar{B}}$. Our first claim is that the size of the informed population $x^D(t)$ for $t \geq t_k$ is upperbounded by $\hat{x}(t)$, that is $x^D(t) \leq \hat{x}(t)$ for $t \geq t_k$. Recursively using (6) from t_k to t we get

$$1 - x^D(t+1) = (1 - x^D(t_k))e^{-\lambda b(t_k, t)}, \tag{71}$$

for $t \geq t_k$, where

$$\begin{aligned} b(t_k, t) &= \sum_{\tau=t_k}^{t} p_B b_Y(\tau) + p_{\bar{B}}\bar{b}_Y(\tau) + p_B(1 - p_{\bar{B}})b_Z(\tau) \\ &\leq \hat{p}_B(x^D(t) - x^D(t_k - 1)). \end{aligned} \tag{72}$$

On the other hand, $1 - \hat{x}(t+1) = (1 - \hat{x}(t_k))e^{-\lambda\hat{p}_B(\hat{x}(t) - \hat{x}(t_k - 1))}$. Using a simple induction and the fact that $\hat{x}(t_k - 1) = x^D(t_k - 1)$ and $\hat{x}(t_k) = x^D(t_k)$, we can show that $x^D(t) \leq \hat{x}(t)$ for $t \geq t_k$.

Our next claim is that the profit made by the firm under $u^D(\cdot)$ after t_k is upperbounded by

$$\Pi^D_{>t_k}(u^D(\cdot)) \leq p_1(1 - F(p_1)) \sum_{t=t_k+1}^{\infty} \beta^t \hat{y}(t). \tag{73}$$

To prove this, it apparently suffices to show that $\sum_{t=t_k+1}^{\infty} \beta^t y^D(t) \leq \sum_{t=t_k+1}^{\infty} \beta^t \hat{y}(t)$. This is immediate from the previous result, if we use $y(t+1) = x(t+1) - x(t)$ to rewrite these summations in terms of $x^D(t)$ and $\hat{x}(t)$, with positive coefficients $(\beta^t - \beta^{t+1})$. The dynamics of $\hat{y}(t)$ for $t > t_k$ has the simple form of

$$\hat{y}(t+1) = (1 - \hat{x}(t))(1 - e^{-\lambda \hat{p}_B \hat{y}(t)}), \tag{74}$$

where $\hat{y}(t_k+1) = y^D(t_k+1)$ and $\hat{x}(t_k+1) = x^D(t_k+1)$. Using this, we can easily obtain

$$\hat{y}(t+1) \leq \lambda \hat{p}_B(1 - x^D(t_k+1))\hat{y}(t), \tag{75}$$

for all $t > t_k$. Using this along with (73), the profit of the firm under $u^D(\cdot)$ after t_k can be upperbounded by

$$\Pi^D_{>t_k}(u^D(\cdot)) \leq \frac{\beta^{t_k+1} p_1(1 - F(p_1)) y^D(t_k+1)}{1 - \beta\lambda\hat{p}_B(1 - x^D(t_k+1))}. \tag{76}$$

Next, we compare this profit with that of a modified policy $\tilde{u}(\cdot)$ having the same value as $u^D(\cdot)$ for $t \leq t_k$ and a fixed value $u^c > p_1$ for $t > t_k$. The profit of the firm for policy $\tilde{u}(\cdot)$ after t_k is given by

$$\Pi^D_{>t_k}(\tilde{u}(\cdot)) = u^c(1 - F(u^c)) \sum_{t=t_k+1}^{\infty} \beta^t \tilde{y}(t). \tag{77}$$

The dynamics of $\tilde{y}(t)$ for $t > t_k$ is given by

$$\tilde{y}(t+1) = (1 - \tilde{x}(t))(1 - e^{-\lambda^c \tilde{y}(t)}), \tag{78}$$

where $\lambda^c = \lambda(p_B(1 - F(u^c)) + p_{\bar{B}}F(u^c))$, $\tilde{y}(t_k+1) = y^D(t_k+1)$ and $\tilde{x}(t_k+1) = x^D(t_k+1)$. We next aim to lowerbound $\tilde{y}(t)$ for $t > t_k$ with a geometric sequence, in order to find a closed form lowerbound for the profit of the firm for $\tilde{u}(\cdot)$ after t_k given by (77). This can be easily done by rewriting (78) as

$$\begin{aligned}
\tilde{y}(t+1) &= (1 - \tilde{x}(t+1))(e^{\lambda^c \tilde{y}(t)} - 1) \\
&\geq \lambda^c(1 - \tilde{x}(t+1))\tilde{y}(t) \\
&> \lambda^c(1 - \tilde{q})\tilde{y}(t),
\end{aligned} \tag{79}$$

where $\tilde{q}$ is the asymptotic size of the informed population under $\tilde{u}(\cdot)$. The profit of the firm after t_k for $\tilde{u}(\cdot)$ can thus be lowerbounded by

$$\Pi^D_{>t_k}(\tilde{u}(\cdot)) > \frac{\beta^{t_k+1}u^c(1-F(u^c))y^D(t_k+1)}{1-\beta\lambda^c(1-\tilde{q})}. \tag{80}$$

The asymptotic size $\tilde{q}$ satisfies $1-\tilde{q} = (1-x^D(t_k+1))e^{-\lambda^c(\tilde{q}-x^D(t_k))} > (1-x^D(t_k+1))e^{-\lambda p_B(\tilde{q}-x^D(t_k))}$. Shifting $t \to \infty$ in (71) and noting that the optimal policy drops the price to zero infinitely often, we can show that $1-q^D \leq (1-x^D(t_k+1))e^{-\lambda p_B(q^D-x^D(t_k))}$. From this we can get $\tilde{q} < q^D$, and hence

$$\Pi^D_{>t_k}(\tilde{u}(\cdot)) > \frac{\beta^{t_k+1}u^c(1-F(u^c))y^D(t_k+1)}{1-\beta\lambda^c(1-q^D)}. \tag{81}$$

Noting that $u^D(\cdot)$ is the optimal policy, we should have $\Pi^D_{>t_k}(u^D(\cdot)) \geq \Pi^D_{>t_k}(\tilde{u}(\cdot))$ for all choices of u^c. This, along with (76) and (81) yields

$$\frac{p_1(1-F(p_1))}{1-\beta\lambda\hat{p}_B(1-x^D(t_k+1))} > \frac{u^c(1-F(u^c))}{1-\beta\lambda^c(1-q^D)}, \tag{82}$$

for any choices of k and u^c. Shifting $k \to \infty$, we can obtain

$$\frac{p_1(1-F(p_1))}{1-\beta\lambda\hat{p}_B(1-q^D)} > \frac{u^c(1-F(u^c))}{1-\beta\lambda^c(1-q^D)}. \tag{83}$$

To get rid of q^D in above, we note that $\frac{1-\beta\lambda\hat{p}_B(1-q^D)}{1-\beta\lambda^c(1-q^D)}$ is increasing with q^D since $\lambda\hat{p}_B > \lambda^c$. Thus, (83) yields

$$p_1(1-F(p_1)) > \frac{u^c(1-F(u^c))(1-\beta\lambda\hat{p}_B(1-q_B^0))}{1-\beta\lambda^c(1-q_B^0)}, \tag{84}$$

for every $u^c \in \mathcal{U}\setminus\{p_1\}$, since $q^D \geq q_B^0$ according to Theorem 1. This completes the proof. ■

Proof of Proposition 3. Similar to the proof of Theorem 1, let $p_r \in \mathcal{U}$ denote the smallest price level in $\mathcal{U}$ which holds infinitely often for the optimal pricing policy $u^D(\cdot)$. Since any price level below p_r is used only finitely by $u^D(\cdot)$, there exists $T \geq 0$ such that $u^D(t) \geq p_r$ for all $t \geq T$. It clearly suffices to show that $p_r = 0$, that is $r = 0$. Therefore, to prove the proposition, we assume $r \geq 1$ and try to reach contradiction by showing that the firm can increase its profit by deviating from the price path $u^D(\cdot)$.

Let $Y^D_r(T) \subset Y^D(T)$ denote those freshly informed agents at time T whose valuations are below p_r, i.e. $Y^D_r(T) = \{i \in Y^D(T) | 0 \leq \theta_i < p_r\}$, with a size of $y^D_r(T) = F(p_r)y^D(T)$.

None of the agents in $\cup_{j=1}^{r} Z_j^D(T) \cup Y_r^D(T)$ will ever buy the product under the pricing policy $u^D(\cdot)$, where $\cup_{j=1}^{r} Z_j^D(T)$ is the set of those previously informed agents at time T whose valuations are below p_r. Now, consider the set of agents that will remain uniformed under $u^D(\cdot)$. The size of this set is clearly $1 - q^D$, where q^D is the asymptotic size of the informed population under $u^D(\cdot)$, i.e., $q^D = q(x_0, p_B, p_{\bar{B}}; u^D(\cdot))$. Define Δ_r as the subset of these agents who have at least a friend in $\cup_{j=1}^{r} Z_j^D(T) \cup Y_r^D(T)$. As in Theorem 1, the number of friends of an uninformed agent among $\cup_{j=1}^{r} Z_j^D(T) \cup Y_r^D(T)$ has a Poisson distribution with mean $\lambda(1 - p_{\bar{B}})(F(p_r)y^D(T) + \sum_{j=1}^{r} z_j^D(T))$. Therefore, by zeroing the price at any time $t > T$ we can reach out a subset of Δ_r with the size of

$$\delta_r = (1 - q^D)(1 - e^{-\lambda p_B (1 - p_{\bar{B}})(F(p_r) y^D(T) + \sum_{j=1}^{r} z_j^D(T))}), \tag{85}$$

from which clearly $\delta_r > 0$. Now, the idea is to show that after a while there is so little profit left to be made in future under $u^D(\cdot)$ that it is profitable to zero the price to reach out these agents in Δ_r. Let t_k, $k = 1, 2, \ldots$, denote the k-th price drop to p_r after time T under the optimal policy $u^D(\cdot)$. If a strategic agent $i \in X^D(t_k)$ does not buy the product at this time, neither will she buy it in future, as this is the cheapest offer she will ever get. This means that agents in $X^D(t_k)$ do not contribute to the set of buyers $B^D(t)$ for $t > t_k$. Therefore, the size of the buyers from time $t_k + 1$ can be upperbounded by

$$\sum_{t=t_k+1}^{\infty} b^D(t) \leq q^D - x^D(t_k). \tag{86}$$

Thus, the contribution of the buyers to the firm's profit after t_k can be upperbounded by

$$\Pi_{>t_k}^D(u^D(\cdot)) \leq \beta^{t_k+1} p_m (q^D - x^D(t_k)). \tag{87}$$

Now, consider the following deviation $\tilde{u}(\cdot)$ from the price path $u^D(\cdot)$ only at times $t_k + 1$ and $t_k + 2$. Let $\tilde{u}(t_k + 1) = 0$ and $\tilde{u}(t_k + 2) = \hat{u}$, where $\hat{u}(1 - F(\frac{\hat{u}}{1-\beta_c})) = \max_{u \in \mathcal{U}} u(1 - F(\frac{u}{1-\beta_c}))$. Given that $\beta_c < 1 - p_1$, $\hat{u}$ is always nonzero. Note that a subset of agents in Δ_r with size δ_r as in (85) are among the freshly informed agents $\tilde{Y}(t_k + 2)$. Those of them with valuations $\theta > \frac{\hat{u}}{1-\beta_c}$ buy the product since $\theta - \hat{u} > \beta_c \theta$, meaning that the payoff of a purchase today is higher than even a free purchase tomorrow. The discounted profit made from these newly informed agents in Δ_r at time $t_k + 2$ is hence lowerbounded by $\beta^{t_k+2} \hat{u}(1 - F(\frac{\hat{u}}{1-\beta_c}))\delta_r$. Considering that $x^D(t) \to q^D$ as $t \to \infty$, we can choose k large enough such that

$$q^D - x^D(t_k) < \frac{\beta \delta_r}{p_m} \hat{u}(1 - F(\frac{\hat{u}}{1 - \beta_c})), \tag{88}$$

in which case the profit resulted from $\tilde{u}(\cdot)$ will be clearly higher than that from $u^D(\cdot)$. This contradicts the optimality of $u^D(\cdot)$, hence rejecting the initial assumption of $p_r \neq 0$, which completes the proof. ■

Proof of Proposition 4. First, we show that under the local homophily condition, there will always be a nonzero mass of low-valued informed non-buyers around, with valuations below $\delta_1 = \min(\frac{\delta}{2}, p_1)$. More precisely, we claim that

$$M_Y(\delta_1, t)y(t) + M_Z(\delta_1, t)z(t) > 0. \tag{89}$$

First note that $M_Y(\delta_1, 0)y(0) = F(\delta_1)x_0 > 0$. As long as the offered price $u(t) \geq p_1$, our claim is apparent. If the price becomes zero ($u(t) = 0$), then for an uninformed agent with $\theta' \leq \delta_1$, we have

$$\lambda(\theta', t) \geq \lambda \underline{p}(1 - \bar{p})(M_Y(\delta_1, t)y(t) + M_Z(\delta_1, t)z(t)). \tag{90}$$

Denoting the RHS with $\underline{\lambda}(t) > 0$ along with (34) and (37), we get

$$M_Y(\delta_1, t+1)y(t+1) \geq e^{-\lambda q}(1 - e^{-\underline{\lambda}(t)})F(\delta_1)(1 - x_0) > 0, \tag{91}$$

completing the proof of our claim of having a nonzero measure of low-valued informed non-buyers around at all times. Using a similar approach we can show that if there is a mass w of informed non-buyers with valuations in the range of $[\underline{\theta}, \bar{\theta}]$ at time t, where $|\underline{\theta} - \bar{\theta}| \leq \frac{\delta}{2}$, then by dropping the price to zero we can reach out a set of agents with valuations in the range of $[\bar{\theta}, \bar{\theta} + \frac{\delta}{2}]$ whose size $\tilde{w}$ satisfies

$$\tilde{w} \geq e^{-\lambda q^0}(1 - e^{-\lambda \underline{p}(1-\bar{p})w})(F(\bar{\theta} + \frac{\delta}{2}) - F(\bar{\theta}))(1 - x_0) > 0, \tag{92}$$

where we have also used the fact that the asymptotic size of the informed population is upperbounded by q^0 as defined in Section 3. With these results at hand, we now use an approach similar to that of Theorem 1 to complete the proof.

Let $p_r \in \mathcal{U}$ denote the smallest price level in $\mathcal{U}$ which holds infinitely often for the optimal pricing policy $u^D(\cdot)$. Since any price level below p_r is used only finitely by $u^D(\cdot)$, there exists $T \geq 0$ such that $u^D(t) \geq p_r$ for all $t \geq T$. It clearly suffices to show that $p_r = 0$, that is $r = 0$. Therefore, to prove the proposition, we assume $r \geq 1$ and try to reach contradiction by showing that the firm can increase its profit by deviating from the price path $u^D(\cdot)$.

Let t_k, $k = 1, 2, \ldots$, denote the k-th price drop to p_r after time T under the optimal policy $u^D(\cdot)$. The contribution of the buyers to the firm's profit after t_k can be upperbounded by

$$\Pi^D_{>t_k}(u^D(\cdot)) \le \beta^{t_k+1} p_m (q^D - x^D(t_k)). \tag{93}$$

Denote with $W^D(T)$ the set of non-buyer agents at time T whose valuations are below δ_1, and its size with $w^D(T)$. Agents in $W^D(T)$ will never buy the product under the optimal policy $u^D(\cdot)$, and hence $w^D(t) \ge w^D(T)$ for all $t \ge T$. Now, consider a new policy $\tilde{u}(\cdot)$ having the same values as $u^D(\cdot)$ at all times except $t_k + 1 \le t \le t_k + s + 1$, where s is the smallest integer for which $\delta_1 + \frac{\delta(s-1)}{2} \ge p_1$. Let $\tilde{u}(t_k+1) = \ldots = \tilde{u}(t_k+s) = 0$ and $\tilde{u}(t_k+s+1) = p_1$. From above, we already know that $w^D(T) > 0$. Dropping the price to zero s times in a row, we can reach a nonzero measure population with valuations in $[\delta_1 + \frac{\delta(s-1)}{2}, \delta_1 + \frac{\delta s}{2}]$. By recursively using (92), we can obtain a lower bound $\underline{w} > 0$ for the size of this set. [35] The discounted profit made from these newly informed agents at time $t_k + s + 1$ is lowerbounded by $\beta^{t_k+s+1} p_1 \underline{w}$. Considering that $x^D(t) \to q^D$ as $t \to \infty$, we can choose k large enough such that

$$q^D - x^D(t_k) < \frac{\beta^s p_1 \underline{w}}{p_m}, \tag{94}$$

in which case the profit resulted from $\tilde{u}(\cdot)$ will be clearly higher than that from $u^D(\cdot)$. This contradicts the optimality of $u^D(\cdot)$, hence rejecting the initial assumption of $p_r \ne 0$, which completes the proof. ■

Distribution of WOM Engagement among buyers and non-buyers in Section 4.3. To find the update rule for $Z(\theta^a, t+1)$, we need to figure out how many friends an agent $i \in Z^\tau(t+1)$ interacts with among the new buyers $B(t)$.

$$\begin{aligned}
\text{Prob}(d_i^{WOM}(B_0^{\tau+1}(t)) = d | i \in Z^\tau(t+1)) &\sim \text{Poiss}(\lambda p_B (1 - p_{\bar{B}} + \frac{p_{\bar{B}}}{1 - e^{-\lambda p_{\bar{B}} \bar{b}_Y(\tau)}}) b_0^{\tau+1}(t)) \\
\text{Prob}(d_i^{WOM}(B_{>0}^{\tau+1}(t)) = d | i \in Z^\tau(t+1)) &\sim \text{Poiss}(\lambda p_B b_{>0}^{\tau+1}(t)) \\
\text{Prob}(d_i^{WOM}(B^{\tau}(t)) = d | i \in Z^\tau(t+1)) &\sim \text{Poiss}(\lambda p_B b^{\tau}(t)) \\
\text{Prob}(d_i^{WOM}(B^{\tau-1}(t)) = d | i \in Z_{>0}^\tau(t+1)) &\sim \text{Poiss}(\lambda p_B b^{\tau-1}(t)),
\end{aligned} \tag{95}$$

where $b^{-1}(0) = b^{t+1}(t) = 0$. The number of the interactions between $i \in Z_0^\tau(t+1)$ and new buyers that were informed at $\tau - 1$, also depends on the number of interactions between i

[35]Note that this lower bound is in terms of $w^D(T)$ and is independent of t_k.

and those previous buyers that were informed at the same time $\tau-1$. More precisely,

$$\text{Prob}\left(d_i^{WOM}(B^{\tau-1}(t))=d|i\in Z_0^{\tau}(t+1)\wedge d_i^{WOM}(\cup_{t'=\tau}^{t-1}B^{\tau-1}(t'))=k\right)= \frac{e^{-\lambda p_B b^{\tau-1}(t)}\frac{(\lambda p_B b^{\tau-1}(t))^d}{d!}(1-e^{-\lambda p_{\bar{B}}(\bar{b}_Y(\tau-1)-p_B\sum_{t'=\tau}^{t}b^{\tau-1}(t'))}(1-p_{\bar{B}})^{d+k})}{1-e^{-\lambda p_{\bar{B}}(\bar{b}_Y(\tau-1)-p_B\sum_{t'=\tau}^{t-1}b^{\tau-1}(t'))}(1-p_{\bar{B}})^{k}}. \tag{96}$$

Finally, for $\tau'\notin\{\tau-1,\tau,\tau+1\}$, we have

$$\text{Prob}(d_i^{WOM}(B^{\tau'}(t))=d|i\in Z^{\tau}(t+1))\sim\text{Poiss}(\lambda p_B(1-p_{\bar{B}})b^{\tau'}(t)). \tag{97}$$

■

Proof of Proposition 5. We first prove the following lemma:

Lemma 1. *Consider a price function $u(\cdot)$ with finite number of drops to zero, that is, there exists $T\geq 0$ such that $u(t)\neq 0$ for $t\geq T$. Then, for any $t>\tau\geq T$ we have*

$$z_0^{\tau}(p_1,t)\geq(1-p_B)e^{-\lambda(p_B+p_{\bar{B}})}y_0(p_1,\tau). \tag{98}$$

Using only the term corresponding to $d'=0$ for $d=0$ in (45), we can obtain

$$z_0^{\tau}(p_1,0,\tau+1)\geq\frac{1-e^{-\lambda p_{\bar{B}}(\bar{b}_Y(\tau-1)-p_B b^{\tau-1}(\tau))}}{1-e^{-\lambda p_{\bar{B}}\bar{b}_Y(\tau-1)}}e^{-\lambda^{\tau}_{>0}(\tau)}y_0(p_1,\tau). \tag{99}$$

Using this along with recursive use of (44) for $d=0$ using only the term corresponding to $d'=0$, we get

$$z_0^{\tau}(p_1,0,t+1)\geq\frac{1-e^{-\lambda p_{\bar{B}}(\bar{b}_Y(\tau-1)-p_B\sum_{t'=\tau}^{t}b^{\tau-1}(t'))}}{1-e^{-\lambda p_{\bar{B}}\bar{b}_Y(\tau-1)}}e^{-\sum_{t'=\tau}^{t}\lambda^{\tau}_{>0}(t')}y_0(p_1,\tau), \tag{100}$$

for $t\geq\tau$. We now note that

$$\sum_{t'=\tau}^{t}b^{\tau-1}(t')<\bar{b}_Y(\tau-1). \tag{101}$$

On the other hand, using (41) we can easily show that

$$\lambda^{\tau}_{>0}(t')\leq\lambda p_B b(t')+\lambda p_B p_{\bar{B}}\frac{e^{-\lambda p_{\bar{B}}\bar{b}_Y(\tau)}}{1-e^{-\lambda p_{\bar{B}}\bar{b}_Y(\tau)}}y_0^{\tau+1}(t')\leq\lambda p_B b(t')+p_B\frac{y_0^{\tau+1}(t')}{\bar{b}_Y(\tau)}, \tag{102}$$

which yields

$$\sum_{t'=\tau}^{t} \lambda_{>0}^{\tau}(t') \leq \lambda p_B q + p_B \frac{y_0(\tau+1)}{\bar{b}_Y(\tau)} \leq \lambda p_B + \lambda p_{\bar{B}}(1-x(\tau)), \tag{103}$$

where we have used the fact that $q \leq 1$, and that $y_0(\tau+1) \leq \lambda p_{\bar{B}} \bar{b}_Y(\tau)(1-x(\tau))$ (this is quite straightforward using (38)). Using (100),(101), and (103), we get

$$\begin{aligned} z_0^{\tau}(p_1, 0, t+1) &\geq \frac{1-e^{-\lambda p_{\bar{B}}(1-p_B)\bar{b}_Y(\tau-1)}}{1-e^{-\lambda p_{\bar{B}}\bar{b}_Y(\tau-1)}} e^{-\lambda(p_B+p_{\bar{B}}(1-x(\tau)))} y_0(p_1,\tau) \\ &\geq \frac{e^{\lambda p_{\bar{B}}(1-p_B)\bar{b}_Y(\tau-1)}-1}{1-e^{-\lambda p_{\bar{B}}\bar{b}_Y(\tau-1)}} e^{-\lambda(p_B+p_{\bar{B}}(1-x(\tau)+\bar{b}_Y(\tau-1)))} y_0(p_1,\tau) \\ &\geq (1-p_B)e^{-\lambda(p_B+p_{\bar{B}})} y_0(p_1,\tau), \end{aligned} \tag{104}$$

which completes the proof of the lemma. We now get back to the proof of the proposition. Following the same line as of the proof of Theorem 1, we assume a finite number of drops to zero under the optimal policy $u^D(\cdot)$ and try to reach contradiction by constructing a new policy with a profit higher than that of $u^D(\cdot)$.

Let t_0 be the last drop of the price to zero (we study the case that there is no drop to zero at all later). So, $u^D(t_0) = 0$ and $u(t) > 0$ for $t > t_0$. Our first claim is that, under assumption (48), there exists $t_1 > t_0$ such that $u^D(t_1) > \alpha$., that is the price path cannot stay below α after the last drop to zero. The proof is very similar to that of Proposition 2. We assume $u^D(t) \leq \alpha$ for $t > t_0$ and try to reach contradiction by finding a more profitable price function. Let $u^{\alpha} = \max\{u \in \mathcal{U} | u \leq \alpha\}$. Then, $u^D(t) \leq \alpha$ for $\alpha > t_0$ indeed implies $u^D(t) = u^{\alpha}$ for $\alpha > t_0$. Note that at this price after a drop to zero every body that hears about the product has at least a friend among the buyers and hence her augmented valuation is at least α. Therefore, there is no need to use a price less than u^{α} even if we want to keep it below α. Using almost the same approach as in Proposition 2, we can find an upperbound similar to (76) for the profit of the firm from $t' > t_0$ afterwards as

$$\Pi_{>t'}^{D}(u^D(\cdot)) \leq \frac{\beta^{t'+1} u^{\alpha} y^D(t'+1)}{1-\beta\lambda p_B(1-x^D(t'+1))}. \tag{105}$$

Next, we compare this profit with that of a modified policy $\tilde{u}(\cdot)$ having the same value as $u^D(\cdot)$ for $t \leq t'$ and a fixed value u^c for $t > t'$. Considering only the part of the spread by buyers and the effect of only one of the buyer friends in augmented valuation, we can lowerbound the profit of the firm after t', similar to (81), by

$$\Pi^D_{>t'}(\tilde{u}(\cdot)) > \frac{\beta^{t'+1}u^c(1-F(u^c-\alpha))y^D(t'+1)}{1-\beta\lambda p_B(1-F(u^c-\alpha))(1-\tilde{q})}. \tag{106}$$

Noting that $u^D(\cdot)$ is the optimal policy, we should have $\Pi^D_{>t'}(u^D(\cdot)) \geq \Pi^D_{>t'}(\tilde{u}(\cdot))$ for all choices of u^c and all $t' > t_0$. Therefore,

$$\frac{u^\alpha}{1-\beta\lambda p_B(1-q^D)} \geq \frac{u^c(1-F(u^c-\alpha))}{1-\beta\lambda p_B(1-F(u^c-\alpha))(1-\tilde{q})}. \tag{107}$$

Noting that $q^D \geq q^0_B$ and $\tilde{q} \leq \hat{q}^0_B$, and that $u^\alpha \leq \alpha$ we should thus have

$$\frac{\alpha(1-\beta\lambda p_B(1-F(u^c-\alpha))(1-\hat{q}^0_B))}{u^c(1-F(u^c-\alpha))} \geq 1-\beta\lambda p_B(1-q^0_B), \tag{108}$$

for all $u^c \in \mathcal{U}$which contradicts (48). This proves our first claim, that is, there exists $t_1 > t_0$ such that $u^D(t_1) > \alpha$. Apparently, we can assume that t_1 is the earliest time after t_0 for which $u^D(t_1) > \alpha$. Thus, for $t_0 \leq t < t_1$, the offered price is below α and all the freshly informed agents in $y(t)$ buy the product. Let $u(t_1) = p_r > \alpha$. Consider $Y^D(p_r, t_1)$, the set of freshly informed agents at time t_1 whose augmented valuations are below p_r. We can lowerbound the size of this set using only the term corresponding to $d = 1$ in (38), as

$$y^D(p_r, t_1) \geq (1-x^D(t_1-1))e^{-\lambda p_B y^D(t_1-1)}(\lambda p_B y^D(t_1-1))F(p_r-\alpha), \tag{109}$$

which implies $y^D(p_r, t_1) > 0$ using the assumption $p_r > \alpha$. These agents will not buy the product but will inform friends with a lower probability $p_{\bar{B}} > 0$. Using (38) now yields

$$y^D_0(p_1, t_1+1) = (1-x^D(t_1))e^{-\lambda^D_B(t_1)}(1-e^{-p_{\bar{B}}\lambda \bar{b}^D_Y(t_1)})F(p_1). \tag{110}$$

Using this along with $\bar{b}^D_Y(t_1) \geq y^D(p_r, t_1) > 0$, we get $y^D_0(p_1, t_1+1) > 0$, a nonzero measure set of low-valued agents with no buyer friend yet (in case that $u^D(\cdot)$ does not drop the price to zero at all, we can choose $t_1 = 0$ for which we can show similar to the above that $y^D_0(p_1, t_1+1) > 0$). Due to network externalities, agents in this set may eventually buy the product if many friends do so, elevating their augmented valuations above p_1. However, Lemma 98 guarantees that a nonzero fraction of them will never buy the product, thus,

$$z^D_0(p_1, t) \geq (1-p_B)e^{-\lambda(p_B+p_{\bar{B}})}y^D_0(p_1, t_1+1), \tag{111}$$

for all $t > t_1$.

Now, consider a new policy $\tilde{u}(\cdot)$ having the same value as $u^D(\cdot)$ at all times except t and $t+1$, where we will specify $t > t_1$ later. Let $\tilde{u}(t) = 0$ and $\tilde{u}(t+1) = u^*$, where $u^*(1 - F(u^* - \alpha)) = \max_{u \in \mathcal{U}} u(1 - F(u - \alpha))$. Agents in $z_0^D(p_1, t)$ will buy the product and inform their friends with probability p_B, giving a lowerbound on the size of the freshly informed agents at time $t+1$ as

$$\begin{aligned}\tilde{y}(t+1) &\geq (1 - q^D)(1 - e^{-\lambda p_B (1-p_{\bar{B}}) z_0^D(p_1,t)}) \\ &\geq (1 - q^D)(1 - e^{-\lambda p_B (1-p_B)(1-p_{\bar{B}}) e^{-\lambda(p_B + p_{\bar{B}})} y_0^D(p_1, t_1+1)}),\end{aligned} \tag{112}$$

leading to a profit of at least $u^*(1 - F(u^* - \alpha)) * y(t+1)$ at time $t+1$. Using an approach similar to that used in the proof of Theorem 1, it is quite straightforward to show that by a proper choice of t, this new policy will yield a profit higher than that of $u^D(\cdot)$. Firm's profit from time t onward under the optimal policy can be upperbounded by

$$\begin{aligned}\Pi^D_{\geq t}(u^D(\cdot)) &= \sum_{\tau=t}^{\infty} \beta^\tau u^D(\tau) b^D(\tau) \\ &\leq \beta^t p_m \sum_{\tau=t}^{\infty} b^D(\tau).\end{aligned} \tag{113}$$

Noting that $\sum_{\tau=0}^{\infty} b^D(\tau) \leq q^D$, we can choose t large enough to ensure that

$$\sum_{\tau=t}^{\infty} b^D(\tau) < \frac{\beta u^*(1 - F(u^* - \alpha)) y(t+1)}{p_m}, \tag{114}$$

in which the profit from $\tilde{u}(\cdot)$ will be clearly higher than that of $u^D(\cdot)$. This contradicts the optimality of $u^D(\cdot)$, hence rejecting the initial assumption of finite drops to zero under the optimal policy. ■

Proof of Proposition 6. The proof is very similar to that of Theorem 1 so we only point out the necessary changes here. The first differnece is the size δ_r in (16). That is, the size of the agents that can only be reached via low-valuation agents by offering them the product for free. With only a fraction p_A of low-valuation agents noticing the free-offer, the size of this set becomes

$$\delta_r = (1 - q^D)(1 - e^{-\lambda p_A p_B (1-p_{\bar{B}})(F(p_r) y^D(T) + \sum_{j=1}^{r} z_j^D(T))}). \tag{115}$$

The other difference is the lowerbound on the size of the future buyers in (18) which becomes

$$\sum_{t=t_k+1}^{\infty} b^D(t) \leq q^D - x^D(t_{k-k'}) + (1-p_A)^{k'+1} x^D(t_{k-k'})$$
$$\leq (1+(1-p_A)^{k'+1}) q^D - x^D(t_{k-k'}), \tag{116}$$

for any $1 \leq k' \leq k$. The rough idea is that while bounding the size of the future buyers, we should keep in mind the extra mass of informed non-buyers that have been inattentive to the previous drops of the price to p_r. Using this, the contribution of the buyers to the firm's profit after t_k can be now upperbounded by

$$\Pi^D_{>t_k}(u^D(\cdot)) \leq \beta^{t_k+1} p_m((1+(1-p_A)^{k'+1}) q^D - x^D(t_{k-k'})), \tag{117}$$

for any $1 \leq k' \leq k$. To make the deviation $\tilde{u}(\cdot)$, as defined in the proof of Theorem 1, preferable over $u^D(\cdot)$ it suffices to have

$$(1+(1-p_A)^{k'+1}) q^D - x^D(t_{k-k'}) < \beta\delta_r \frac{u^*(1-F(u^*))}{p_m}. \tag{118}$$

Considering that $x^D(t) \to q^D$ as $t \to \infty$, and that $(1-p_A)^{k'+1} \to 0$ as $k' \to \infty$, we can choose k' and $k-k'$ large enough to satisfy this condition, hence completing the proof. ■

Proof of Theorem 2. Below, we provide more details on parts of the proof of Theorem 2.
i) *Proof of the lowerbound on $\Pi^{ND}_{\geq t_0}(\tilde{u}(\cdot))$ given by (56):* For $t > t_0$, we have $\tilde{x}(t) \geq x^{ND}(t-1)$ and $\tilde{u}(t) = u^{ND}(t-1)$. Also, $\tilde{x}(t_0) = x^{ND}(t_0)$ and $\tilde{u}(t_0) = u^*$. Therefore, we can write

$$\Pi^{ND}_{\geq t_0}(\tilde{u}(\cdot)) = \sum_{t=t_0}^{\infty} \beta^t \tilde{u}(t)(1-F(\tilde{u}(t)))\tilde{x}(t)$$
$$\geq \beta^{t_0}\tilde{u}(t_0)(1-F(\tilde{u}(t_0)))\tilde{x}(t_0) + \sum_{t=t_0+1}^{\infty} \beta^t u^{ND}(t-1)(1-F(u^{ND}(t-1)))x^{ND}(t-1)$$
$$= \beta^{t_0} u^*(1-F(u^*))x^{ND}(t_0) + \beta\Pi^{ND}_{\geq t_0}(u^{ND}(\cdot)). \tag{119}$$

ii) *Proof of the upperbound on $\Pi^{ND}_{\geq t_0}(u^{ND}(\cdot))$ given by (58):* Noting that $x^{ND}(t) \leq q^{ND}$, we have

$$\begin{aligned}\Pi^{ND}_{\geq t_0}(u^{ND}(\cdot)) &= \sum_{t=t_0}^{\infty} \beta^t u^{ND}(t)(1-F(u^{ND}(t)))x^{ND}(t) \\ &\leq \beta^{t_0} u^{ND}(t_0)(1-F(u^{ND}(t_0)))x^{ND}(t_0) + \sum_{t=t_0+1}^{\infty} \beta^t u^*(1-F(u^*))q^{ND} \\ &= \beta^{t_0} u^{ND}(t_0)(1-F(u^{ND}(t_0)))x^{ND}(t_0) + \frac{\beta^{t_0+1}}{1-\beta} u^*(1-F(u^*))q^{ND}. \qquad (120)\end{aligned}$$

iii) *Proof of the inequality given by* (57)*:* Applying the upperbound given by (58) and simple algebra, we can easily see that this inequality is satisfied if

$$(1-\beta)u^{ND}(t_0)(1-F(u^{ND}(t_0)))x^{ND}(t_0)+\beta u^*(1-F(u^*))q^{ND} < u^*(1-F(u^*))x^{ND}(t_0), \quad (121)$$

or equivalently

$$x^{ND}(t_0) > \frac{\beta u^*(1-F(u^*))}{u^*(1-F(u^*)) - (1-\beta)u^{ND}(t_0)(1-F(u^{ND}(t_0)))} q^{ND}, \qquad (122)$$

where $u^{ND}(t_0) < u^*$ (this follows from the assumption $u^{ND}(t_0) \neq u^*$ and that $u^{ND}(t_0) \leq u^*$). The above follows then from the fact that $x^{ND}(t_0) > \gamma q^{ND}$ and the definition of γ in (54). ■

Proof of Proposition 7. The proof is by induction. However, in order to use induction, we will need a more accurate but dirtier version of the proposition as follows.
Claim: Choose some λ_1 satisfying $\lambda^* < \lambda_1 < \frac{\lambda(\lambda^*-1)}{\lambda^*} + 1$. Note that RHS is greater than LHS since $\lambda^* < \lambda$. For any such λ_1, there exists $x^c > 0$ such that for $x^{ND}(t) < x^c$:
i) $u^{ND}(t)u^{ND}(t+1) = 0$,
ii) if $u^{ND}(t) = 0$ then $x^{ND}(t+1) > \lambda_1 x^{ND}(t)$. And, if $u^{ND}(t) \neq 0$ then $x^{ND}(t+1) > \lambda^* x^{ND}(t)$.
Note that in either case in ii) we have $x^{ND}(t+1) > \lambda^* x^{ND}(t)$ since $\lambda_1 > \lambda^*$.

Define

$$g_1(x) = 1-(1-x)e^{-\frac{\lambda^*(\lambda_1-1)}{\lambda_1}x} - \lambda^* x, \qquad (123)$$

$$g_2(x) = 1-(1-x)e^{-\frac{\lambda(\lambda^*-1)}{\lambda^*}x} - \lambda_1 x, \qquad (124)$$

$$g_3(x) = (1-x)(e^{-\frac{\lambda(\lambda_1-1)(1-F(u))}{\lambda_1}x} - e^{-\frac{\lambda(\lambda_1-1)}{\lambda_1}x}) - \frac{\lambda(\lambda^*-1)F(u)}{\lambda^*}x, \qquad (125)$$

where in $g_3(x)$, u is a nonzero price level in $\mathcal{U}$. We can easily verify that the derivatives of these functions at $x = 0$ are all positive. This implies that all the three functions are strictly increasing in a vicinity of $x = 0$. Using this along with the fact that $g_1(0) = g_2(0) = g_3(0) = 0$, we can conclude that there exists $\tilde{x}^c > 0$ such that $g_1(x) > 0, g_2(x) > 0,$ and $g_3(x) > 0$ for all $0 < x < \tilde{x}^c$. Also, define

$$h(y) = e^{-\lambda(1-F(u))y} - e^{-\lambda y}. \tag{126}$$

It is quite straightforward to show that h is strictly increasing for $0 < y < \frac{-\ln(1-F(u))}{\lambda F(u)}$, and that $\frac{1}{\lambda} < \frac{-\ln(1-F(u))}{\lambda F(u)}$ for $0 < F(u) < 1$. Thus, h is strictly increasing for $0 < y < \frac{1}{\lambda}$ for any nonzero price level $u \in \mathcal{U}$. We set $x^c = \min\{\tilde{x}^c, \frac{1}{\lambda}\}$ and prove the claim above for this choice of x^c with induction.

We start with the transition part of the induction. Assuming that the claim holds for $t-1$, we try to prove it for t. Suppose that $x^{ND}(t) < x^c$. First of all note that $x^{ND}(t-1) < x^c$ since $x^{ND}(t-1) < x^{ND}(t)$. We first tackle part ii). If $u^{ND}(t) \neq 0$, then $u^{ND}(t-1) = 0$ since, according to the assumption of induction, $u^{ND}(t-1)u^{ND}(t) = 0$. This implies $Z^{ND}(t) = \emptyset$, and hence $b^{ND}(t) = (1 - F(u^{ND}(t)))y^{ND}(t)$ from (52). Using (6), we have

$$x^{ND}(t+1) = 1 - (1 - x^{ND}(t))e^{-\lambda(1-F(u^{ND}(t)))y^{ND}(t)}. \tag{127}$$

Now, we try to lowerbound $y^{ND}(t)$ in terms of $x^{ND}(t)$. From the assumption of induction and that $u^{ND}(t-1) = 0$, we get $x^{ND}(t) > \lambda_1 x^{ND}(t-1)$, which along with $y^{ND}(t) = x^{ND}(t) - x^{ND}(t-1)$ yields

$$y^{ND}(t) > \frac{\lambda_1 - 1}{\lambda_1} x^{ND}(t). \tag{128}$$

This, along with (127) and that $g_1(x^{ND}(t)) > 0$ since $x^{ND}(t) < x^c$, we find

$$\begin{aligned} x^{ND}(t+1) &> 1 - (1 - x^{ND}(t))e^{-\frac{\lambda(1-F(u^{ND}(t)))(\lambda_1-1)}{\lambda_1}x^{ND}(t)} \\ &\geq 1 - (1 - x^{ND}(t))e^{-\frac{\lambda(1-F(u^*))(\lambda_1-1)}{\lambda_1}x^{ND}(t)} \\ &> \lambda^* x^{ND}(t), \end{aligned} \tag{129}$$

where we have also used the fact that for the optimal policy $u^{ND}(t) \leq u^*$. This proves that if $u^{ND}(t) \neq 0$ then $x^{ND}(t+1) > \lambda^* x^{ND}(t)$. If, on the other hand, $u^{ND}(t) = 0$, then $b^{ND}(t) \geq y^{ND}(t)$ from (52), and we can use (6) to obtain

$$x^{ND}(t+1) \geq 1 - (1 - x^{ND}(t))e^{-\lambda y^{ND}(t)}. \tag{130}$$

Using the assumption of induction, $x^{ND}(t) > \lambda^* x^{ND}(t-1)$, which along with $y^{ND}(t) = x^{ND}(t) - x^{ND}(t-1)$ yields

$$y^{ND}(t) > \frac{\lambda^* - 1}{\lambda^*} x^{ND}(t). \tag{131}$$

Using (130) and (131), and that $g_2(x^{ND}(t)) > 0$, we get

$$\begin{aligned} x^{ND}(t+1) &> 1 - (1 - x^{ND}(t)) e^{-\frac{\lambda(\lambda^*-1)}{\lambda^*} x^{ND}(t)} \\ &> \lambda_1 x^{ND}(t), \end{aligned} \tag{132}$$

which completes the proof of part ii) for t. Now, we get to the proof of part i). Assume that $u^{ND}(t) u^{ND}(t+1) \neq 0$, and construct a new policy $\tilde{u}(\cdot)$ that is obtained from $u^{ND}(\cdot)$ by only changing $u^{ND}(t)$ to 0. We claim that the new policy will result in a profit higher than that of $u^{ND}(\cdot)$. First of all, note that for all times $\tau \geq 0$, $X^{ND}(\tau) \subseteq \tilde{X}(\tau)$, thus $x^{ND}(\tau) \leq \tilde{x}(\tau)$. In particular, we are interested in calculating $\tilde{x}(t+1) - x^{ND}(t+1)$. From the assumption of induction, we should have $u^{ND}(t-1) u^{ND}(t) = 0$. Therefore, $u^{ND}(t-1) = 0$ since $u^{ND}(t) \neq 0$. This implied that $Z^{ND}(t) = \emptyset$. Hence, using (52) and (6) we get

$$\tilde{x}(t+1) - x^{ND}(t+1) = (1 - x^{ND}(t)) (e^{-\lambda(1 - F(u^{ND}(t))) y^{ND}(t)} - e^{-\lambda y^{ND}(t)}). \tag{133}$$

It follows from $u^{ND}(t-1) = 0$ and the assumption of induction that $x^{ND}(t) > \lambda_1 x^{ND}(t-1)$, which in turn implies

$$y^{ND}(t) > \frac{\lambda_1 - 1}{\lambda_1} x^{ND}(t). \tag{134}$$

Now, considering that $h(y)$ defined in (126) is strictly increasing for $0 < y < x^c$, (133) yields

$$\begin{aligned} \tilde{x}(t+1) - x^{ND}(t+1) &> (1 - x^{ND}(t)) (e^{-\frac{\lambda(\lambda_1 - 1)(1 - F(u^{ND}(t)))}{\lambda_1} x^{ND}(t)} - e^{-\frac{\lambda(\lambda_1-1)}{\lambda_1} x^{ND}(t)}) \\ &> \frac{\lambda(\lambda^* - 1) F(u^{ND}(t))}{\lambda^*} x^{ND}(t), \end{aligned} \tag{135}$$

where the last inequality comes from $g_3(x^{ND}(t)) > 0$.

In order to show that the new policy $\tilde{u}(\cdot)$ results in a higher profit, it suffices to show that

$$\begin{aligned} u^{ND}(t)(1 - F(u^{ND}(t))) x^{ND}(t) + \beta u^{ND}(t+1)(1 - F(u^{ND}(t+1))) x^{ND}(t+1) < \\ \beta u^{ND}(t+1)(1 - F(u^{ND}(t+1))) \tilde{x}(t+1), \end{aligned} \tag{136}$$

or equivalently,

$$u^{ND}(t)(1-F(u^{ND}(t)))x^{ND}(t) < \beta u^{ND}(t+1)(1-F(u^{ND}(t+1)))(\tilde{x}(t+1)-x^{ND}(t+1)). \quad (137)$$

Applying (135) and some simplifications, we can see that the above is satisfied if

$$\frac{u^{ND}(t)(1-F(u^{ND}(t)))}{F(u^{ND}(t))} < \beta u^{ND}(t+1)(1-F(u^{ND}(t+1)))\frac{\lambda(\lambda^*-1)}{\lambda^*}. \quad (138)$$

Therefore, it suffices to have

$$1 < \beta u^F \frac{\lambda(\lambda^*-1)}{\lambda^*}, \quad (139)$$

which holds if $\beta > \frac{1-F(u^*)}{u^F(\lambda^*-1)}$, noting the definition of u^F in (59). This shows that $\tilde{u}(\cdot)$ has a higher profit that $u^{ND}(\cdot)$ which contradicts its optimality. This completes the proof of part i).

The only thing which is left is to verify the base of the induction, that is to prove the claim for $t = 0$. For part ii), it is easy to see that all the relations (127)-(132) also hold for $t = 0$, noting that $y^{ND}(0) = x^{ND}(0)$ and $Z(0) = \emptyset$. Similar story holds for part i). ■

B Upperbound on p_1 Assuring Infinitely Many Jumps above p_1 in Example 1

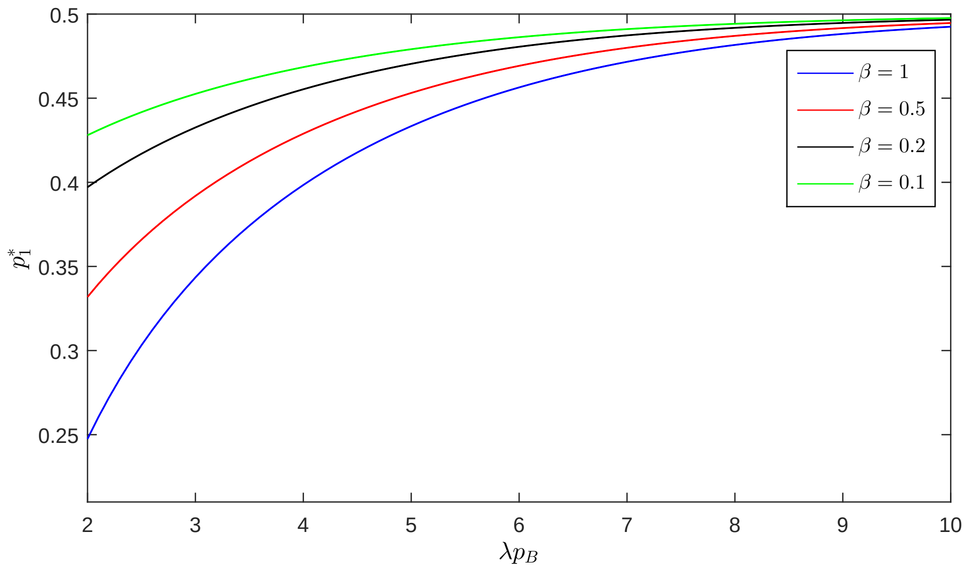

Figure 3: Values of p_1 that guarantee frequent jumps above p_1 for different values of β and λp_B, using $u = 0.5$ as a (sub)maximizer. Having $p_1 < p_1^*$ guarantees infinitely many jumps above p_1 under the optimal policy.

C More Evidence on Price Drops from App Market

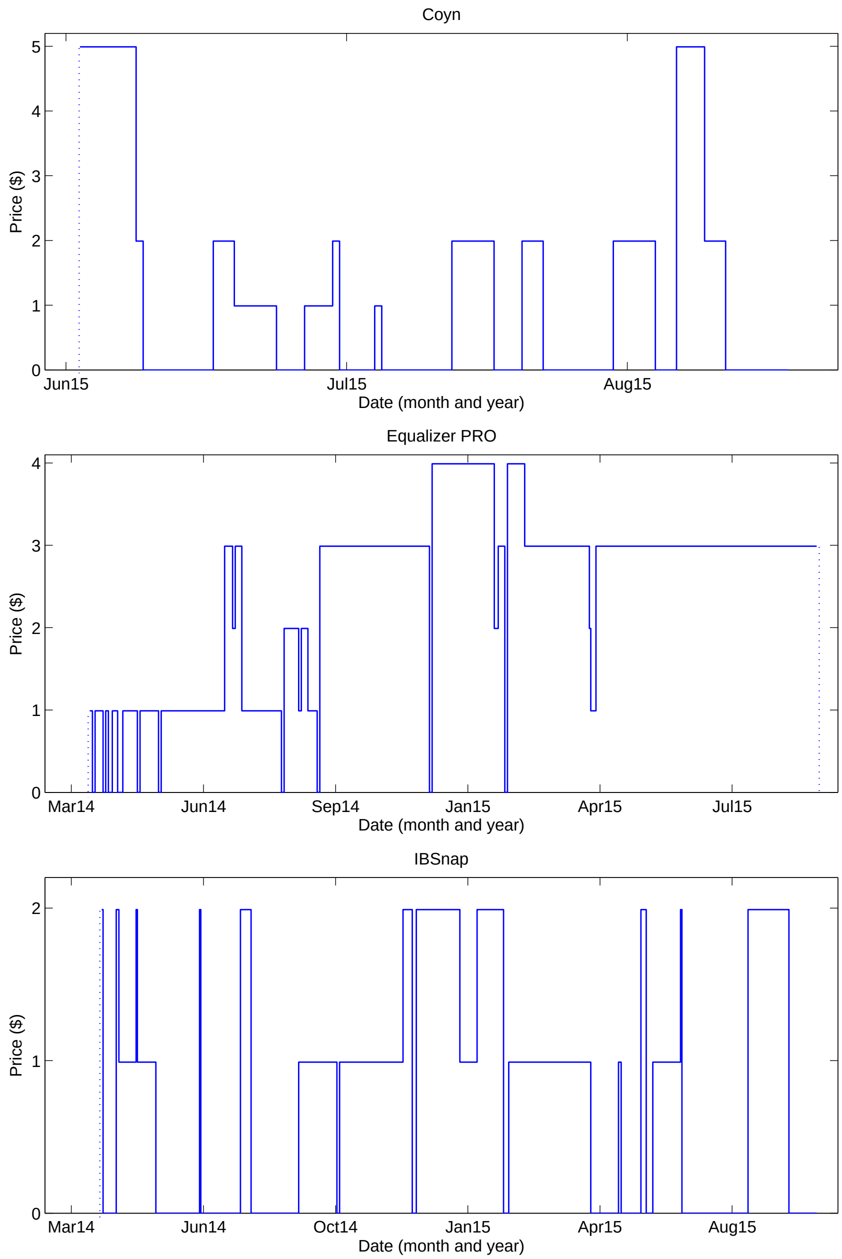

Figure 4: Price histories of three other apps (Coyn, Equalizer PRO, and IBSnap) from the time they were debuted to September 2015.

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