

A TWO-RESOURCE ALLOCATION PROBLEM SOLVABLE IN LINEAR TIME*

NIMROD MEGIDDO^{†‡} AND TETSUO ICHIMORI[§]

The allocation problem discussed here is as follows. The processing time of a task is a linear function $t-ax-by$ of amounts x, y of resources allocated to it, where t, a, b are positive constants. Given the amounts X and Y of available resources, we wish to allocate them among n independent tasks so as to minimize the maximal completion time. We solve this problem in linear time.

Introduction. The complexity of the linear programming problem is a central question in the interface between operations research and computer science. It is still an open question whether an $m \times n$ problem can be solved within a polynomial number $p(m, n)$ of arithmetic operations. Another direction of research is that of exploring the complexities of structured linear programming problems (see [M1, M2, M3]). It turns out that for many subproblems there exist algorithms which are more efficient than the simplex algorithm and investigating such algorithms may eventually lead to an alternative for the simplex algorithm in the general case.

In this paper we present a linear-time algorithm for a class of linear programming problems which may be described as follows.

Suppose we have n independent "tasks" denoted $1, 2, \dots, n$, whose processing times depend on the amounts of two resources allocated to them. More specifically, assume we have X units of the first resource and Y units of the second resource. Given the initial time t_i , if we allocate x_i units of the first resource and y_i units of the second resource to task i , then task i will process in $T_i = \max\{t_i - a_i x_i - b_i y_i, d_i\}$ time units ($t_i, a_i, b_i, d_i > 0$). Suppose we wish to minimize makespan [B], given all tasks start at time zero. Then our problem is to minimize $\text{Max}\{T_i\}$ subject to $\sum x_i = X$, $\sum y_i = Y$ and $x_i, y_i \geq 0$. It is easy to see that it is sufficient to minimize $\text{Max}\{t_i - a_i x_i - b_i y_i\}$. For some references on resource allocation consult [E], [GM] and [KIM1], [KIM2].

Recent results of Megiddo [M1], [M2] establish that whenever the number of variables (or, equivalently, the number of constraints) is fixed, a linear programming problem is solvable in linear time. However, the problem we discuss in this paper does not fall in this category. The methods we use are related to those used in [M1, M2]. Our problem is also related but not equivalent to the multiple-choice linear knapsack problem [IHTI], [Z1] recently revisited by Dyer [D3] and Zemel [Z2], the latter two stemming from the results independently obtained by Megiddo [M1], [M2] and Dyer [D1], [D2]. Thus, it turns out that these new methods lead to better algorithms than previous ones. Apparently, the "sorting" step can be avoided in these special linear programming problems and successive reductions in the size of the problem yield

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[†] Carnegie-Mellon University.

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[§] Hiroshima University.

better performance. In a certain sense these methods are “greedy.” The methods may also apply to linear programs arising in preemptive scheduling problems where linear programming is a fundamental tool.

Finally, we note that even though our linear-time argument follows from finding medians in linear time (which may not be practical), the method is very practical if instead of the exact median we pick a random element (or the median of a random sample of three).

Some formulation considerations. We solve the following problem

$$\begin{aligned} \text{minimize}_{x, y} \text{maximum}_{1 \leq i \leq n} \{c_i - a_i x_i - b_i y_i\} \quad \text{s.t.} \quad & \sum_{i=1}^n x_i = X, \quad \sum_{i=1}^n y_i = Y, \quad (\text{P1}) \\ & x_i, y_i \geq 0 \quad (i = 1, \dots, n), \end{aligned}$$

where $X, Y, a_i, b_i, c_i > 0$, $i = 1, \dots, n$. The problem can be formulated as a linear programming problem:

$$\begin{aligned} \text{minimize}_{x, y, t} \quad & t \quad \text{s.t.} \quad t + a_i x_i + b_i y_i \geq c_i \quad (i = 1, \dots, n), \quad (\text{P2}) \\ & \sum_{i=1}^n x_i = X, \quad \sum_{i=1}^n y_i = Y, \\ & x_i, y_i \geq 0 \quad (i = 1, \dots, n). \end{aligned}$$

We will concentrate on the dual problem:

$$\begin{aligned} \text{maximize}_{z, u, v} \quad & \sum_{i=1}^n c_i z_i + Xu + Yv \quad (\text{P3}) \\ & u + a_i z_i \leq 0 \quad (i = 1, \dots, n), \\ & v + b_i z_i \leq 0 \quad (i = 1, \dots, n), \\ & \sum_{i=1}^n z_i = 1, \quad z_i \geq 0 \quad (i = 1, \dots, n). \end{aligned}$$

Let us consider the latter as the problem of maximizing a function $f(u, v)$ of two variables, namely

$$f(u, v) = Xu + Yv + \text{Max} \left\{ \sum_{i=1}^n c_i z_i : \sum_{i=1}^n z_i = 1, 0 \leq z_i \leq \min \left(-\frac{u}{a_i}, -\frac{v}{b_i} \right), i = 1, \dots, n \right\}$$

(assuming $\text{Max } \emptyset = -\infty$). It is easy (and well known) to *evaluate* $f(u, v)$ in $O(n)$ time. This can be accomplished by searching (employing repeated median-finding) for the optimal value of the dual variable associated with the equality constraint. However, we will *maximize* $f(u, v)$ in $O(n)$ time. The procedure is based on repeated reduction in the size of the problem.

An overview of the algorithm. The function $f(u, v)$ is piecewise linear and concave. In order to understand the algorithm for maximizing f , we first discuss the piecewise linear structure of this function. The problem

$$\begin{aligned} \text{maximize}_z \quad & \sum_{i=1}^n c_i z_i \quad \text{s.t.} \quad \sum_{i=1}^n z_i = 1, \quad (\text{P4}) \\ & 0 \leq z_i \leq Z_i \quad (i = 1, \dots, n), \end{aligned}$$

is easy and its optimal solution can be described as follows. Assume, without loss of generality, that $c_1 \geq c_2 \geq \dots \geq c_n$ and $\sum_{i=1}^n Z_i \geq 1$. Then there is a unique k , $0 \leq k \leq n$ such that $\sum_{i=1}^k Z_i \leq 1 < \sum_{i=1}^{k+1} Z_i$ (assuming, for convenience, $Z_{n+1} = 2$). The solution is then $z = Z_i, i = 1, \dots, k, Z_{k+1} = 1 - \sum_{i=1}^k Z_i$, and $z_i = 0, i = k + 2, \dots, n$. It follows that $f(u, v)$ has the following description:

$$f(u, v) = \sum_{i=1}^k c_i \min\{-u/a_i, -v/b_i\} + c_{k+1} \left(1 - \sum_{i=1}^k \min\{-u/a_i, -v/b_i\} \right) + Xu + Yv$$

for u and v such that

$$\sum_{i=1}^k \min\{-u/a_i, -v/b_i\} \leq 1 < \sum_{i=1}^{k+1} \min\{-u/a_i, -v/b_i\}.$$

The domains of linearity of $f(u, v)$ are as shown in Figure 1. The lines forming these domains are as follows. First, there are the lines $u/a_i = v/b_i, i = 1, \dots, n$. Along any line $v = \alpha u$ ($\alpha > 0$) the function $f(u, v)$ breaks as we describe below. Let $LE = \{i : b_i \leq \alpha a_i\}$ and $G = \{i : b_i > \alpha a_i\}$. Thus, for $i \in LE$ (assuming $v = \alpha u$), $-u/a_i \leq -v/b_i$, and for $i \in G, -u/a_i > -v/b_i$. A critical value of u on our line must be such that for some k ($1 \leq k \leq n$) $-u \cdot \sum_{i=1}^k \min(1/a_i, \alpha/b_i) = 1$, i.e.,

$$u = - \left[\sum_{\substack{i=1 \\ i \in LE}}^k 1/a_i + \alpha \sum_{\substack{i=1 \\ i \in G}}^k 1/b_i \right]^{-1}.$$

However, given k , the curve $C_k = \{(u, v) : \sum_{i=1}^k \min(-u/a_i, -v/b_i) = 1\}$ (which has the form of a broken line) breaks only at points where $u/a_i = v/b_i$ for some $i, 1 \leq i \leq k$. This implies that the map of linearity domains of $f(u, v)$ is as shown in Figure 1. See also Figure 2 for the curves C_k .

In the subsequent sections we develop procedures for the following problems. Let (u^*, v^*) denote a maximum of $f(u, v)$. First, given a straight line $v = \alpha u$ we will recognize in $O(n)$ time whether $v^* < \alpha u^*, v^* = \alpha u^*$, or $v^* > \alpha u^*$. Secondly, given one

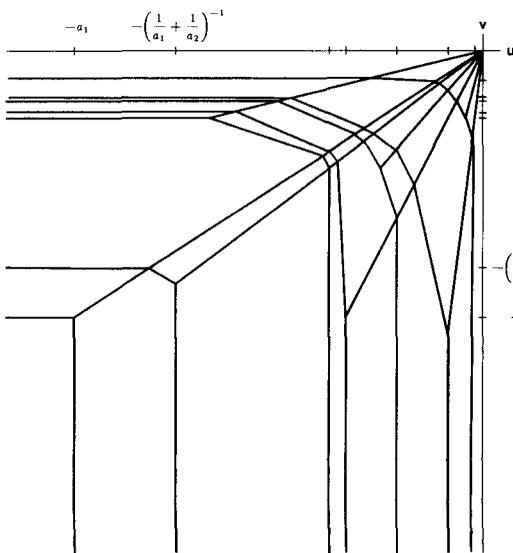


FIGURE 1

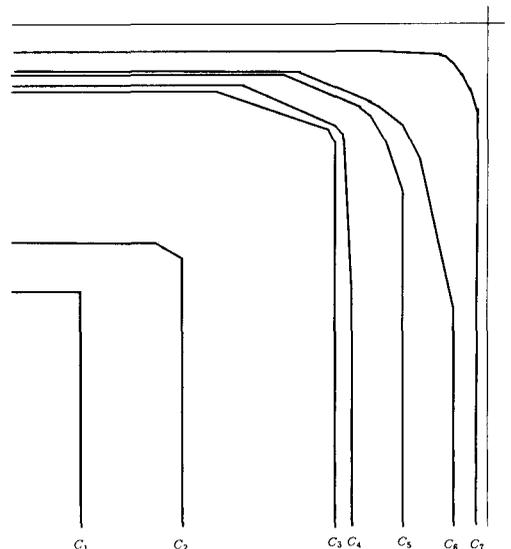


FIGURE 2

of the curves C_k ($1 \leq k \leq n$), we will recognize in $O(n)$ time the position of (u^*, v^*) relative to C_k , i.e., whether $\sum_{i=1}^k \min(-u^*/a_i, -v^*/b_i)$ is less than, equal to, or greater than 1.

The tests we have just defined enable us to reduce the problem in the following way. First, assume $c_1 \geq c_2 \geq \dots \geq c_n$. This assumption is only for the convenience of notation. We start by finding the median $c_{n/2}$ (assume, for convenience, n is even) and identifying the sets $\{c_1, \dots, c_{n/2}\}$, $\{c_{n/2+1}, \dots, c_n\}$. We now test the curve $C_{n/2}$. If we find that $\sum_{i=1}^{n/2} \min(-u^*/a_i, -v^*/b_i) \geq 1$ then the corresponding solution $z^* = (z_1^*, \dots, z_n^*)$ must satisfy $z_i^* = 0$, $i = n/2 + 1, \dots, n$. We thus reduce problem (P3) to the same one with $n/2$ replacing n . The case of an equality is particularly easy since then we actually obtain u^* and v^* . If $\sum_{i=1}^{n/2} \min(-u^*/a_i, -v^*/b_i) < 1$ then $z_i^* = \min(-u^*/a_i, -v^*/b_i)$, $i = 1, \dots, n/2$. In this case our problem is reduced to the following:

$$\begin{aligned} & \text{maximize} && \sum_{i=n/2+1}^n c_i z_i + Xu + Yv + \sum_{i=1}^{n/2} c_i \min(-u/a_i, -v/b_i) && \text{(P5)} \\ & \text{s.t.} && \sum_{i=n/2+1}^n z_i = 1 - \sum_{i=1}^{n/2} \min(-u/a_i, -v/b_i), \\ & && 0 \leq z_i \leq \min(-u/a_i, -v/b_i) \quad (i = n/2 + 1, \dots, n). \end{aligned}$$

This is obviously more complicated than problem (P3) with $n/2$ variables since both the function and sum of z_i 's contain a piecewise linear function of u and v . However, we also work on the front of this complication in the following way. We find the median α_m of the ratios b_i/a_i , $i = 1, \dots, n/2$. Next, we test the line $v = \alpha_m u$, i.e., determine whether $v^* < \alpha_m u^*$, $v^* = \alpha_m u^*$, or $v^* > \alpha_m u^*$. Assume, for example, $v^* \geq \alpha_m u^*$. In this case for every i such that $b_i/a_i \geq \alpha_m$ we have $-u^*/a_i \geq -v^*/b_i$. Denoting $T = \{1 \leq i \leq n/2 : b_i/a_i \geq \alpha_m\}$ we have to solve an easier problem:

$$\begin{aligned} & \text{maximize} && \sum_{i=n/2+1}^n c_i z_i + Xu + \left(Y - \sum_{i \in T} \frac{c_i}{b_i} \right) v + \sum_{\substack{i=1 \\ i \notin T}}^{n/2} c_i \min(-u/a_i, -v/b_i) && \text{(P6)} \\ & \text{s.t.} && \sum_{i=n/2+1}^n z_i = 1 + \left(\sum_{i \in T} \frac{1}{b_i} \right) v - \sum_{\substack{i=1 \\ i \notin T}}^{n/2} \min\{-u/a_i, -v/b_i\}, \\ & && 0 \leq z_i \leq \min(-u/a_i, -v/b_i), \quad i = n/2 + 1, \dots, n. \end{aligned}$$

This problem is obviously not harder than

$$\begin{aligned} & \text{maximize} && \sum_{i \notin T} c_i z_i + Xu + \left(Y - \sum_{i \in T} \frac{c_i}{b_i} \right) v && \text{s.t.} && \sum_{i \notin T} z_i = 1 + \left(\sum_{i \in T} 1/b_i \right) v, && \text{(P7)} \\ & && 0 \leq z_i \leq \min\{-u/a_i, -v/b_i\} && (i \notin T). \end{aligned}$$

However, the latter is of "size" not greater than $3/4$ of the original problem since $|T| \geq n/4$. The only difference is the linear term on the right-hand side of the constraint on the sum of z_i 's. However, the latter does not affect our ability to test lines and curves as described above in linear time. The fact that the problem is reduced with a linear-time effort to a problem of the same kind whose size is not more than $3/4$ of the original one implies that the total effort is linear in n . In view of the discussion above we need to consider a more general problem obtained by adding a linear

function of u and v to $\sum_{i=1}^n z_i$; however, we later discuss the solution of problems of the type of (P8) below only when they are derived from the original problem.

$$\begin{aligned} \text{maximize}_{z, u, v} \quad & \sum_{i=1}^n c_i z_i + Xu + Yv \quad \text{s.t.} \quad \sum_{i=1}^n z_i + Au + Bv = 1, & \text{(P8)} \\ & u + a_i z_i \leq 0 \quad (i = 1, \dots, n), \\ & v + b_i z_i \leq 0 \quad (i = 1, \dots, n), \\ & z_i \geq 0 \quad (i = 1, \dots, n) \end{aligned}$$

(where $A, B < 0$). The dual of the latter is:

$$\begin{aligned} \text{minimize}_{x, y, t} \quad & t \quad \text{s.t.} \quad t + a_i x_i + b_i y_i \geq c_i \quad (i = 1, \dots, n), & \text{(P9)} \\ & \sum_{i=1}^n x_i + At = X, \quad \sum_{i=1}^n y_i + Bt = Y, \\ & x_i, y_i \geq 0 \quad (i = 1, \dots, n). \end{aligned}$$

Testing a ray. In this section we develop a procedure for testing the following: Given the function

$$\begin{aligned} f(u, v) = Xu + Yv + \max \left\{ \sum_{i=1}^n c_i z_i : \sum_{i=1}^n z_i = 1 - Au - Bv, 0 \leq z_i \right. \\ \left. \leq \min(-u/a_i, -v/b_i) \right\} \end{aligned}$$

and a ray $v = \alpha u$ ($\alpha > 0$), recognize whether the maximum of f , (u^*, v^*) , satisfies $v^* < \alpha u^*$, $v^* = \alpha u^*$, or $v^* > \alpha u^*$.

The procedure starts by maximizing f on the ray $v = \alpha u$. Several quantities defined later are dependent on α ; however, we omit α from their notational description for simplicity. Thus, we now consider the following problem

$$\begin{aligned} \text{maximize}_{z, u} \quad & \sum_{i=1}^n c_i z_i + (X + \alpha Y)u \quad \text{s.t.} \quad \sum_{i=1}^n z_i = 1 - (A + \alpha B)u, & \text{(P10)} \\ & 0 \leq z_i \leq -u \cdot \min(1/a_i, \alpha/b_i) \quad (i = 1, \dots, n). \end{aligned}$$

This problem is quite easy to solve. Consider the function

$$\begin{aligned} g(u) = (X + \alpha Y)u + \max \left\{ \sum_{i=1}^n c_i z_i : \sum_{i=1}^n z_i = 1 - (A + \alpha B)u, \right. \\ \left. 0 \leq z_i \leq -u \cdot \min(1/a_i, \alpha/b_i) \right\}. \end{aligned}$$

The function $g(u)$ is piecewise linear and concave. Its breakpoints are where $-u \sum_{i=1}^k \min(1/a_i, \alpha/b_i) = 1 - (A + \alpha B)u$ for some k , $0 \leq k \leq n$ (assuming $c_1 \geq \dots \geq c_n$). Let u^0 denote a maximum of g . Given any value u' of u we can determine whether $u' < u^0$, $u' = u^0$, or $u' > u^0$ in $O(n)$ time as follows. First, compute $g(u')$. This is essentially like solving (P4). Let k be such that

$$-u' \sum_{i=1}^k \min(1/a_i, \alpha/b_i) \leq 1 - (A + \alpha B)u' < -u' \sum_{i=1}^{k+1} \min(1/a_i, \alpha/b_i), \quad 0 \leq k \leq n$$

(assume for convenience $\min(1/a_{n+1}, \alpha/b_{n+1}) = \infty$). We know that such k can be found in $O(n)$ time (even if the c_i 's are not sorted) by repeated median finding. Knowing the critical index k , we can express $g(u')$ as follows:

$$g(u') = -u' \sum_{i=1}^k c_i \min(1/a_i, \alpha/b_i) + \left(1 - (A + \alpha B)u' + u' \sum_{i=1}^k \min(1/a_i, \alpha/b_i) \right) c_{k+1} \\ + (X + \alpha Y)u'$$

(assuming $c_{n+1} = 0$). If $-u' \sum_{i=1}^k \min(1/a_i, \alpha/b_i) < 1 - (A + \alpha B)u'$ then g is linear in the neighborhood of u' , with a slope of $X + \alpha Y - (A + \alpha B)c_{k+1} - \sum_{i=1}^k (c_i - c_{k+1}) \cdot \min(1/a_i, \alpha/b_i)$. The sign of this slope tells us whether $u' < u^0$, $u' = u^0$, or $u' > u^0$. If $-u' \sum_{i=1}^k \min(1/a_i, \alpha/b_i) = 1 - (A + \alpha B)u'$ then g breaks at u' . The left-hand side derivative at u' equals the slope we have calculated in the previous case (since any decrease in the value of u , which is assumed to be negative in any case, takes us back to that previous case). The right-hand side derivative is similar but with $k-1$ replacing k , i.e.,

$$X + \alpha Y - (A + \alpha B)c_k - \sum_{i=1}^{k-1} (c_i - c_k) \min(1/a_i, \alpha/b_i).$$

Given the one-sided derivatives of g at u' , it is easy to recognize whether $u' < u^0$, $u' = u^0$, or $u' > u^0$.

Now, maximizing $g(u)$ is carried out by a binary search. We first let u' be equal to the median of critical values of u , i.e., where $-u \sum_{i=1}^{n/2} \min(1/a_i, \alpha/b_i) = 1 - (A + \alpha B)u$; namely,

$$u' = \left[A + \alpha B - \sum_{i=1}^{n/2} \min(1/a_i, \alpha/b_i) \right]^{-1}.$$

If $u' = u^0$ then we are, of course, done. If $u' > u^0$ then our maximization problem is now reduced to a similar problem with only $n/2$ variables:

$$\text{maximize } \sum_{i=1}^{n/2} a_i z_i + (X + \alpha Y)u \quad \text{s.t.} \quad \sum_{i=1}^{n/2} z_i = 1 - (A + \alpha B)u, \\ 0 \leq z_i \leq -u \cdot \min(1/a_i, \alpha/b_i), \quad i = 1, \dots, n/2.$$

If $u' < u^0$ then the variables $z_1, \dots, z_{n/2}$ are determined to be equal to their respective upper-bounds at the optimal solution of the problem corresponding to $g(u^0)$. In other words, our problem is then reduced to:

$$\text{maximize } \sum_{i=n/2+1}^n c_i z_i + \left[X + \alpha Y - \sum_{i=1}^{n/2} c_i \min(1/a_i, \alpha/b_i) \right] u \\ \text{s.t.} \quad \sum_{i=n/2+1}^n z_i = 1 - \left[A + \alpha B - \sum_{i=1}^{n/2} \min(1/a_i, \alpha/b_i) \right] u \\ 0 \leq z_i \leq u \cdot \min(1/a_i, \alpha/b_i) \quad (i = n/2 + 1, \dots, n).$$

It follows that the maximum of $g(u)$ is found with a total effort of $O(n)$.

We now turn to the question of recognizing the side of the ray $v = au$ which contains the maximum (u^*, v^*) of $f(u, v)$. It follows from the concavity of $f(u, v)$ that it is sufficient to look in the neighborhood of the point $(u^0, \alpha u^0)$. More specifically, denote $h(\alpha) = \text{Max}_u f(u, \alpha u)$. We are seeking $\text{Max } h(\alpha)$. Knowing that h increases at α

tells us that $v^*/u^* > \alpha$, and knowing that h decreases at α tells us that $v^*/u^* < \alpha$; if h is maximized at α then $v^*/u^* = \alpha$ so that $u^* = u^0$ and $v^* = \alpha u^0$. The problem is therefore to determine the behavior of h at α . Consider first the dependence of u^0 on α . Since $g(u)$ is piecewise linear it follows that it attains its maximum, u^0 , at a breakpoint. In other words, there exists k , $0 \leq k \leq n$ such that

$$u^0 = \left[A + \alpha B - \sum_{i=1}^k \min(1/a_i, \alpha/b_i) \right]^{-1}.$$

Hence,

$$\begin{aligned} g(u^0) &= \sum_{i=1}^k c_i \min(-u^0/a_i, -\alpha u^0/b_i) + (X + \alpha Y)u^0 \\ &= \left[X + \alpha Y - \sum_{i=1}^k c_i \min(1/a_i, \alpha/b_i) \right] / \left[A + \alpha B - \sum_{i=1}^k \min(1/a_i, \alpha/b_i) \right]. \end{aligned}$$

It is only a technical matter to determine the one-sided derivatives with respect to α of the quantity on the right-hand side of the latter equation. This finally tells us the side of the ray on which (u^*, v^*) lies.

Testing a curve. We describe in this section a procedure for determining the side of the curve

$$C_k = \left\{ (u, v) : \sum_{i=1}^k \min(-u/a_i, -v/b_i) = 1 - Au - Bv \right\}$$

which contains (u^*, v^*) . We describe a procedure that, given (u, v) , determines if $Au + Bv + \sum_{i=1}^k \min(-u/a_i, -v/b_i)$ is less than, equal to, or greater than 1. To that end, consider the dual problems (P8), (P9). Let t^* denote their optimal value. We first note that if $t^* < c_i$ ($1 \leq i \leq n$) then at an optimal solution either $x_i > 0$ or $y_i > 0$. By the complementary slackness conditions, we have either $u + a_i z_i = 0$ or $v + b_i z_i = 0$. In other words, $t^* < c_i$ implies $z_i = \min(-u/a_i, -v/b_i)$. Now, consider the following problem:

$$\begin{aligned} \text{minimize}_{x, y, t} \quad & t \quad \text{s.t.} \quad t + a_i x_i + b_i y_i = c_i \quad (i = 1, \dots, n), \end{aligned} \quad (\text{P11})$$

$$\sum_{i=1}^n x_i + At = X, \quad \sum_{i=1}^n y_i + Bt = Y, \quad x_i, y_i \geq 0 \quad (i = 1, \dots, n),$$

and its dual

$$\begin{aligned} \text{maximize}_{z, u, v} \quad & \sum_{i=1}^n c_i z_i + Xu + Yv \quad \text{s.t.} \quad \sum_{i=1}^n z_i + Au + Bv = 1, \\ & u + a_i z_i \leq 0 \quad (i = 1, \dots, n), \\ & v + b_i z_i \leq 0 \quad (i = 1, \dots, n). \end{aligned} \quad (\text{P12})$$

We will later discuss an algorithm for (P11). However, we first argue that solving (P11) is the key to testing a curve. Let us denote by $\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}, \tilde{v}$ optimal values for (P11)–(P12) and let $t^*, x^*, y^*, z^*, u^*, v^*$ denote optimal values for (P8)–(P9). Since (P11) is more constrained than (P9), it follows that $\tilde{t} \geq t^*$. Suppose $\tilde{t} < c_n$ (and recall that $c_n = \min\{c_i : i = 1, \dots, n\}$). It follows that $t^* < c_n$ and hence for every i , $i = 1, \dots, n$, either $x_i^* > 0$ or $y_i^* > 0$. By complementary slackness, we have $z_i^* = \min(-u^*/a_i, -v^*/b_i)$, $i = 1, \dots, n$. Conversely, suppose $\tilde{t} \geq c_n$. Since $\tilde{t} = c_n - a_n \tilde{x}_n - b_n \tilde{y}_n$, our assumption here is equivalent to either $\tilde{t} = c_n$ or (P11) infeasible. Consider, again, (P8)–(P9). If these problems are derived from the original (P2)–(P3) (hence, the values of n as well as other indexes may have been updated) then,

necessarily, $u^*, v^* < 0$. (Here u^* and v^* are identical to those of (P2)–(P3).) This is because in (P3) we have $z_i^* > 0$ for some i . Now, if $t^* < c_n$ (in (P8)) then we know that $z_i^* = \min(-u^*/a_i, -v^*/b_i) > 0, i = 1, \dots, n$. Using complementary slackness again, we have $t^* + a_i x_i^* + b_i y_i^* = c_i, i = 1, \dots, n$. The interesting conclusion is that $t^* < c_n$ implies (t^*, x^*, y^*) is feasible not only in (P9) but also in (P11) and therefore $\tilde{t} = t^* < c_n$.

Equivalently, if $\tilde{t} \geq c_n$ then $t^* \geq c_n$. Moreover, we may assume in this case $x_n^* = y_n^* = 0$. This is true not only for n but also for every i such that $c_i = c_n$.

The algorithmic implications of the above observations can now be described. Given that we need to test the curve C_k , solve the problem (P12) with $n = k$. In other words, we restrict $z_i = 0$ for $i > k$. Let \tilde{t} denote the optimal value. If $\tilde{t} < c_k$ then we know that $z_i^* = \min(-u^*/a_i, -v^*/b_i)$ for $i = 1, \dots, k$. If $\tilde{t} \geq c_k$ then for all i such that $c_i \leq c_k$, $x_i^* = y_i^* = 0$. In this case, denoting $G = \{i : c_i > c_k\}$, problem (P9) reduces to

$$\begin{aligned} \text{minimize}_{x, y, t} \quad & t \quad \text{s.t.} \quad t + a_i x_i + b_i y_i \geq c_i \quad (i \in G) \\ & \sum_{i \in G} x_i + At = X, \quad \sum_{i \in G} y_i + Bt = Y, \\ & x_i, y_i \geq 0 \quad (i \in G), \end{aligned}$$

and the dual is

$$\begin{aligned} \text{maximize}_{z, u, v} \quad & \sum_{i \in G} c_i z_i + Xu + Yv, \quad \text{s.t.} \quad \sum_{i \in G} z_i + Au + Bv = 1, \\ & u + a_i z_i \leq 0 \quad (i \in G), \\ & v + b_i z_i \leq 0 \quad (i \in G), \\ & z_i \geq 0 \quad (i \in G). \end{aligned}$$

Thus, testing a curve can be accomplished by solving (P11).

We finally consider solving (P11). Actually, we are interested only in \tilde{t} so we may, instead, solve (P12). Consider a simplified problem

$$\begin{aligned} \text{maximize}_z \quad & \sum_{i=1}^n c_i z_i \quad \text{s.t.} \quad \sum_{i=1}^n z_i = 1, \\ & z_i \leq Z_i \quad (i = 1, \dots, n). \end{aligned} \quad (\text{P13})$$

The analysis of (P13) is easy: If $\sum_{i=1}^n Z_i < 1$ then the problem is infeasible. Otherwise (assuming $c_n = \min\{c_i : i = 1, \dots, n\}$), an optimal solution is $z_i = Z_i, i = 1, \dots, n-1, z_n = 1 - \sum_{i=1}^{n-1} Z_i$, the optimal value being equal to $c_n + \sum_{i=1}^{n-1} (c_i - c_n) Z_i$. It now follows that in (P12) $Au + Bv + \sum_{i=1}^n \min(-u/a_i, -v/b_i) \geq 1$. The optimal z_i -values given u and v are therefore (in (P12)):

$$z_i = \min(-u/a_i, -v/b_i), \quad i = 1, \dots, n-1,$$

$$z_n = 1 - Au - Bv - \sum_{i=1}^{n-1} \min(-u/a_i, -v/b_i),$$

and the objective function value is

$$\phi(u, v) = (X - Ac_n)u + (Y - Bc_n)v + c_n + \sum_{i=1}^{n-1} (c_i - c_n) \min(-u/a_i, -v/b_i).$$

Now problem (P12) is equivalent to

$$\text{maximize} \quad \phi(u, v) \quad \text{s.t.} \quad Au + Bv + \sum_{i=1}^n \min(-u/a_i, -v/b_i) \geq 1. \quad (\text{P14})$$

It is essential to observe that $\phi(u, v)$ is concave. By maximizing $\phi(u, v)$ on any straight line we can tell (by looking in the neighborhood of the relative maximum) on which side of the line the global maximum lies. We now apply this idea to a line of the form $v = \alpha u$ where $\alpha > 0$. Maximizing on such a line amounts to

$$\begin{aligned} & \underset{u}{\text{maximize}} \quad u \left[X - Ac_n + \alpha(Y - Bc_n) - \sum_{i=1}^{n-1} (c_i - c_n) \min(1/a_i, \alpha/b_i) \right] + c_n \quad (\text{P15}) \\ & \text{s.t.} \quad u \left[A + \alpha B - \sum_{i=1}^n \min(1/a_i, \alpha/b_i) \right] \geq 1. \end{aligned}$$

It is only a technical matter to determine at the maximum $\bar{u} = \bar{u}(\alpha)$ whether the function $\psi(\alpha) = \phi(\bar{u}(\alpha), \alpha\bar{u}(\alpha))$ increases, decreases, or attains its maximum. This lets us maximize $\phi(u, v)$ in $O(n)$ time as follows. First, find the median of the ratios b_i/a_i , $i = 1, \dots, n$. Call it α_m and determine the behavior of $\psi(\alpha)$ at α_m . If, for example, $\psi(\alpha)$ is increasing at α_m then at the maximum $\bar{\alpha}$ ($\bar{\alpha} > \alpha_m$) we have $1/a_i < \bar{\alpha}/b_i$ for all i such that $\alpha_m \geq b_i/a_i$. Let $L = \{i : b_i/a_i < \alpha_m\}$, $E = \{i : b_i/a_i = \alpha_m\}$ and $G = \{i : b_i/a_i > \alpha_m\}$. Note that $|L \cup E|, |G \cup E| \geq n/2$. In the present case we can update the function $\phi(u, v)$:

$$\begin{aligned} \phi(u, v) = & \left[X - Ac_n - \sum_{\substack{i=1 \\ i \in L \cup E}}^{n-1} (c_i - c_n)/a_i \right] u + (Y - Bc_n)v + c_n \\ & + \sum_{\substack{i=1 \\ i \in G}}^{n-1} (c_i - c_n) \min(-u/a_i, -v/b_i). \end{aligned}$$

Moreover, we need to maximize $\phi(u, v)$ subject to

$$\left[A - \sum_{\substack{i=1 \\ i \in L \cup E}}^{n-1} 1/a_i \right] u + Bv + \sum_{\substack{i=1 \\ i \in G}}^{n-1} \min(-u/a_i, -v/b_i) = 1.$$

If $\psi(\alpha)$ attains its maximum at α_m then we are done. The case of $\psi(\alpha)$ decreasing at α_m is handled in an analogous way. In any case, with an $O(n)$ effort we reduce (P4) to the same problem with no more than $n/2$ critical ratios, accumulating at least $n/2$ functions in the linear components. This establishes a linear-time algorithm for (P14).

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MEGIDDO: IBM RESEARCH K51/281, 5600 COTTLE ROAD, SAN JOSE, CALIFORNIA 95193
ICHIMORI: SYSTEMS AND INDUSTRIAL ENGINEERING, HIROSHIMA UNIVERSITY,
HIROSHIMA 724, JAPAN