Continuous-Time Red and Black:

How to Control a Diffusion to a Goal

Victor C. Pestien
University of Miami
and
William D. Sudderth*
University of Minnesota

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Abstract

A player starts at x in $(0,1)$ and tries to reach 1. The process $\{X_t, t \geq 0\}$ of his positions moves according to a diffusion process (or, more generally, an Ito process) whose infinitesimal parameters μ, σ are chosen by the player at each instant of time from a set depending on his current position. To maximize the probability of reaching 1, the player should choose the parameters so as to maximize μ/σ^2, at least when the maximum is achieved by bounded, measurable functions. This implies that bold (timid) play is optimal for subfair (superfair), continuous-time red-and-black. Furthermore, in superfair red-and-black, the strategy which maximizes the drift coefficient of $\{\log X_t\}$ minimizes the expected time to reach 1.

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1. Introduction.

One of the most interesting discrete-time, stochastic control problems is the game of Red-and-Black, which inspired Dubins and Savage to write their fundamental book [5] on sequential gambling problems. The game goes as follows: a player starts at $x \in (0,1)$ and wants to reach 1. The player can stake any amount s_0, $0 \leq s_0 \leq x$, and will win the stake with a fixed probability p and lose it with probability $1-p$. The player can then make another stake s_1, $0 \leq s_1 \leq X_1$ where X_1 is the position after the first bet. And so on.

Here is another description of the game which suggests a continuous-time version. Let $Y_1, Y_2, \ldots$ be independent random variables such that $P[Y_n=1] = p = 1 - P[Y_n=-1]$. The process $X_0 = x, X_1, X_2, \ldots$ of the gambler's fortunes can be described in terms of its increments

$$X_{n+1} - X_n = s_n Y_n$$

where $s_n = s_n(X_0, \ldots, X_n) \in [0,X_n]$. If Y_n is regarded as being the n th increment of a simple random walk, then the natural continuous-time analogue is a stochastic differential

$$X_0 = x, \quad dX_t = s(t)dB_t \quad (t \geq 0)$$

where $B = \{B_t\}$ is a Brownian motion process with drift λ and $s(t)$ is a non-anticipative function restricted to lie in an interval $[s_1(X_t), s_2(X_t)]$ depending on the current state X_t.

Dubins and Savage [5] proved that, in discrete-time, subfair (i.e. $p < \frac{1}{2}$) Red-and-Black, the strategy which maximizes the probability of reaching 1 is bold play in which the player makes the maximum possible stake short of overshooting the goal (i.e. $s_n = \min(X_n, 1-X_n)$). Analogously, if the continuous-time game is subfair in the sense that $\lambda < 0$, then it is optimal to take $s(t) = s_2(X_t)$, at least if s_2 is a bounded, Borel measurable function on [0,1] with a positive infimum and $s_1 \geq 0$. If $\lambda > 0$ and s_1 is bounded, Borel measurable, and has a positive infimum on [0,1], it is optimal to take $s(t) = s_1(X_t)$. There is a comparable result in discrete-time when the state space is a discrete grid rather than [0,1] (Ross [15]).

A discrete-time game which is more general than Red-and-Black and much more difficult is Roulette. In Roulette a gambler has two choices at each stage — the size of the stake s and what event to bet on. For a given stake s, all bets have the same mean, but they may have different variances. It has been shown (Smith [18], Dubins [4]) that, in order to maximize the probability of reaching a goal, it is optimal to choose that bet which, for a given stake, has the largest variance and then play boldly. Here is an analogous continuous-time result. Suppose the processes at $x \in (0,1)$ satisfy

$$X_0 = x, \qquad dX_t = s(t)(\lambda dt + \sigma(t)dW_t)$$

where $W = \{W_t\}$ is standard Brownian motion, $\lambda < 0$, and s and σ are non-anticipative functions such that

$$0 \leq s_1(X_t) \leq s(t) \leq s_2(X_t)$$

and

$$0 \leq \sigma_1(X_t) \leq \sigma(t) \leq \sigma_2(X_t).$$

If s_1 and σ_1 are bounded, Borel, and have positive infima, then it is optimal to take $s(t) = s_2(X_t)$ and $\sigma(t) = \sigma_2(X_t)$.

Continuous-time Red-and-Black and Roulette are special cases of the problem of controlling a process $\{X_t\}$ given by a stochastic differential

$$X_0 = x, \qquad dX_t = \mu(t)dt + \sigma(t)dW_t$$

where the non-anticipative functions μ and σ satisfy certain integrability requirements together with the condition that $(\mu(t),\sigma(t))$ must lie in a control set $C(X_t)$ depending on the current position X_t. The results stated above follow from Theorem 1 in section 3 which says that if $\mu_0:[0,1]\longrightarrow R$, $\sigma_0:[0,1]\longrightarrow(0,\infty)$ are bounded, Borel functions such that $\inf\sigma_0>0$, and for all x,

$$\mu_0(x)/\sigma_0(x)^2 = \sup\{\mu/\sigma^2:\ (\mu,\sigma)\in C(x)\},$$

and

$$(\mu_0(x),\sigma_0(x))\in C(x),$$

then a process $\{X_t\}$ for which $\mu(t)=\mu_0(X_t)$ and $\sigma(t)=\sigma_0(X_t)$ reaches 1 with maximum probability.

In discrete-time, superfair (i.e. $p>\frac{1}{2}$) Red-and-Black, it is possible to reach 1 with probability 1. An interesting open problem (cf. Breiman [2]) is to determine the strategy which minimizes the expected time to the goal. In continuous-time, superfair (i.e. $\lambda>0$) Red-and-Black, it is also possible to reach 1 with probability 1. Furthermore, among all non-anticipative, non-negative s for which $\int_0^t Es(r)^2dr<\infty$ for all $t>0$, the expected time to 1 is minimized when $s(t)=\lambda X_t$. This result is a special case of Theorem 4 in section 4 which gives the optimal strategy to minimize expected time to the goal for a class of problems which also includes superfair, continuous-time Roulette.

The next section gives a careful formulation of the problems to be treated and establishes some verification lemmas. Section 3 studies how to maximize the probability of reaching a goal; section 4 treats the problem of reaching a goal in minimum expected time.

2. Verification lemmas.

A <u>continuous-time gambling problem</u> is a triple (F, Σ, u) where

(2.1) the <u>state space</u> F is Polish (i.e. F can be metrized so as to be complete and separable),

(2.2) the <u>gambling house</u> Σ is a mapping which assigns to each $x \in F$ a non-empty collection of processes $X = \{X_t, t \geq 0\}$ with state space F such that $X_0 = x$ and X has right-continuous paths with left-limits,

(2.3) the <u>utility function</u> u is a Borel function from F to the real line.

A process $X \in \Sigma(x)$ is said to be <u>available</u> at x. Each available X is defined on some probability space $(\Omega, \mathcal{F}, P)$ and is adapted to an increasing filtration $\{\mathcal{F}_t, t \geq 0\}$ of complete sub-sigma fields of $\mathcal{F}$. The probability space and filtration may depend on X. (This allows us to use 'weak' solutions to stochastic differential equations below.) When there is a danger of confusion, superscripts will be used and, for example, $\mathcal{F}_t^X$ will be written instead of $\mathcal{F}_t$.

A player, starting at position $x \in F$, selects a process $X \in \Sigma(x)$ and receives the payoff $u(X)$ defined by

$$u(X) = E[\limsup_{t\to\infty} u(X_t)].$$

The expectation occurring on the right is assumed to be well-defined for every available process X.

The payoff $u(X)$ is, in view of the Fatou equation (Corollary 2.1, Pestien [14]), the continuous-time analogue of the payoff function of Dubins and Savage [5]. Although this payoff may appear to be quite special, most of the payoff functions studied in control theory can be reduced to this one by a change of coordinates. An example of this occurs in section 4 where the payoff is the expected time to reach a goal.

The *value function* V is defined by

$$V(x) = \sup\{u(X): X \in \Sigma(x)\}$$

for every $x \in F$. A process $X \in \Sigma(x)$ is *optimal* at x if

$$u(X) = V(x).$$

Here is, in outline form, a standard technique for proving optimality which goes back to Dubins and Savage [5]. First guess an optimal X at x. (This is the hard part!) Define $Q(x) = u(X)$. Obviously $Q \leq V$; so what is needed are conditions to guarantee that $Q \geq V$. Such conditions will be established in the rest of this section.

Let $Q: F \longrightarrow R$ be Borel measurable. For every available X, let $\mathcal{T}(X)$ be the collection of $\{\mathcal{F}_t^X\}$-stopping times τ which are almost surely finite. The function Q is called *excessive* if for every $x \in F$, $X \in \Sigma(x)$, and $\tau \in \mathcal{T}(X)$, the expectation of $Q(X_\tau)$ is well-defined and satisfies

$$EQ(X_\tau) \leq Q(x). \tag{2.4}$$

Set

$$Q(X) = E[\limsup_{t\to\infty} Q(X_t)].$$

Our first lemma is a descendant of Theorem 2.12.1 of Dubins and Savage [5] and of Theorem 7 of Heath and Sudderth [8]. It is almost a consequence of Proposition 3.4 of Pestien [14].

Lemma 1. Suppose Q is excessive, and for every available X, $Q(X)$ is well-defined and $Q(X) \geq u(X)$. Then $Q(x) \geq V(x)$ for every $x \in F$.

<u>Proof</u>: For $x \in F$ and $X \in \Sigma(x)$,

$$\begin{aligned} Q(x) &\geq \sup\{EQ(X_\tau): \tau \in \mathcal{T}(X)\} \\ &\geq Q(X) \\ &\geq u(X). \end{aligned}$$

The first and last inequalities are true by hypothesis; the middle one is a consequence of Theorem 2.2 of Pestien [14].

Now take the sup over $X \in \Sigma(x)$. □

If certain natural conditions are imposed on Σ, then V is excessive and $V(X) \geq u(X)$ for all available X. Thus, by Lemma 1, V is the smallest function with these properties (cf. Proposition 3.4 of Pestien [14]).

From now on, each process $X = \{X_t\}$ under consideration will have values in a Euclidean space R^d and will be an <u>Ito process</u> of the form

$$X_t = x + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW_s \tag{2.5}$$

where $W = \{W_t\}$ is a standard m-dimensional Brownian motion process on (Ω, F, P) adapted to $\{F_t\}$. Assume also that F_t is independent of $\{W_{t+s} - W_t,\ s \geq 0\}$ and contains all P-null sets. The function $\alpha = \alpha(t,\omega)$ is to be R^d-valued, jointly measurable, adapted to $\{F_t\}$ and such that

$$\int_0^t |\alpha(s)|ds < \infty \quad \text{a.s.} \quad \text{for all } t. \tag{2.6}$$

The function $\beta = \beta(t,\omega)$ has as values real $d \times m$ matrices, is jointly measurable, adapted to $\{F_t\}$, and satisfies

$$E\int_0^t |\beta(s)|^2 ds < \infty \text{ for all } t. \tag{2.7}$$

(The notation '$|\cdot|$' is for the Euclidean norm.) As before, the space (Ω, F, P)

and filtration $\{\mathcal{F}_t\}$ and now also the Brownian motion W are allowed to vary with X.

For each pair (a,b), where $a \in R^d$ is a $d\times 1$ vector and b is a $d\times m$ real-valued matrix, define the differential operator $D(a,b)$ for sufficiently smooth functions $Q:R^d \longrightarrow R$ by

$$D(a,b)Q(y) =$$

$$Q_x(y)a + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} Q_{x_i x_j}(y)(bb')_{ij}$$

where

$$Q_x = \left(\frac{\partial Q}{\partial x_1},\ldots,\frac{\partial Q}{\partial x_d}\right),$$

$$Q_{x_i x_j} = \frac{\partial^2 Q}{\partial x_i \partial x_j},$$

and b' is the transpose of b.

Suppose now that the state space F of the gambling problem is a Borel subset of R^d and has non-empty interior F^o. All available processes are assumed to be Ito processes as in (2.5) and can be specified in terms of the possible values for the infinitesimal parameters α and β. To make this specification, suppose that, for each $x \in F$, $C(x)$ is a non-empty set of pairs (a,b) where $a \in R^d$ and b is a real $d\times m$ matrix. (The idea is that $C(x)$ is the set from which a player at state x may choose the value of (α,β).) Assume also that every available process X is absorbed at the time τ_X of its first exit from F^o. These conditions define a function Σ_C on F where $\Sigma_C(x)$ is the collection of all processes X having paths in F and satisfying (2.5),(2.6),(2.7) together with

(2.8) $\qquad (\alpha(t,\omega),\beta(t,\omega)) \in C(X_t(\omega))$ for all (t,ω),

(2.9) $\qquad (\alpha(t,\omega),\beta(t,\omega)) = (0,0)$ for $t \geq \tau_X(\omega)$,

$$(2.10) \qquad C(x) = \{(0,0)\} \text{ for } x \in F - F^{o}.$$

(Here '0' is used to denote both a zero vector and a zero matrix.)

Let Σ be a gambling house such that $\Sigma(x) \subset \Sigma_C(x)$ for every $x \in F$. (Recall that each $\Sigma(x)$ is assumed to be non-empty. It could happen that, for some highly irregular C, $\Sigma_C(x)$ is empty for some x. We are excluding such uninteresting cases.)

In the next two lemmas, G is assumed to be an open subset of R^d which contains F.

Lemma 2. Suppose $Q:G \longrightarrow R$ has continuous second-order derivatives. Assume the following conditions for every $x \in F^{o}$ and every $X \in \Sigma(x)$:

(i) $D(a,b)Q(x) \leq 0$ for all $(a,b) \in C(x)$,

(ii) $E\int_0^t |Q_x(X_s)\beta(s)|^2 ds < \infty$ for all $t \geq 0$,

(iii) there is an integrable random variable Y such that $Q(X_t) \geq E[Y|F_t]$ for all $t \geq 0$.

Then Q is excessive.

Proof: Let $x \in F$, $X \in \Sigma(x)$, and $\tau \in T(X)$. If $x \in F-F^{o}$, then $P[X_t = x \text{ for all } t] = 1$ and, hence, $EQ(X_\tau) = Q(x)$. So assume $x \in F^{o}$. By Ito's Lemma,

$$Q(X_t) = Q(x) + \int_0^t D(\alpha(s),\beta(s))Q(X_s)ds + \int_0^t Q_x(X_s)\beta(s)dW_s.$$

where α and β are as in (2.5). By (i), the first integral on the right is a decreasing process. By (ii), the second integral is a martingale. It now follows from (iii) that $\{Q(X_t)\}$ is a supermartingale to which the optional sampling theorem (cf. Dellacherie and Meyer [3], Theorems VI.3 and VI.10) can be applied to yield $EQ(X_\tau) \leq Q(x)$. □

The next lemma gives a verification result that can be used for a function Q which is not smooth, but can be approximated by smooth functions.

Lemma 3. Suppose $Q:G \longrightarrow R$ and $Q_n:G \longrightarrow R$ for $n = 1,2,\cdots$. Suppose also that each Q_n has continuous second order derivatives on G, and that

(i) $\lim_{n\to\infty} Q_n(x) = Q(x)$ for every $x \in F$.

Assume the following conditions for every $x \in F^o$ and every $X \in \Sigma(x)$:

(ii) $\limsup_{n\to\infty} D(a,b)Q_n(x) \leq 0$ for all $(a,b) \in C(x)$,

(iii) $E\int_0^t |(Q_n)_x(X_s)\beta(s)|^2 ds < \infty$ for all n,

(iv) there is an integrable random variable Y such that $Q_n(X_t) \geq Y$ for all n and all $t \geq 0$,

(v) there is a measurable process $Z = \{Z_s\}$ such that

$$D(\alpha(s),\beta(s))Q_n(X_s) \leq Z_s$$

for all n and all $s \geq 0$, and

$$E\int_0^t |Z_s| ds < \infty$$

for all $t \geq 0$.

Then Q is excessive.

Proof. Let $x \in F^o$, $X \in \Sigma(x)$, and $\tau \in \mathcal{T}(X)$. It suffices to check inequality (2.4). (As in the proof of Lemma 2, the case that $x \in F - F^o$ is trivial.) By conditions (i) and (iv), $Q(X_t) \geq Y$ for all t. So, by Fatou's inequality,

$$EQ(X_\tau) \leq \liminf_{n\to\infty} EQ(X_{\tau\wedge n}).$$

Consequently, it suffices to check (2.4) for a bounded $\tau \in \mathcal{T}(X)$.

Let X satisfy (2.5) and use Ito's Lemma to write

$$Q_n(X_t) = Q_n(x) + \int_0^t D(\alpha(s),\beta(s))Q_n(X_s)ds + \int_0^t (Q_n)_x(X_s)\beta(s)dW_s. \tag{2.11}$$

By (iii), the final term on the right is a martingale. Now calculate.

$$\begin{aligned}
EQ(X_\tau) &= E[\lim_{n\to\infty} Q_n(X_\tau)] \\
&\leq \liminf_{n\to\infty} EQ_n(X_\tau) \\
&= Q(x) + \liminf_{n\to\infty} E\int_0^\tau D(\alpha(s),\beta(s))Q_n(X_s)ds \\
&\leq Q(x) + E\int_0^\tau \limsup_{n\to\infty} D(\alpha(s),\beta(s))Q_n(X_s)ds \\
&\leq Q(x).
\end{aligned}$$

The successive lines are, respectively, by (i) and (iv); by Fatou and (iv); by (2.11), (i), and the optional sampling theorem; by Fatou and (v); and by (ii). □

Remarks.

1. The usual formulations of stochastic control problems, as, for example, in Fleming and Rishel [6] or Krylov [12], use stochastic differential equations rather than Ito processes. Of course, solutions to stochastic differential equations of the form

$$X_0 = x$$

$$dX_t = \hat{\alpha}(t,X_t)dt + \hat{\beta}(t,X_t)dW_t$$

are Ito processes. So the simpler formulation used here allows for a more general class of processes. In the specific problems considered below, the optimal processes turn out to be diffusion processes which are solutions of stochastic differential equations.

2. The usual formulations have the controller select a control function which determines the infinitesimal parameters α and β rather than have the controller select α and β directly as we do. This difference is essentially the same as the difference between the discrete-time theories of dynamic programming, where a player chooses an action which determines the distribution of the next state, and gambling, where a player chooses the distribution of the next state directly. For most purposes, this difference is of no consequence, but there are some measure-theoretic subtleties (cf. Blackwell [1]).

3. Lemma 2 is analogous to other verification lemmas in the stochastic control literature such as Theorem VI.4.1 of Fleming and Rishel [6] and Theorem 1.5.4 of Krylov [12]. One trivial, but useful, difference is that Lemma 2 applies to functions Q which are not solutions of the Hamilton-Jacobi-Bellman equation. (This is needed in section 4.) Also, no assumptions are made that the processes are non-degenerate or exit from F^o in a finite amount of time. Finally, the use of Ito processes rather than stochastic differential equations allows us to avoid the smoothness assumptions usually made about the coefficients.

4. One could try to establish a result similar to Lemma 3 by using Krylov's generalization ([12], Theorem 2.10.1) of Ito's Lemma, which applies to certain non-smooth functions Q. However, Krylov's result requires that the processes be uniformly non-degenerate, which is not assumed here.

3. Maximizing the probability of reaching a goal.

Consider a gambling problem with state space $F = [0,1]$ and utility function u = the indicator function of $\{1\}$. All available processes $X = \{X_t\}$ will be absorbed at the endpoints 0 and 1, and hence,

$$u(X) = E[\limsup_{t\to\infty} u(X_t)] \tag{3.1}$$

$$= P[X \text{ reaches } 1] \;.$$

In the notation of the previous section, $d = m = 1$ and, for each $x \in F$, $C(x)$ is a non-empty subset of $R\times[0,\infty)$. A typical element of $C(x)$ will be written (μ,σ) to emphasize that it is a possible value for the infinitesimal mean and standard deviation of a process starting from x. The assumptions of the previous section are in force, and, in particular, by (2.10), $C(0) = C(1) = \{(0,0)\}$. Assume that $\Sigma_C(x)$ is non-empty for every x so that Σ_C is a gambling house.

Example 1. Continuous-time Red-and-Black.

Let $\lambda \in R$; let $s_i:[0,1]\longrightarrow [0,\infty)$ $(i=1,2)$ be bounded, Borel mappings such that $s_1 \le s_2$. Define

$$C(x) = \{(s\lambda, s): s_1(x) \le s \le s_2(x)\}.$$

Example 2. Continuous-time roulette.

let λ, s_1, s_2 be as in the previous example; let $\sigma_i:[0,1]\longrightarrow [0,\infty)$ $(i=1,2)$ be bounded, Borel mappings such that $\sigma_1 \le \sigma_2$. Define

$$C(x) = \{(s\lambda, s\sigma): s_1(x) \le s \le s_2(x),\ \sigma_1(x) \le \sigma \le \sigma_2(x)\}.$$

Return now to the general goal problem and define, for $0 < x < 1$,

$$\rho(x) = \sup\{\mu/\sigma^2 : (\mu,\sigma) \in C(x)\}. \tag{3.2}$$

(Here, 0/0 is taken to be $-\infty$.)

The ratio μ/σ^2 has a history in discrete-time gambling theory where it provides a measure of superfairness (cf. Dubins and Savage [5], pp. 167-168). The function ρ is crucial here and the following assumption is made.

<u>Assumption</u> <u>A</u>. The function ρ is of the form

$$\rho(x) = \mu_0(x)/\sigma_0^2(x) \ , \ 0 < x < 1, \tag{3.3}$$

where μ_0 and σ_0 are bounded, Borel-measurable functions on (0,1) and $\inf \sigma_0 > 0$.

Consider now a diffusion process X starting at $x \in (0,1)$ which is absorbed at the endpoints 0 and 1 and which solves the stochastic differential equation

$$\begin{aligned} X_0 &= x \\ dX_t &= \mu_0(X_t)dt + \sigma_0(X_t)dW_t \ . \end{aligned} \tag{3.4}$$

It follows from Krylov ([12], Theorem 2.6.1, p.87) or Ikeda and Watanabe ([9], Section IV.4) that such an X exists.

The probability

$$Q(x) = P[X \text{ reaches } 1]$$

depends only on ρ and x. In fact, let γ be any bounded, measurable function on (0,1) and define

$$Q_\gamma(x) = \frac{S\gamma(x)}{S\gamma(1)} \tag{3.5}$$

where

$$S_\gamma(x) = \int_0^x \xi_\gamma(y)dy, \quad \xi_\gamma(x) = \exp\{-2\int_0^x \gamma(y)dy\}. \tag{3.6}$$

Then

$$Q(x) = Q_\rho(x). \tag{3.7}$$

This formula for Q is well-known when the functions μ_0 and σ_0 of (3.3) are continuous (cf. Karlin and Taylor [10], pp.191-195). The proof in the general case is the same as that in Gihman and Skorohod ([7], Theorem 3.15.4) except that Krylov's generalization of Ito's Lemma ([12], Theorem 2.10.1) must be used.

The process X of (3.4) will belong to $\Sigma_C(x)$ under the following assumption.

<u>Assumption</u> <u>B</u>. $(\mu_0(x),\sigma_0(x)) \in C(x)$, $0 < x < 1$.

Let V be the value function for the problem (F, Σ_C, u) defined in the first paragraph of this section.

<u>Theorem</u> <u>1</u>. If A holds, then $V \le Q$. If A and B hold, then $V = Q$ and the diffusion process X defined by (3.4) is optimal at x.

<u>Proof</u>: If B holds, then the process X of (3.4) is an element of $\Sigma_C(x)$ and so $Q(x) = u(X) \le V(x)$. Thus it suffices to prove the first assertion.

It follows from the Vitali-Caratheodory Theorem (Rudin [17], Theorem 2.24) that there is a decreasing sequence $\{\gamma_n\}$ of bounded, lower semicontinuous functions such that $\gamma_n(x) \ge \rho(x)$ for every n and every $x \in (0,1)$, and $\gamma_n(x) \longrightarrow \rho(x)$ for Lebesgue almost every x. By the monotone convergence theorem, (3.5), (3.6), and (3.7), $Q_{\gamma_n}(x) \longrightarrow Q_\rho(x) = Q(x)$ for every $x \in [0,1]$. Thus, to show $Q \ge V$, it is enough to prove the following lemma:

<u>Lemma</u> <u>4</u>. If γ is a bounded, lower semicontinuous function defined on $[0,1]$ and $\gamma \geq \rho$ on $(0,1)$, then $Q_\gamma \geq V$.

<u>Proof</u>: Q_γ is bounded, Borel-measurable, and $Q_\gamma \geq u$. Thus $Q_\gamma(X)$ is well-defined and $Q_\gamma(X) \geq u(X)$ for every available X. By Lemma 1, it is enough to show Q_γ is excessive. We will use Lemma 3, with Q_γ playing the role of Q, to establish this last fact.

Because γ is bounded and lower semicontinuous, there is a sequence $\{\rho_n\}$ of bounded, continuous functions which converge up to γ pointwise on $[0,1]$ (cf. Royden [16], Problem 2.49). Let $Q_n = Q_{\rho_n}$. Notice, because each ρ_n is continuous, that each Q_n has a continuous second derivative and can be extended smoothly to a fixed open interval G containing $[0,1]$. Furthermore, by (3.5) and (3.6), Q_n satisfies

$$\tfrac{1}{2}Q_n'' + \rho_n Q_n' = 0 \tag{3.8}$$

on $(0,1)$. We are now ready to check the conditions of Lemma 3.

Condition (i). $\lim_n Q_n(x) = Q_\gamma(x)$ for $0 \leq x \leq 1$ by the monotone convergence theorem.

Condition (ii). Let $0 < x < 1$ and $(\mu,\sigma) \in C(x)$. Then

$$\begin{aligned} D(\mu,\sigma)Q_n(x) &= \mu Q_n'(x) + \tfrac{1}{2}\sigma^2 Q_n''(x) \\ &= \mu Q_n'(x) + \tfrac{1}{2}\sigma^2 Q_n''(x) - \sigma^2[\tfrac{1}{2}Q_n''(x)+\rho_n(x)Q_n'(x)] \\ &= (\mu-\sigma^2\rho_n(x))Q_n'(x). \end{aligned} \tag{3.9}$$

Hence,

$$\limsup_{n\to\infty} D(\mu,\sigma)Q_n(x) = (\mu - \sigma^2\gamma(x))\limsup_{n\to\infty} Q_n'(x)$$

$$\leq (\mu - \sigma^2\rho(x))\limsup_{n\to\infty} Q_n'(x)$$

$$\leq 0$$

by (3.2) and the fact that $Q_n' \geq 0$ on $(0,1)$ for every n.

Condition (iii). Q_n' is continuous and, therefore, bounded on $[0,1]$. So this condition is a consequence of (2.7).

Condition (iv). Take Y to be the constant 0.

Condition (v). By (3.9), for $\sigma \neq 0$,

$$D(\mu,\sigma)Q_n(x) = \sigma^2(\frac{\mu}{\sigma^2} - \rho_n(x))Q_n'(x)$$

$$\leq \sigma^2(\rho(x) - \rho_n(x))Q_n'(x).$$

Now $\rho(x)$ is bounded by assumption A; the ρ_n are uniformly bounded above by the bounded function γ and below by the bounded function ρ_1; and the Q_n' can be seen to be uniformly bounded from (3.5) and (3.6). Also, if $\sigma = 0$ and $(\mu,\sigma) \in C(x)$, then $\mu \leq 0$. (Otherwise, $\rho(x) = +\infty$.) So, in this case, $D(\mu,\sigma)Q_n(x) = \mu Q_n'(x) \leq 0$. Therefore, there is a positive constant B such that

$$D(\mu,\sigma)Q_n(x) \leq B\sigma^2$$

for $0 < x < 1$ and $(\mu,\sigma) \in C(x)$. Condition (v) now follows from (2.7).

The proofs of Lemma 4 and Theorem 1 are now complete. □

It can easily happen that the optimal process in Theorem 1 is not uniquely so. For example, the supremum in (3.2) could be achieved by another pair of functions μ_1 and σ_1. Or, if $(0,0) \in C(x)$, there is no harm in using $(0,0)$ as the control for a time and then switching to (μ_0,σ_0).

There are general gambling techniques which make it possible to characterize the class of all optimal processes. (For the discrete-time case, see Chapter 3 of Dubins and Savage [5] or Sudderth [19].) We plan to write another paper on this general subject.

<u>Example 1</u> (continued). Suppose $\lambda < 0$ so that the game is subfair and suppose $\inf s_2 > 0$. Then $\rho(x) = \sup\{\lambda/s: s_1(x) \leq s \leq s_2(x)\} = \lambda/s_2(x)$ and, by Theorem 1, the optimal process corresponds to bold play: $s(t) = s_2(X_t)$ for all t. If $\lambda > 0$, and $\inf s_1 > 0$, a similar argument shows timid play ($s(t) = s_1(X_t)$ for all t) is optimal. The case when $s_1 = 0$ is discussed in the next section.

<u>Example 2</u> (continued). Suppose $\lambda < 0$, and the functions s_2, σ_2 have positive infima. Then $\rho(x) = \lambda/(s_2(x)\sigma_2(x))$ and the optimal controls are $s(t) = s_2(X_t)$, $\sigma(t) = \sigma_2(X_t)$ for all t. Similarly, if $\lambda > 0$ and s_1, σ_1 have positive infima, then $s(t) = s_1(X_t)$, $\sigma(t) = \sigma_1(X_t)$ are optimal.

Turn now to the problem of reaching a goal on a half-line. Take $\underset{\sim}{F} = (-\infty,0]$ and $\underset{\sim}{u}$ = the indicator function of $\{0\}$. Let $\underset{\sim}{C}(x)$ be a non-empty subset of $R\times[0,\infty)$ for $x < 0$ and $\underset{\sim}{C}(0) = \{(0,0)\}$. Define

$$\underset{\sim}{\rho}(x) = \sup\{\mu/\sigma^2: (\mu,\sigma) \in \underset{\sim}{C}(x)\}, \quad x > 0.$$

<u>Assumption $\underset{\sim}{A}$</u>. The function $\underset{\sim}{\rho}$ is of the form

$$\underset{\sim}{\rho}(x) = \mu_0(x)/\sigma_0(x), \quad -\infty < x < 0$$

where μ_0 and σ_0 are bounded, Borel-measurable functions on $(-\infty,0)$ and $\inf \sigma_0 > 0$.

Assumption $\underset{\sim}{B}$. $(\mu_0(x),\sigma_0(x)) \in \underset{\sim}{C}(x)$, $-\infty < x < 0$.

Let $\underset{\sim}{V}$ be the value function for the problem $(\underset{\sim}{F},\underset{\sim}{u}, \Sigma_{\underset{\sim}{C}})$. For each $x < 0$, let X be a diffusion on $(-\infty,0]$ which is absorbed at 0 and satisfies

$$X_0 = x, \qquad dX_t = \mu_0(X_t)dt + \sigma_0(X_t)dW_t. \tag{3.10}$$

Let

$$\underset{\sim}{Q}(x) = P[X \text{ reaches } 0].$$

The next result can be proved directly or derived from Theorem 1.

Theorem $\underset{\sim}{2}$. If $\underset{\sim}{A}$ holds, then $\underset{\sim}{V} \leq \underset{\sim}{Q}$. If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ hold, then $\underset{\sim}{V} = \underset{\sim}{Q}$ and the process defined by (3.10) is optimal at x. □

Of course, there is nothing special about the goal being 0 in Theorem 3. A process which maximizes the critical ratio μ/σ^2 is most likely to reach any goal to the right of the initial position. This suggests the following comparison result.

Theorem $\underset{\sim}{3}$. Consider two diffusion processes

$$X_0^i = x^i, \quad dX_t^i = \mu_i(X_t^i)dt + \sigma_i(X_t^i)dW_t$$

with μ_i and σ_i bounded, Borel-measurable and $\inf \sigma_i > 0$ for $i = 1,2$. If $x^2 \leq x^1$ and $\mu_2/\sigma_2^2 \leq \mu_1/\sigma_1^2$, then $\sup_t X_t^2$ is stochastically smaller than $\sup_t X_t^1$.

<u>Proof</u>. Fix g where $x^1 \leq g < \infty$. Consider the problem: $F=(-\infty,g]$, u = the indicator function of $\{g\}$, $\Sigma = \Sigma_C$ where $C(g) = \{(0,0)\}$ and $C(x) = \{(\mu_i(x),\sigma_i(x)): i=1,2\}$ for $x < g$. By Theorem 2, the optimal process at x^1 is X^1. It follows that

$$P[\sup X_t^1 \geq g] \geq P[\sup X_t^2 \geq g] . \square$$

The comparison theorem of Ikeda and Watanabe ([9], Section VI.1) has the stronger conclusion that $X_t^2 \leq X_t^1$ for every t with probability one. It is easy to give examples to see that this need not follow from the hypotheses of Theorem 3.

4. Minimizing the expected time to the goal.

If arbitrarily small positive stakes are permitted in superfair Red-and-Black, then, as is shown below, it is possible to reach the goal with probability 1. The next problem is how to minimize the expected time to reach the goal. The theorem of this section gives the solution for a class of gambling problems which includes superfair Red-and-Black and Roulette when arbitrary positive stakes are allowed.

The formulation uses two-dimensional processes $X = \{X(t)\}$ where

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}.$$

The first coordinate X_1 corresponds to the player's position in $(0,1]$; the second coordinate X_2 is the time, starting from x_2, prior to absorption of X_1 at 1. The state space is

$$F = \{x \in R^2 \colon 0 < x_1 \leq 1,\ x_2 \in R^1\}.$$

(Notice that every real number x_2 is a possible starting time.) Let C_0 be a fixed, nonempty subset of $R \times [0,\infty)$ and define, for $x \in F$,

$$\begin{aligned} C(x) &= \left\{\left(\begin{bmatrix} s\mu \\ 1 \end{bmatrix}, \begin{bmatrix} s\sigma \\ 0 \end{bmatrix}\right) \colon (\mu,\sigma) \in C_0,\quad s \geq 0\right\} \quad \text{if } x_1 < 1, \\ &= \left\{\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)\right\} \quad \text{if } x_1 = 1. \end{aligned}$$

A process $X \in \Sigma_C(x)$ can be expressed by a stochastic differential

$$X(0) = x, \quad dX(t) = \begin{bmatrix} s(t)\mu(t)dt + s(t)\sigma(t)dW(t) \\ dt \end{bmatrix}$$

prior to the absorption of X_1 at 1. Here W is a one-dimensional Brownian motion. (In the notation of section 2, $d=2$ and $m=1$.)

Because the object is to minimize expected time, set

$$u(x) = -x_2 \text{ for } x \in F.$$

Then, for $X \in \Sigma_C(x)$,

$$(4.1) \qquad u(X) = -ET - x_2 ,$$

where x_2 is the starting time and $T = \inf\{t \geq 0: X_1(t) = 1\}$.

Examples. If $C_0 = \{(\lambda,1)\}$, then the processes X_1 correspond to those available in continuous-time Red-and-Black when arbitrary non-negative stakes are allowed. If $C_0 = \{(\lambda,\sigma): \sigma_1 \leq \sigma \leq \sigma_2\}$, the processes X_1 correspond to those available in a version of continuous-time Roulette.

Consider now the general problem (F, Σ_C, u). Assume that the problem is superfair in the sense that there is an element $(\mu,\sigma) \in C_0$ for which $\mu > 0$. Define

$$\Sigma(x) = \{X \in \Sigma_C(x): u(X) > -\infty\}$$

$$= \{X \in \Sigma_C(x): ET < \infty\}$$

To see that $\Sigma(x)$ is not empty, suppose $x_1 < 1$ and fix $(\mu,\sigma) \in C_0$ with $\mu > 0$. Consider the proportional strategy at x based on (μ,σ) and c for which

$$s(t) = cX_1(t)$$

and

$$dX_1(t) = c\mu X_1(t)dt + c\sigma X_1(t)dW(t).$$

Use Ito's formula to check that, for $t \leq T$,

$$X_1(t) = e^{Y(t)}$$

where

$$Y(t) = \log x_1 + mt + c\sigma W(t)$$

is a Brownian motion with drift

$$m = m(\mu,\sigma,c) = c\mu - \frac{1}{2}c^2\sigma^2.$$

This drift coefficient is positive if $0 < c < \frac{2\mu}{\sigma^2}$. So, for c in this interval, Y reaches 0 almost surely and, consequently, X_1 reaches 1 almost surely. That is, $P[T < \infty] = 1$. Furthermore it is easy to show that ET is finite. So, by Wald's identity for Brownian motion (Liptser and Shiryayev [13], Lemma 4.8),

$$E\, W(T) = 0 \,.$$

But $Y(T) = 0$ a.s.. Thus

$$0 = EY(T) = \log x_1 + mET$$

and

$$ET = -\frac{\log x_1}{m} \,. \tag{4.2}$$

In particular, $X \in \Sigma(x)$.

Our guess of an optimal strategy is inspired by the 'Kelly criterion' [11], which, as Breiman [2] showed, often leads to good strategies for discrete-time, superfair problems. The criterion says to bet so as to maximize the expected log of your next fortune. There are difficulties with overshooting when the object is to reach a goal quickly and variables are discrete. Thus Breiman conjectured that an optimal plan would follow the criterion up to some point and then switch to smaller bets to avoid overshooting the goal. The continuous processes considered here cannot overshoot and so it is natural to consider that strategy which always maximizes the drift of $\log X_1(t)$ prior to reaching the goal.

For fixed $(\mu,\sigma) \varepsilon C_0$ with $\mu > 0$, $0 < \sigma < \infty$, $m(\mu,\sigma,c)$ is a maximum when $c = \mu/\sigma^2$, and $m(\mu,\sigma,\mu/\sigma^2) = \mu^2/2\sigma^2$. Define

$$M = \sup\{\mu^2/2\sigma^2 : (\mu,\sigma) \varepsilon C_0, \ \mu > 0\}. \tag{4.3}$$

Let V be the value function for the gambling problem (F, Σ, u).

<u>Theorem</u> 4. $V(x) = \dfrac{\log x_1}{M} - x_2$.

If $M = \mu_0{}^2/2\sigma_0{}^2$ for some $(\mu_0,\sigma_0) \varepsilon C_0$ with $\mu_o > 0$, then the proportional strategy based on (μ_0,σ_0) and $c = \mu_0/\sigma_0{}^2$ is optimal at every x.

<u>Proof</u>: The second assertion follows from (4.1) and (4.2) together with the first. So it suffices to prove the equality. Set

$$Q(x) = \frac{\log x_1}{M} - x_2 .$$

It is clear from (4.1), (4.2), (4.3) and the definition of V that $Q \leq V$. It remains to prove the opposite inequality. If $M = \infty$, the inequality is clear. So assume $M < \infty$.

Let $\varepsilon > 0$ and define

$$Q^{\varepsilon}(x) = \frac{\log(x_1+\varepsilon)}{M} - x_2 .$$

We will show $Q^{\varepsilon} \geq V$. Because $Q^{\varepsilon} \longrightarrow Q$ as $\varepsilon \longrightarrow 0$, this will be sufficient.

Let $x \varepsilon F$ and $X \varepsilon \Sigma(x)$. To see that $Q^{\varepsilon}(X) \geq u(X)$, calculate:

$$Q^{\varepsilon}(X) = E[\limsup_{t\to\infty}(\frac{\log(X_1(t)+\varepsilon)}{M} - X_2(t))]$$

$$= \frac{\log(1+\varepsilon)}{M} - ET - x_2$$

$$\geq - ET - x_2$$

$$= u(X)$$

To finish proving that $Q^{\varepsilon} \geq V$, it suffices by Lemma 1 to show Q^{ε} is excessive. We now check the conditions of Lemma 2. Take the open set G to be $\{x \in R^2: x_1 > 0, x_2 \in R^1\}$.

Condition (i): Let $(a,b) = (\begin{bmatrix} s\mu \\ 1 \end{bmatrix}, \begin{bmatrix} s\sigma \\ 0 \end{bmatrix}) \in C(x)$. Then

$$D(a,b)Q^{\varepsilon}(x) = \left[\frac{1}{(x\ +\varepsilon)M}, -1\right]\begin{bmatrix} s\mu \\ 1 \end{bmatrix} - \frac{s^2\sigma^2}{2(x_1+\varepsilon)^2M}$$

$$= \frac{s\mu}{(x_1+\varepsilon)M} - 1 - \frac{s^2\sigma^2}{2(x_1+\varepsilon)^2M}$$

$$\leq \frac{\sqrt{2}s\sigma}{(x_1+\varepsilon)\sqrt{M}} - 1 - \frac{s^2\sigma^2}{2(x_1+\varepsilon)^2M}$$

$$= -(1 - \frac{s\sigma}{(x_1+\varepsilon)\sqrt{2M}})^2$$

$$\leq 0.$$

The first inequality holds because $\mu \leq \sigma\sqrt{2M}$ by (4.3).

Condition (ii):

$$|Q^{\varepsilon}_x(X(t))\beta(t)| = \left|\left[\frac{1}{(X_1(t)+\varepsilon)M}, -1\right]\begin{bmatrix} s(t)\sigma(t) \\ 0 \end{bmatrix}\right|$$

$$\le \frac{1}{\varepsilon M}|s(t)\sigma(t)|$$

$$= \frac{1}{\varepsilon M}|\beta(t)|.$$

The condition is thus a consequence of assumption (2.7).

Condition (iii):

$$Q^{\varepsilon}(X(t)) = \frac{\log(X_1(t)+\varepsilon)}{M} - X_2(t)$$

$$\ge \frac{\log \varepsilon}{M} - x_2 - T.$$

The right side is integrable by the definition of Σ.

Thus Lemma 2 applies, Q^{ε} is excessive, and the proof of Theorem 4 is complete. □

Examples (continued). If, corresponding to Red-and-Black, $C_0 = \{(\lambda,1)\}$ where $\lambda > 0$, then, by Theorem 4, the proportional strategy given by $s(t) = \lambda X_t$ is optimal. If, as in roulette, $C_0 = \{(\lambda,\sigma): \sigma_1 \le \sigma \le \sigma_2\}$, then $s(t) = (\lambda/\sigma_1^2)X_t$ is optimal.

Consider now the problem of reaching 0 in minimum expected time from a position in $(-\infty,0]$ when the control set is constant. Formally, take

$$\underset{\sim}{F} = \{x \in R^2: x_1 \le 0,\ x_2 \in R^1\},$$

$$\underset{\sim}{u}(x) = -x_2.$$

Let $C_0 \subset R\times[0,\infty)$ and suppose $\mu > 0$ for some $(\mu,\sigma) \in C_0$. Define

$$\underset{\sim}{C}(x) = \{(\begin{bmatrix}\mu\\1\end{bmatrix},\begin{bmatrix}\sigma\\0\end{bmatrix}):(\mu,\sigma)\in C_0\} \quad \text{if } x_1 < 0$$

$$= \{(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix})\} \quad \text{if } x_1 = 0.$$

Let

$$\underset{\sim}{\Sigma}(x) = \{X \in \Sigma_{\underset{\sim}{C}}(x)\colon u(X) > -\infty\},$$

and let $\underset{\sim}{V}$ be the value function for the problem $(\underset{\sim}{F},\underset{\sim}{\Sigma},\underset{\sim}{u})$.

This problem is essentially the log of the problem considered in Theorem 4. So the next theorem is not surprising.

Define

$$\underset{\sim}{\mu} = \sup\{\mu\colon \exists\, \sigma \ni (\mu,\sigma) \in C_0\}.$$

<u>Theorem 5</u>. $\underset{\sim}{V}(x) = x_1/\underset{\sim}{\mu} - x_2$. If $(\underset{\sim}{\mu},\underset{\sim}{\sigma}) \in C_0$ for some $\underset{\sim}{\sigma}$, then the process X, for which

$$X_1(t) = x_1 + \underset{\sim}{\mu} t + \underset{\sim}{\sigma} W(t),$$

is optimal at x.

<u>Proof</u>: Apply Lemma 2. □

If the control set C_0 for X_1 depends on the position, the minimum expected time problem seems to be more difficult. This is because the optimal control at position x_1 may depend on other things than just the set $C_0(x_1)$. To see this, suppose that $C_0(x_1) = \{(0,0)\}$ for $x_1 \leq -1$ or $x_1 = 0$ and $C_0(x_1) = \{(s\lambda,s)\colon s \geq 0\}$ for $-1 < x_1 < 0$. The problem of reaching 0 in minimum time from a starting point in (-1,0) is just the Red-and-Black problem translated to the interval (-1,0). So $s(t) = \lambda(X_t+1)$ is optimal. However, if $C_0(x_1) = \{(s\lambda,s)\colon s \geq 0\}$ for all $x_1 < 0$, Theorem 5 applies to show $\underset{\sim}{V} = -x_2$ and s(t) should be taken very large.

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