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Quasi-Product Forms for Lévy-Driven Fluid Networks

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We study stochastic tree fluid networks driven by a multidimensional Lévy process. We are interested in (the joint distribution of) the steady-state content in each of the buffers, the busy periods, and the idle periods. To investigate these fluid networks, we relate the above three quantities to fluctuations of the input Lévy process by solving a multidimensional Skorokhod reflection problem. This leads to the analysis of the distribution of the componentwise maximums, the corresponding epochs at which they are attained, and the beginning of the first last-passage excursion. Using the notion of splitting times, we are able to find their Laplace transforms. It turns out that, if the components of the Lévy process are "ordered," the Laplace transform has a so-called quasi-product form.

The theory is illustrated by working out special cases, such as tandem networks and priority queues.

Key words: n-dimensional Lévy process; splitting time; quasi-product form; multidimensional Skorokhod problem; Lévy-driven fluid network; tree fluid network; buffer content; busy period; idle period
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1. Introduction. Prompted by a series of papers by Kella and Whitt (Kella [20, 22], Kella and Whitt [25, 26]), there has been considerable interest in multidimensional generalizations of the classical storage model with nondecreasing Lévy input and constant release rate (Prabhu [34, Ch. 4]). In the resulting networks, often called *stochastic fluid networks*, the input into the buffers is governed by a multidimensional Lévy process. Recently, motivated by work of Harrison and Williams on diffusion approximations (Harrison and Williams [16, 17]), the presence of product forms has been investigated (Kella [21, 23], Konstantopoulos et al. [28], Piera et al. [33]). Recall that the stationary buffer-content vector has a product form if it has independent components, meaning that the distribution of this vector is a product of the marginal distributions.

The results in these papers show that, apart from trivial cases, the stationary buffer-content vector of stochastic fluid networks *never* has a product form. Despite this "negative" result, we show that it may still be possible to express the joint distribution of the buffer content in terms of the marginal distributions. This is most evident in the Laplace domain. For certain tandem queues, for instance, the Laplace transform is a product that cannot be "separated"; we then say that the buffer-content vector has a *quasi-product form*.

In the literature on stochastic fluid networks, there has been a focus on the stationary buffer-content vector W or one of its components. Here, we are also interested in the stationary distribution of vector of ages of the busy periods B and idle periods I. The age of a busy (or idle) period is the amount of time that the buffer content has been positive (or zero) without being zero (positive). Knowing these, it is also possible to find the distribution of the remaining length of the busy (or idle) period and the total length of these periods.

We are interested in W, B, and I for a class of Lévy-driven fluid networks with a tree structure, which we therefore call *tree fluid networks*. Our analysis of these networks relies on a detailed study of a related multidimensional Skorokhod reflection problem (see, e.g., Robert [35]). Using its explicit solution, we relate the triplet of vectors (W, B, I) to the fluctuations of a multidimensional Lévy process X. We also prove that the stationary distribution of the buffer-content vector is unique.

Since our analysis of fluid tree networks is based on fluctuations of the process X, this paper also contributes to fluctuation theory for multidimensional Lévy processes. Supposing that each of the components of X drifts to $-\infty$, we write $\bar{X}$ for the (vector of) componentwise maximums of X, G for the corresponding epochs at which they are attained, and H for the beginning of the first last-passage excursion. Under a certain independence assumption, if the components of G are "ordered," we express the Laplace transform of $(\bar{X}, G)$ in terms of the transforms of the marginals $(\bar{X}_j, G_j)$. Since X_j is a real-valued Lévy process, the Laplace transform of $(\bar{X}_j, G_j)$ is known if X_j has one-sided jumps; see, for instance, Bertoin [4, Thm. VII.4].

We also examine the distribution of H under the measure $\mathbb{P}_k^{\downarrow}$, which is the law of X given that the process X_k stays nonpositive. There exists a vast body of literature on (one-dimensional) Lévy processes conditioned to stay nonpositive (or nonnegative); see the recent paper by Chaumont and Doney [5] for references. Under the measure $\mathbb{P}_k^{\downarrow}$, we also find the transform of $(\bar{X}, G)$. As a special case, we establish the Laplace transform of the maximum of a Lévy process conditioned to stay below a subordinator, such as a (deterministic) positive-drift process.

By exploiting the solution of the aforementioned Skorokhod problem, the results that we obtain for the process X can be cast immediately into the fluid-network setting. For instance, the knowledge of $(\bar{X}, G)$ allows us to derive the Laplace transform of the stationary distribution of (W, B) in a tandem network and a priority system if there are only positive jumps, allowing Brownian input at the "root" station. That is, we characterize the *joint* law of the buffer-content vector and the busy-period vector. With the $\mathbb{P}_k^{\downarrow}$-distribution of H, we establish the transform of the idle-period vector I for a special tandem network. Our formulas generalize all explicit results for tandem fluid networks known to date (in the Laplace domain), such as those obtained by Kella [20] and more recently by Dębicki et al. [6]. Most notably, quasi-products appear in our formulas, even for idle periods.

To derive our results, we make use of the notion of *splitting times*. These essentially allow us to reduce the problem to the one-dimensional case. For real-valued Markov processes, splitting times have been introduced by Jacobsen [18]. Splitting times decompose ("split") a sample path of a Markov process into two independent pieces. A full description of the process *before* and *after* the splitting time can be given. However, since the splitting time is not necessarily a stopping time, the law of the second piece may differ from the original law of the Markov process (refer to Millar [31, 32] for further details and to Kersting and Memişoğlu [27] for a recent contribution).

The idea to use splitting times in the context of stochastic networks is novel. The known results to date are obtained with Itô's formula (Konstantopoulos et al. [28]), a closely related martingale (Kella and Whitt [26]), or differential equations (Piera et al. [33]). Intuitively, these approaches all exploit a certain harmonicity. However, the results of Kyprianou and Palmowski [29] already indicate a relation between these approaches and splitting. Splitting has the advantage that it is insightful and that proofs are short. Moreover, it can also be used for studying more complicated systems (Dieker and Mandjes [8]).

This paper is essentially divided into two parts. In the first part, consisting of §§2–4, we analyze the fluctuations of an n-dimensional Lévy processes X. The notion of splitting times is formalized in §2. These splitting times are first used to study the distribution of $(\bar{X}, G)$ in §3, and then to analyze the distribution of H under $\mathbb{P}_k^{\downarrow}$ in §4. The second part of this paper deals with fluid networks. Section 5 ties these networks to fluctuations of X, so that the theory of the first part can be applied in §6. Finally, in Appendix A, we derive some results for compound Poisson processes with negative drift. They are used in §4.

2. Splitting times. This paper relies on the application of *splitting times* to a multidimensional Lévy process. After splitting times have been introduced, we study splitting at the maximum (§2.1) and splitting at a last-passage excursion (§2.2).

Throughout, let $X = (X_1, \ldots, X_n)'$ be an n-dimensional Lévy process, that is, a càdlàg process with stationary, independent increments such that $X(0) = 0 \in \mathbb{R}^n$. Without loss of generality, as in Bertoin [4], we work with the canonical measurable space $(\Omega, \mathcal{F}) = (D([0,\infty), \mathbb{R}^d \cup \{\partial\}), \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-field generated by the Skorokhod topology, and ∂ is an isolated point that serves as a cemetery state. In particular, X is the coordinate process. Unless otherwise stated, "almost surely" refers to $\mathbb{P}$. All vectors are column vectors.

The following assumption is used extensively throughout this paper:

D $X_k(t) \to -\infty$ almost surely, for every k.

We emphasize that a dependence between components is allowed. In the sequel, $\bar{X}_k(t)$ (or $\underline{X}_k(t)$) is shorthand for $\sup_{s\le t} X_k(s)$ (or $\inf_{s\le t} X_k(s)$). Due to **D**, $\bar{X}_k := \bar{X}_k(\infty)$ is well-defined and almost surely finite for every k. Furthermore, we write $\bar{X} = (\bar{X}_1, \ldots, \bar{X}_n)'$.

The following two definitions are key to further analysis. The second definition is closely related to the first, but somewhat more care is needed on a technical level. Intuitively, for the purposes of this paper, there is no need to distinguish the two definitions.

Definition 2.1. We say that a random time T is a *splitting time* for X under $\mathbb{P}$ if the two processes $\{X(t): 0 \le t \le T\}$ and $\{X(T+t) - X(T): t \ge 0\}$ are independent under $\mathbb{P}$. We say that T is a *splitting time from the left* for X under $\mathbb{P}$ if the two processes $\{X(t): 0 \le t < T\}$ and $\{X(T+t) - X(T-): t \ge 0\}$ are independent under $\mathbb{P}$.

Note that if X is a Lévy process under $\mathbb{P}$ with respect to some filtration $\mathcal{F}$ that includes the natural filtration, any $\mathcal{F}$-stopping time τ is a splitting time for X under $\mathbb{P}$. In fact, the Lévy assumption implies that $\{X(\tau + t) - X(\tau): t \ge 0\}$ is not only independent of $\{X(t): 0 \le t \le \tau\}$, but also that it has the same distribution as $\{X(t): t \ge 0\}$.

We need some notions related to the initial behavior of X. For $k=1,\dots,n$, set $\bar{R}_k=\inf\{t>0\colon X_k(t)=\bar{X}_k(t)\}$. Since $\{\bar{X}_k(t)-X_k(t)\colon t\ge 0\}$ is a Markov process under $\mathbb{P}$ with respect to the filtration generated by X (see Proposition VI.1 of Bertoin [4]), the Blumenthal zero-one law shows that either $\bar{R}_k>0$ almost surely (0 is then called *irregular* for $\{\bar{X}_k(t)-X_k(t)\colon t\ge 0\}$) or $\bar{R}_k=0$ almost surely (0 is then called *regular* for $\{\bar{X}_k(t)-X_k(t)\colon t\ge 0\}$). We also set $\underline{R}_k=\inf\{t>0\colon X_k(t)=\underline{X}_k(t)\}$, and define regularity of 0 for $\{X_k(t)-\underline{X}_k(t)\colon t\ge 0\}$ similarly as for $\{\bar{X}_k(t)-X_k(t)\colon t\ge 0\}$. If $\bar{R}_k=0$ almost surely, we introduce

$$\bar{S}_k=\bar{S}_k^X:=\inf\{t>0\colon X_k(t)\neq \bar{X}_k(t)\}.$$

Again, either $\bar{S}_k=0$ almost surely (0 is then called an *instantaneous point* for $\{\bar{X}_k(t)-X_k(t)\colon t\ge 0\}$) or $\bar{S}_k>0$ almost surely (0 is then called a *holding point* for $\{\bar{X}_k(t)-X_k(t)\colon t\ge 0\}$). One defines $\underline{S}_k$, instantaneous points, and holding points for $\{X_k(t)-\underline{X}_k(t)\colon t\ge 0\}$ similarly if $\underline{R}_k=0$.

2.1. Splitting at the maximum under $\mathbb{P}$. Let $G_k=G_k^X:=\inf\{t\ge 0\colon X_k(t)=\bar{X}_k \text{ or } X_k(t-)=\bar{X}_k\}$ be the (first) epoch that X_k "attains" its maximum, and write $G=(G_1,\dots,G_n)'$. Observe that G_k is well-defined and almost surely finite for every k by **D**.

Lemma 2.1. *Consider a Lévy process X that satisfies* **D**.

(i) *If $\bar{R}_k>0$ $\mathbb{P}$-almost surely or X_k is a compound Poisson process, then G_k is a splitting time for X under $\mathbb{P}$.*

(ii) *If $\bar{R}_k=0$ $\mathbb{P}$-almost surely but X_k is not a compound Poisson process, then G_k is a splitting time from the left for X under $\mathbb{P}$.*

Proof. We use ideas of Lemma VI.6 of Bertoin [4], who proves the one-dimensional case under exponential killing.

We start with the first case, in which the ascending ladder set is discrete. Set $\tau_0=0$ and define the stopping times $\tau_{n+1}=\inf\{t>\tau_n\colon \bar{X}_k(t)>\bar{X}_k(t-)\}$ for $n>0$. Write $N=\sup\{n\colon \tau_n<\infty\}$. Note that **D** implies that $N<\infty$ almost surely.

Let F and K be bounded functionals. Apply the Markov property to see that for $n\in\mathbb{Z}_+$,

$$\begin{aligned}
&\mathbb{E}[F(X(t),0\le t\le G_k)K(X(G_k+t)-X(G_k),t\ge 0);N=n]\\
&\quad=\mathbb{E}\big[F(X(t),0\le t\le \tau_n)1_{\{N\ge n\}}K(X(\tau_n+t)-X(\tau_n),t\ge 0)1_{\{\sup_{t\ge\tau_n}X_k(t)=X_k(\tau_n)\}}\big]\\
&\quad=\mathbb{E}\big[F(X(t),0\le t\le \tau_n)1_{\{N\ge n\}}\big]\mathbb{E}\big[K(X(\tau_n+t)-X(\tau_n),t\ge 0)1_{\{\sup_{t\ge\tau_n}X_k(t)=X_k(\tau_n)\}}\big]\\
&\quad=\mathbb{E}\big[F(X(t),0\le t\le \tau_n)1_{\{N\ge n\}}\big]\mathbb{E}\big[K(X(t),t\ge 0)1_{\{\sup_{t\ge 0}X_k(t)=0\}}\big].
\end{aligned}$$

Summing over n shows that the processes $\{X(t)\colon 0\le t\le G_k\}$ and $\{X(G_k+t)-X(G_k)\colon t\ge 0\}$ are independent.

The argument in the case $\bar{R}_k=0$ is more technical but essentially the same. The idea is to discretize the ladder height structure, for which we use the local time $\bar{\ell}_k$ at zero of the process $\{\bar{X}_k(t)-X_k(t)\colon t\ge 0\}$; see Bertoin [4, Ch. IV] for definitions. Note that $\bar{\ell}_k(\infty)<\infty$ almost surely by Assumption **D**.

Therefore, we fix some $\epsilon>0$ and denote the integer part of $\epsilon^{-1}\bar{\ell}_k(\infty)$ by $n=\lfloor \epsilon^{-1}\bar{\ell}_k(\infty)\rfloor$. A variation of the argument for $\bar{R}_k>0$ (using the additivity of the local time) shows that $\{X(t)\colon 0\le t\le \bar{\ell}_k^{-1}(n\epsilon)\}$ and $\{X(\bar{\ell}_k^{-1}(n\epsilon)+t)-X(\bar{\ell}_k^{-1}(n\epsilon))\colon t\ge 0\}$ are independent. According to Bertoin [4, Prop. IV.7(iii)], $\bar{\ell}_k^{-1}(n\epsilon)\uparrow G_k$ as $\epsilon\downarrow 0$, which proves the lemma. □

2.2. Splitting at a last-passage excursion under $\mathbb{P}_k^{\downarrow}$. Let $H_k=H_k^X:=\inf\{t\ge 0\colon \sup_{s\ge t}X_k(s)\neq X_k(t)\}$ be the beginning of the first last-passage excursion, and write $H=(H_1,\dots,H_n)'$.

In this subsection, we study the splitting properties of H_k for some fixed $k=1,\dots,n$. We suppose that 0 is a holding point for $\{X_k(t)-\underline{X}_k(t)\colon t\ge 0\}$, i.e., that $\underline{R}_k=0$ and $\underline{S}_k>0$ $\mathbb{P}$-almost surely. Under this condition, the event $\{\bar{X}_k=0\}$ has strictly positive probability. Therefore, one can straightforwardly define the conditional law $\mathbb{P}_k^{\downarrow}$ of X given $\bar{X}_k=0$.

It is our aim to investigate splitting of H_k under $\mathbb{P}_k^{\downarrow}$, but we only have knowledge of X under $\mathbb{P}$. As a first step, it is therefore useful to give a *sample path construction* of the law $\mathbb{P}_k^{\downarrow}$ on the canonical measurable space $(\Omega,\mathcal{F})$. For this, we define a process $X^{k\downarrow}$ by

$$X^{k\downarrow}(t)=\begin{cases}X(t) & \text{if } t\in\big[\underline{R}_k^{(j)},\underline{S}_k^{(j)}\big);\\ X\big(\underline{R}_k^{(j)}\big)-X\big(\big(\underline{R}_k^{(j)}+\underline{S}_k^{(j)}-t\big)-\big) & \text{if } t\in\big[\underline{S}_k^{(j)},\underline{R}_k^{(j)}\big),\end{cases}\tag{1}$$

where $\underline{R}_k^{(0)} = 0$, and for $j \geq 1$,

$$\underline{S}_k^{(j)} := \inf\{t > \underline{R}_k^{(j-1)}\colon \underline{X}_k(t) \neq X_k(t)\}, \qquad \underline{R}_k^{(j)} := \inf\{t > \underline{S}_k^{(j)}\colon \underline{X}_k(t) = X_k(t)\}.$$

In other words, $X^{k\downarrow}$ is constructed from the coordinate process X by "reverting" the excursions of $\{X_k(t) - \underline{X}_k(t)\colon t \geq 0\}$.

We have the following interesting lemma, which is the key to all results related to $\mathbb{P}_k^{\downarrow}$. For the random-walk analogue, refer to Doney [9].

Lemma 2.2. *Consider a Lévy process X that satisfies* **D**. *If $\underline{R}_k = 0$ and $\underline{S}_k > 0$ $\mathbb{P}$-almost surely, then $X^{k\downarrow}$ has law $\mathbb{P}_k^{\downarrow}$ under $\mathbb{P}$.*

Proof. Observe that $\bar{R}_k > 0$, and that the postmaximum process $\{X(G_k + t) - X(G_k)\colon t \geq 0\}$ has distribution $\mathbb{P}_k^{\downarrow}$ (a proof of this uses similar arguments as in the proof of Lemma 2.1; see Millar [31, 32] for more details).

Fix some $q > 0$ and let e_q be an exponentially distributed random variable, independent of X (obviously, one must then enlarge the probability space). The first step is to construct the law of $\{X(G_k^q + t) - X(G_k^q)\colon 0 \leq t < e_q - G_k^q\}$, where $G_k^q := \inf\{t < e_q\colon X_k(t) = \bar{X}_k(e_q) \text{ or } X_k(t-) = \bar{X}_k(e_q)\}$. By the time-reversibility of X (Bertoin [4, Lem. II.2]), it is equivalent to construct the law of $\{X(F_k^q) - X((F_k^q - t)-)\colon 0 \leq t < F_k^q\}$, where $F_k^q := \sup\{t < e_q\colon X_k(t) = \underline{X}_k(e_q) \text{ or } X_k(t-) = \underline{X}_k(e_q)\}$.

To do so, we use ideas from Greenwood and Pitman [13]. Let $\underline{\ell}_k$ be the local time of $\{X_k(t) - \underline{X}_k(t)\colon t \geq 0\}$ at zero (since $\underline{R}_k = 0$, $\bar{S}_k > 0$, we refer to Bertoin [4, Sec. IV.5] for its construction). Its right-continuous inverse is denoted by $\underline{\ell}_k^{-1}$. The X-excursion at local time s, denoted by X^s, is the càdlàg process defined by

$$X^s(u) := X((\underline{\ell}_k^{-1}(s-) + u) \wedge \underline{\ell}_k^{-1}(s)) - X(\underline{\ell}_k^{-1}(s-)-), \quad u \geq 0.$$

If $\underline{\ell}_k^{-1}(s-) = \underline{\ell}_k^{-1}(s)$, then we let X^s be ∂, the zero function that serves as a cemetery. Since $\{X^s\colon s > 0\}$ is a càdlàg-valued Poisson point process as a result of **D**, one can derive (e.g., with the arguments of Lemma II.2 and Lemma VI.2 of Bertoin [4]) that the process

$$W := \{W(s) = (D(s), X^s)\colon s > 0\}$$

is time-reversible, where $D(s) := X(\underline{\ell}_k^{-1}(s))$. After setting $\sigma_q := \underline{\ell}_k^{-1}(e_q)$, it can be seen that this implies that $\{(D(s), X^s)\colon 0 < s < \sigma_q\}$ and $\{(D(\sigma_q-) - D((\sigma_q - s)-), X^{\sigma_q - s})\colon 0 < s < \sigma_q\}$ have the same distribution. In other words, one can construct the law of $\{X(F_k^q) - X((F_k^q - t)-)\colon 0 \leq t < F_k^q\}$ from the law of $\{X(t)\colon 0 \leq t < F_k^q\}$ by "reverting" excursions as in (1).

It remains to show that this construction is "consistent" in the sense of Kolmogorov, so that one can let $q \to 0$ to obtain the claim. For this, note that the family $\{\sigma_q\}$ can be coupled with a single random variable through $\sigma_q = \underline{\ell}_k^{-1}(e_1/q)$. $\square$

We now study the splitting properties of H_k using the alternative construction of $\mathbb{P}_k^{\downarrow}$ given in Lemma 2.2. Since $\underline{S}_k^{(1)}$ is a $\mathbb{P}$-stopping time with respect to the (completed) natural filtration of X, the Markov property of X under $\mathbb{P}$ with respect to this filtration (Bertoin [4, Prop. I.6]) immediately yields the following analogue of Lemma 2.1.

Lemma 2.3. *Consider a Lévy process X that satisfies* **D**. *If $\underline{R}_k = 0$ and $\underline{S}_k > 0$ $\mathbb{P}$-almost surely, then H_k is a splitting time for X under $\mathbb{P}_k^{\downarrow}$. Moreover, it has an exponential distribution under $\mathbb{P}_k^{\downarrow}$.*

We remark that the construction and analysis of $\mathbb{P}_k^{\downarrow}$ is the easiest under the assumption that $\underline{R}_k = 0$ and $\underline{S}_k > 0$ $\mathbb{P}$-almost surely, which is exactly what we need in the remainder. A vast body of literature is devoted to the case $n = 1$, and the measure $\mathbb{P}_1^{\downarrow}$ is then studied under the assumption that $\bar{R}_1 = 0$. This is challenging from a theoretical point of view, since the condition that the process stays negative has $\mathbb{P}$-probability zero. Therefore, much more technicalities are needed to treat this case. We refer to Bertoin [3] and Doney [9] for more details. See also Chaumont and Doney [5].

3. The $\mathbb{P}$-distribution of $(\bar{X}, G)$.

The aim of this section is to find the Laplace transform of the distribution of $(\bar{X}, G)$, assuming some additional structure on the process X. Thus, in the sequel we write $X_k \prec X_j$ if there exists some $K_{kj} > 0$ such that $X_j - K_{kj} X_k$ is nondecreasing almost surely.

Lemma 3.1. *Suppose the Lévy process X satisfies* **D**. *If $X_k \prec X_j$, then $G_k \leq G_j$.*

Proof. First note that $G_k, G_j < \infty$ as a consequence of **D**. To prove the claim, let us assume instead that $G_j < G_k$ while $\widehat{X}(t) := X_j(t) - C X_k(t)$ is nondecreasing for some arbitrary $C > 0$. Suppose that $X_k(G_k) = \bar{X}_k$

and $X_j(G_j)=\bar X_j$; the argument can be repeated if, for instance, $X_k(G_k-)=\bar X_k$. The assumption $G_j<G_k$ implies that

$$0\le \widehat X(G_k)-\widehat X(G_j)=X_j(G_k)-\bar X_j - C[\bar X_k - X_k(G_j)]\le 0,$$

meaning that $\bar X_k = X_k(G_j)$. This contradicts $G_j<G_k$ in view of the definition of G_k. □

The following proposition expresses the distribution of $(\bar X, G)$ in terms of those of $(X(G_k),G_k)$ and $(X(G_k-),G_k)$. We denote the scalar product of x and y in $\mathbb{R}^n$ by $\langle x,y\rangle$, and we write "cpd Ps" for "compound Poisson." Throughout this paper, the expression $\prod_j\alpha_j\times\prod_j\beta_j\times\gamma$ should be read as $(\prod_j\alpha_j)\times(\prod_j\beta_j)\times\gamma$.

Proposition 3.1. *Suppose that X is an n-dimensional Lévy process satisfying* **D** *and that $X_1\prec X_2\prec\cdots\prec X_n$. Then for any $\alpha,\beta\in\mathbb{R}^n_+$,*

$$\mathbb{E}\,e^{-\langle\alpha,G\rangle-\langle\beta,\bar X\rangle} = \prod_{\substack{j=1\\ \bar R_j>0 \text{ or } X_j \text{ cpd Ps}}}^{n-1}\frac{\mathbb{E}\,e^{-[\sum_{\ell=j}^n\alpha_\ell]G_j-\sum_{\ell=j}^n\beta_\ell X_\ell(G_j)}}{\mathbb{E}\,e^{-[\sum_{\ell=j+1}^n\alpha_\ell]G_j-\sum_{\ell=j+1}^n\beta_\ell X_\ell(G_j)}}$$
$$\times\prod_{\substack{j=1\\ \bar R_j=0,\, X_j \text{ not cpd Ps}}}^{n-1}\frac{\mathbb{E}\,e^{-[\sum_{\ell=j}^n\alpha_\ell]G_j-\sum_{\ell=j}^n\beta_\ell X_\ell(G_j-)}}{\mathbb{E}\,e^{-[\sum_{\ell=j+1}^n\alpha_\ell]G_j-\sum_{\ell=j+1}^n\beta_\ell X_\ell(G_j-)}}\times\mathbb{E}\,e^{-\alpha_nG_n-\beta_n\bar X_n}.$$

Proof. First observe that the assumptions imply that the terms $X_\ell(G_j)$ and $X_\ell(G_j-)$ in the formula are nonnegative for $\ell\ge j$, which legitimates the use of the Laplace transforms. Remark also that $\bar R_i=0$ for $i>j$ whenever $\bar R_j=0$, i.e., for some deterministic i_0 we have $\bar R_i>0$ for $i\le i_0$ and $\bar R_i=0$ for $i>i_0$.

Let us first suppose that $\bar R_j>0$ or that X_j is a compound Poisson process. We prove that for $j=1,\ldots,n-1$,

$$\mathbb{E}\,e^{-\sum_{\ell=j}^n\alpha_\ell G_\ell-\sum_{\ell=j}^n\beta_\ell\bar X_\ell}=\frac{\mathbb{E}\,e^{-[\sum_{\ell=j}^n\alpha_\ell]G_j-\sum_{\ell=j}^n\beta_\ell X_\ell(G_j)}}{\mathbb{E}\,e^{-[\sum_{\ell=j+1}^n\alpha_\ell]G_j-\sum_{\ell=j+1}^n\beta_\ell X_\ell(G_j)}}\,\mathbb{E}\,e^{-\sum_{\ell=j+1}^n\alpha_\ell G_\ell-\sum_{\ell=j+1}^n\beta_\ell\bar X_\ell}.$$

The key observations are that $\bar X_j=X_j(G_j)$ and that $G_\ell\ge G_j$ almost surely for $\ell=j,\ldots,n$ by Lemma 3.1. The fact that G_j is a splitting time by Lemma 2.1(i) then yields

$$\begin{aligned}\mathbb{E}\,e^{-\sum_{\ell=j}^n\alpha_\ell G_\ell-\sum_{\ell=j}^n\beta_\ell\bar X_\ell}&=\mathbb{E}\,e^{-[\sum_{\ell=j}^n\alpha_\ell]G_j-\sum_{\ell=j}^n\beta_\ell X_\ell(G_j)}e^{-\sum_{\ell=j+1}^n\alpha_\ell[G_\ell-G_j]-\sum_{\ell=j+1}^n\beta_\ell[\bar X_\ell-X_\ell(G_j)]}\\&=\mathbb{E}\,e^{-[\sum_{\ell=j}^n\alpha_\ell]G_j-\sum_{\ell=j}^n\beta_\ell X_\ell(G_j)}\,\mathbb{E}\,e^{-\sum_{\ell=j+1}^n\alpha_\ell[G_\ell-G_j]-\sum_{\ell=j+1}^n\beta_\ell[\bar X_\ell-X_\ell(G_j)]}.\end{aligned}\tag{2}$$

The latter factor, which is rather complex to analyze directly, can be computed on choosing $\alpha_j=\beta_j=0$ in the above display.

Repeating this argument for the case $\bar R_j=0$ yields with Lemma 2.1(i), provided that X_j is not a compound Poisson process,

$$\mathbb{E}\,e^{-\sum_{\ell=j}^n\alpha_\ell G_\ell-\sum_{\ell=j}^n\beta_\ell\bar X_\ell}=\frac{\mathbb{E}\,e^{-[\sum_{\ell=j}^n\alpha_\ell]G_j-\beta_j\bar X_j-\sum_{\ell=j+1}^n\beta_\ell X_\ell(G_j-)}}{\mathbb{E}\,e^{-[\sum_{\ell=j+1}^n\alpha_\ell]G_j-\sum_{\ell=j+1}^n\beta_\ell X_\ell(G_j-)}}\,\mathbb{E}\,e^{-\sum_{\ell=j+1}^n\alpha_\ell G_\ell-\sum_{\ell=j+1}^n\beta_\ell\bar X_\ell}.$$

It is shown in the proof of Theorem VI.5(i) of Bertoin [4] that $\bar X_j=X_j(G_j-)$ almost surely, and this proves the claim. □

In the rest of this section, the following assumption is imposed.

G For $j=1,\ldots,n-1$, we have

$$X_{j+1}(t)=K_{j+1}X_j(t)+\Upsilon_{j+1}(t),\tag{3}$$

where $(\Upsilon_2,\ldots,\Upsilon_n)$ are mutually independent nonnegative subordinators and $K_2,\ldots,K_n$ are strictly positive.

Note that Assumption **G** implies $X_1\prec X_2\prec\cdots\prec X_n$. Moreover, it entails that for $j=1,\ldots,n-1$ and $\ell\ge j$, we have

$$X_\ell(t)=K_j^\ell X_j(t)+\sum_{i=j+1}^{\ell}K_i^\ell\Upsilon_i(t),$$

where we have set $K_j^\ell=\prod_{i=j+1}^\ell K_i$ and $K_j^j=1$. In other words, X_ℓ can be written as the sum of X_j and $\ell-j$ independent processes, which are all mutually independent and independent of X_j.

The following reformulation of (3) in terms of matrices is useful in §6. Let K be the upper triangular matrix with element $(i,i+1)$ equal to K_{i+1} for $i=1,\ldots,n-1$, and zero elsewhere. Also write $\Upsilon(t):=(\Upsilon_1(t),\ldots,\Upsilon_n(t))'$, where $\Upsilon_1(t)=X_1(t)$. Equation (3) is then nothing else than the identity $X(t)=(I-K')^{-1}\cdot\Upsilon(t)$. The matrix $(I-K')^{-1}$ is lower triangular, and element (i,j) equals K_j^i for $j\ge i$.

The *cumulant* of the subordinator $\Upsilon_j(t)$ is defined as

$$\theta_j^{\Upsilon}(\beta) := -\log \mathbb{E}\,\mathrm{e}^{-\beta\Upsilon_j(1)}$$

for $\beta \geq 0$ and $j = 2, \ldots, n$.

The following theorem expresses the joint Laplace transform of $(\bar{X}, G)$ in terms of its marginal distributions and the cumulants θ^{Υ}. However, except for trivial cases, the Laplace transform is *not* the product of marginal Laplace transforms. Still, it can be expressed in terms of these marginal transforms in a product-type manner. We call this a *quasi-product* form.

Theorem 3.1. *Suppose that X is an n-dimensional Lévy process satisfying* **D** *and* **G**. *Then for any* $\alpha, \beta \in \mathbb{R}_+^n$, *the transform* $\mathbb{E}\,\mathrm{e}^{-\langle \alpha, G\rangle - \langle \beta, \bar{X}\rangle}$ *equals*

$$\prod_{j=1}^{n-1} \frac{\mathbb{E}\,\mathrm{e}^{-[\sum_{\ell=j}^n \alpha_\ell + \sum_{\ell=j+1}^n \theta_\ell^{\Upsilon}(\sum_{k=\ell}^n K_\ell^k \beta_k)]G_j - [\sum_{\ell=j}^n K_j^\ell \beta_\ell]\bar{X}_j}}{\mathbb{E}\,\mathrm{e}^{-[\sum_{\ell=j+1}^n \alpha_\ell + \sum_{\ell=j+1}^n \theta_\ell^{\Upsilon}(\sum_{k=\ell}^n K_\ell^k \beta_k)]G_j - [\sum_{\ell=j+1}^n K_j^\ell \beta_\ell]\bar{X}_j}} \times \mathbb{E}\,\mathrm{e}^{-\alpha_n G_n - \beta_n \bar{X}_n}.$$

Proof. Let j be such that $\bar{R}_j > 0$ or X_j is compound Poisson. By Assumption **G**, we then have for $a \in \mathbb{R}_+$,

$$\begin{aligned}\mathbb{E}\,\mathrm{e}^{-aG_j - \sum_{\ell=j}^n \beta_\ell X_\ell(G_j)} &= \mathbb{E}\,\mathrm{e}^{-aG_j - [\sum_{\ell=j}^n K_j^\ell \beta_\ell]X_j(G_j) - \sum_{\ell=j+1}^n [\sum_{k=\ell}^n K_\ell^k \beta_k]\Upsilon_\ell(G_j)} \\ &= \mathbb{E}(\mathrm{e}^{-aG_j - [\sum_{\ell=j}^n K_j^\ell \beta_\ell]X_j(G_j)}\,\mathbb{E}[\mathrm{e}^{-\sum_{\ell=j+1}^n [\sum_{k=\ell}^n K_\ell^k \beta_k]\Upsilon_\ell(G_j)} \mid G_j]) \\ &= \mathbb{E}\,\mathrm{e}^{-[a + \sum_{\ell=j+1}^n \theta_\ell^{\Upsilon}(\sum_{k=\ell}^n K_\ell^k \beta_k)]G_j - [\sum_{\ell=j}^n K_j^\ell \beta_\ell]X_j(G_j)}.\end{aligned}$$

The claim now follows from Proposition 3.1 and from $X_j(G_j) = \bar{X}_j$ almost surely.

If $\bar{R}_j = 0$ but not a compound Poisson process, the same argument gives the joint transform of $\{X_\ell(G_j-)\colon \ell = j, \ldots, n\}$ and G_j. In the resulting formula, $X_j(G_j-)$ can be replaced by $X_j(G_j)$, as outlined in the proof of Theorem VI.5(i) in Bertoin [4]. $\square$

The following corollary shows that Theorem 3.1 not only completely characterizes the law of $(\bar{X}, G)$ under $\mathbb{P}$, but also its law conditioned on one component to stay nonpositive. Indeed, let $\mathbb{P}_k^{\downarrow}$ be the law of $\{X(G_k + t) - X(G_k)\colon t \geq 0\}$ for $k = 1, \ldots, n$; it can be checked that this measure equals $\mathbb{P}_k^{\downarrow}$ as defined in §2.2 in case $\underline{R}_k = 0$ and $\underline{S}_k > 0$ $\mathbb{P}$-almost surely. Note that $\mathbb{P}_k^{\downarrow}$ can be regarded as the law of X given that X_k stays nonpositive.

Corollary 3.1. *For* $\alpha, \beta \in \mathbb{R}_+^n$, *we have*

$$\mathbb{E}_k^{\downarrow}\,\mathrm{e}^{-\langle \alpha, \bar{X}\rangle - \langle \beta, G\rangle} = \prod_{j=k}^{n-1} \frac{\mathbb{E}\,\mathrm{e}^{-[\sum_{\ell=j+1}^n \alpha_\ell + \sum_{\ell=j+2}^n \theta_\ell^{\Upsilon}(\sum_{i=\ell}^n K_\ell^i \beta_i)]G_{j+1} - [\sum_{\ell=j+1}^n K_{j+1}^\ell \beta_\ell]\bar{X}_{j+1}}}{\mathbb{E}\,\mathrm{e}^{-[\sum_{\ell=j+1}^n \alpha_\ell + \sum_{\ell=j+1}^n \theta_\ell^{\Upsilon}(\sum_{i=\ell}^n K_\ell^i \beta_i)]G_j - [\sum_{\ell=j+1}^n K_j^\ell \beta_\ell]\bar{X}_j}}.$$

Proof. Directly from Theorem 3.1 and (2). $\square$

In particular, this corollary characterizes the law of the maximum of a Lévy process given that it stays below a subordinator. It provides further motivation for studying the law of the vector H under $\mathbb{P}_k^{\downarrow}$.

4. The $\mathbb{P}_k^{\downarrow}$-distribution of H. The aim of this section is to find the Laplace transform of the distribution of H under $\mathbb{P}_k^{\downarrow}$ under the assumption that 0 is a holding point for $\{X_k(t) - \underline{X}_k(t)\colon t \geq 0\}$ under $\mathbb{P}$.

We try to follow the same train of thoughts that led us to the results in §3. This analogy leads to Proposition 4.1, which does not yet give the Laplace transform of the distribution of H under $\mathbb{P}_k^{\downarrow}$. Therefore, we need an auxiliary result, formulated as Lemma 4.1, which relies on Appendix A. Finally, Proposition 4.2 enables us to find the Laplace transform of the distribution of H under $\mathbb{P}_k^{\downarrow}$.

As in the previous section, additional assumptions are imposed on the Lévy process X. Here, they are significantly more restrictive. The following Assumption **H** plays a similar role in the present section as Assumption **G** in §3. Note that it implies $X_1 \prec X_2 \prec \cdots \prec X_n$.

H Let $\Pi = \{\Pi(t)\colon t \geq 0\}$ be a compound Poisson process with positive jumps only. For each $j = 1, \ldots, n$, we have

$$X_j(t) = \Pi(t) - c_j t,$$

where c_j decreases strictly in j.

In the remainder of this section, we write $\lambda \in (0, \infty)$ for the intensity of jumps of Π. We also set $\rho_k^{(n)} := \sup\{\underline{R}_k^{(j)}\colon \underline{R}_k^{(j)} \leq \underline{R}_n^{(1)}\}$ and $\sigma_k^{(n)} := \sup\{\underline{S}_k^{(j)}\colon \underline{S}_k^{(j)} \leq \underline{R}_n^{(1)}\}$. In particular, $\rho_n^{(n)} = \underline{R}_n^{(1)}$ and $\sigma_n^{(n)} = \underline{S}_n^{(1)}$. Also, we write for $\beta \geq 0$ and $i = 1, \ldots, n$,

$$\psi_i(\beta) := \log \mathbb{E}\,\mathrm{e}^{-\beta X_i(1)}$$

for the *Laplace exponent* of $-X_i$. Since we assume **D**, ψ_i is strictly increasing on $\mathbb{R}_+$, see the proof of Corollary VII.2 of Bertoin [4]. Therefore, we can define Φ_i as the inverse of ψ_i. The function Φ_i plays an important role in this section.

Recall that we used n splitting times to arrive at Proposition 3.1. Here, we only know that H_k is a splitting time for X under $\mathbb{P}_k^{\downarrow}$ (see Lemma 2.3). In general, however, H_i $(i<k)$ is not a splitting time under $\mathbb{P}_k^{\downarrow}$, and the similarity with Proposition 3.1 is lost.

Proposition 4.1. *Suppose the Lévy process X satisfies* **D**. *For $\gamma \in \mathbb{R}_+^k$, we have*

$$\mathbb{E}_k^{\downarrow} \mathrm{e}^{-\sum_{j=1}^k \gamma_j H_j} = \frac{\lambda}{\lambda + \sum_{j=1}^k \gamma_j} \mathbb{E}\, \mathrm{e}^{-\sum_{j=1}^{k-1} \gamma_j \left(\rho_k^{(k)} - \rho_j^{(k)}\right)}.$$

Proof. Lemma 2.3 yields

$$\mathbb{E}_k^{\downarrow} \mathrm{e}^{-\sum_{j=1}^k \gamma_j H_j} = \mathbb{E}_k^{\downarrow} \mathrm{e}^{-\left(\sum_{j=1}^k \gamma_j\right) H_k}\, \mathbb{E}_k^{\downarrow} \mathrm{e}^{-\sum_{j=1}^{k-1} \gamma_j (H_j - H_k)}.$$

In the discussion following (1), we have seen that there is a simple sample-path correspondence between the laws $\mathbb{P}_k^{\downarrow}$ and $\mathbb{P}$. This yields immediately that H_k is exponentially distributed under $\mathbb{P}_k^{\downarrow}$ with parameter λ. It also gives that the $\mathbb{P}_k^{\downarrow}$-distribution of $\{H_j - H_k\colon j=1,\dots,k-1\}$ is the same as the $\mathbb{P}$-distribution $\{\rho_k^{(k)} - \rho_j^{(k)}\colon j = 1,\dots,k-1\}$. $\square$

Motivated by the preceding proposition, we now focus on the calculation of the distribution of the $\rho_k^{(k)} - \rho_j^{(k)}$ (that is, their joint Laplace transform). For this, we apply results from Appendix A.

The following lemma is of crucial importance, as it provides a recursion for the transform of $\{\rho_{j+1}^{(i)} - \rho_j^{(i)}\colon j = 1,\dots,i-1\}$ and $\{\rho_j^{(i)} - \sigma_j^{(i)}\colon j=1,\dots,i\}$ in terms of the transform of the same family with superscript $(i-1)$. The transforms of the marginals $\rho_i^{(i)} - \sigma_i^{(i)}$ and $\rho_{i-1}^{(i-1)} - \sigma_{i-1}^{(i-1)}$ also appear in the expression, but these transforms are known: for $\gamma \ge 0$, $i=1,\dots,n$ (cf. the proof of Proposition A.1),

$$\lambda\, \mathbb{E}\, \mathrm{e}^{-\gamma\left(\rho_i^{(i)} - \sigma_i^{(i)}\right)} = \lambda + \gamma - c_i \Phi_i(\gamma). \tag{4}$$

Lemma 4.1. *Suppose that X is an n-dimensional Lévy process satisfying* **D** *and* **H**. *Then for any $i = 2,\dots,n$, $\beta \in \mathbb{R}_+^{i-1}$, $\gamma \in \mathbb{R}_+^i$, we have the following recursion*:

$$\begin{aligned}
&\mathbb{E}\, \mathrm{e}^{-\sum_{j=1}^{i-1} \beta_j \left(\rho_{j+1}^{(i)} - \rho_j^{(i)}\right) - \sum_{j=1}^{i} \gamma_j \left(\rho_j^{(i)} - \sigma_j^{(i)}\right)} \\
&\quad = \frac{\beta_{i-1} + \lambda\, \mathbb{E}\, \mathrm{e}^{-\gamma_i \left(\rho_i^{(i)} - \sigma_i^{(i)}\right)}}{\beta_{i-1} + \lambda\, \mathbb{E}\, \mathrm{e}^{-[((c_{i-1}/c_i)-1)(\lambda+\beta_{i-1}) + (c_{i-1}/c_i)\gamma_i]\left(\rho_{i-1}^{(i-1)} - \sigma_{i-1}^{(i-1)}\right)}} \\
&\qquad \times \mathbb{E}\, \mathrm{e}^{-\sum_{j=1}^{i-2} \beta_j \left(\rho_{j+1}^{(i-1)} - \rho_j^{(i-1)}\right) - \sum_{j=1}^{i-2} \gamma_j \left(\rho_j^{(i-1)} - \sigma_j^{(i-1)}\right) - [((c_{i-1}/c_i)-1)(\lambda+\beta_{i-1}) + (c_{i-1}/c_i)\gamma_i + \gamma_{i-1}]\left(\rho_{i-1}^{(i-1)} - \sigma_{i-1}^{(i-1)}\right)}.
\end{aligned}$$

Proof. Fix some $i = 2,\dots,n$, and consider the process X_{i-1} between $\sigma_i^{(i)}$ and $\rho_i^{(i)}$. There are several excursions (at least one) of the process $\{X_{i-1}(t) - \underline{X}_{i-1}(t)\colon t \ge 0\}$ away from 0 between $\sigma_i^{(i)}$ and $\rho_i^{(i)}$, and we call these excursions the $(i-1)$-subexcursions. Each $(i-1)$-subexcursion contains excursions of the processes $\{X_\ell(t) - \underline{X}_\ell(t)\colon t\ge 0\}$ for $\ell < i-1$; we call these the ℓ-subexcursions. To each $(i-1)$-subexcursion, we assign $2i-4$ marks, namely two for each of the $i-2$ types of further subexcursions. The first mark corresponds to the length of the last ℓ-subexcursion in the $(i-1)$-subexcursion, and the second to the difference between the end of the last ℓ-subexcursion and the end of the $(\ell+1)$-subexcursion. Observe that these marks are independent for every $(i-1)$-subexcursion between $\sigma_i^{(i)}$ and $\rho_i^{(i)}$, and that their distributions are equal to those of $\{\rho_\ell^{(i-1)} - \sigma_\ell^{(i-1)}\colon \ell = 1,\dots,i-2\}$ (the first marks) and $\{\rho_{\ell+1}^{(i-1)} - \rho_\ell^{(i-1)}\colon \ell=1,\dots,i-2\}$ (the second marks).

The idea is to apply Proposition A.1 to the process

$$Z(x) := \inf\left\{t \ge 0\colon X_{i-1}\left(\sigma_i^{(i)}\right) - X_{i-1}\left(\sigma_i^{(i)} + t\right) = x\right\} - \frac{x}{c_{i-1} - c_i},$$

see Figure 1. In this diagram, excursions of $\{X_{i-1}(t) - \underline{X}_{i-1}(t)\colon t\ge 0\}$ correspond to jumps of Z. The relevant information on the subexcursions is incorporated into Z as jump marks.

Observe that Z is a compound Poisson process with negative drift $1/c_{i-1} - 1/(c_{i-1} - c_i)$ and intensity λ/c_{i-1}, starting with a (marked) jump at zero. The jumps of Z correspond to $(i-1)$-excursions, and the above marks are assigned to the each of the jumps. In terms of Proposition A.1, it remains to observe that $\rho_i^{(i)} - \rho_{i-1}^{(i)}$ and $\rho_i^{(i)} - \sigma_i^{(i)}$ correspond to $(\tau_- - T_{N_-})/c_{i-1}$ and $\tau_-/(c_{i-1} - c_i)$, respectively. $\square$

With the recursion of Lemma 4.1, we can find the joint transform of $\rho_k^{(k)} - \rho_j^{(k)}$ for $j = 1,\dots,k-1$, which is required to work out Proposition 4.1. This is done in (14) below. It is equivalent to find the transform of

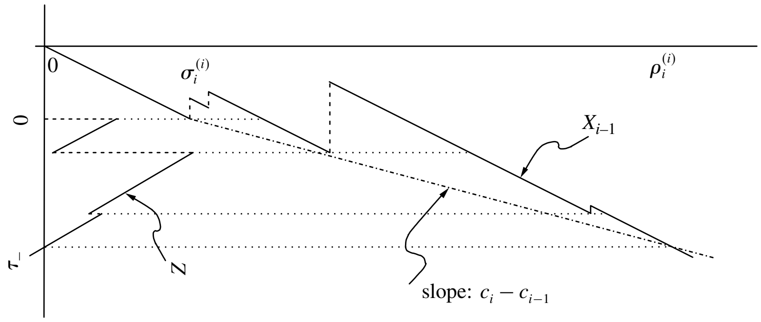

FIGURE 1. Excursions of $\{X_{i-1}(t) - \underline{X}_{i-1}(t)\colon t \geq 0\}$ correspond to jumps of Z.

$\rho_{j+1}^{(k)} - \rho_j^{(k)}$ for $j = 1, \ldots, k-1$, which is the content of the next proposition. We have also added $\rho_k^{(k)} - \sigma_k^{(k)}$ for convenience. The resulting formula has some remarkable features similar to the formula in Theorem 3.1. Most interestingly, a quasi-product form appears here as well.

For $\beta \in \mathbb{R}_+^{k-1} \geq 0$, and $j = 1, \ldots, k-1$, we define

$$\mathscr{C}_j^k(\beta) := c_j \sum_{\ell=j}^{k-1} \left(\frac{1}{c_{\ell+1}} - \frac{1}{c_\ell} \right) (\lambda + \beta_\ell).$$

PROPOSITION 4.2. *Suppose that X is an n-dimensional Lévy process satisfying* **D** *and* **H**. *Then for any $k = 2, \ldots, n$, $\beta \in \mathbb{R}_+^{k-1}$, $\gamma \geq 0$, we have*

$$\mathbb{E}\, e^{-\sum_{j=1}^{k-1} \beta_j \left(\rho_{j+1}^{(k)} - \rho_j^{(k)}\right) - \gamma\left(\rho_k^{(k)} - \sigma_k^{(k)}\right)} = \prod_{j=1}^{k-1} \frac{\beta_j + \lambda\, \mathbb{E}\, e^{-\left[\mathscr{C}_{j+1}^k(\beta) + (c_{j+1}/c_k)\gamma\right]\left(\rho_{j+1}^{(j+1)} - \sigma_{j+1}^{(j+1)}\right)}}{\beta_j + \lambda\, \mathbb{E}\, e^{-\left[\mathscr{C}_j^k(\beta) + (c_j/c_k)\gamma\right]\left(\rho_j^{(j)} - \sigma_j^{(j)}\right)}} \times \mathbb{E}\, e^{-\left[\mathscr{C}_1^k(\beta) + (c_1/c_k)\gamma\right]\left(\rho_1^{(1)} - \sigma_1^{(1)}\right)}.$$

PROOF. Since for $\ell = 2, \ldots, i$, by definition of $\mathscr{C}_\ell^k(\beta)$,

$$\left(\frac{c_{\ell-1}}{c_\ell} - 1 \right) (\lambda + \beta_{\ell-1}) + \frac{c_{\ell-1}}{c_\ell} \mathscr{C}_\ell^k(\beta) = \mathscr{C}_{\ell-1}^k(\beta),$$

it follows from Lemma 4.1 that

$$\frac{\mathbb{E}\, e^{-\sum_{j=1}^{\ell-1} \beta_j \left(\rho_{j+1}^{(\ell)} - \rho_j^{(\ell)}\right) - \left[\mathscr{C}_\ell^k(\beta) + (c_\ell/c_k)\gamma\right]\left(\rho_\ell^{(\ell)} - \sigma_\ell^{(\ell)}\right)}}{\mathbb{E}\, e^{-\sum_{j=1}^{\ell-2} \beta_j \left(\rho_{j+1}^{(\ell-1)} - \rho_j^{(\ell-1)}\right) - \left[\mathscr{C}_{\ell-1}^k(\beta) + (c_{\ell-1}/c_k)\gamma\right]\left(\rho_{\ell-1}^{(\ell-1)} - \sigma_{\ell-1}^{(\ell-1)}\right)}} = \frac{\beta_{\ell-1} + \lambda\, \mathbb{E}\, e^{-\mathscr{C}_\ell^k(\beta,\gamma)\left(\rho_\ell^{(\ell)} - \sigma_\ell^{(\ell)}\right)}}{\beta_{\ell-1} + \lambda\, \mathbb{E}\, e^{-\mathscr{C}_{\ell-1}^k(\beta,\gamma)\left(\rho_{\ell-1}^{(\ell-1)} - \sigma_{\ell-1}^{(\ell-1)}\right)}}.$$

The claim follows from this recursion (start with $\ell = k$ and note that $\mathscr{C}_k^k(\beta) = 0$). $\square$

5. Multidimensional Skorokhod problems. In the next sections, we apply results of the previous sections to the analysis of *fluid networks*. Such networks are closely related to (multidimensional) *Skorokhod reflection problems*, which we describe first. Subject to certain assumptions, we explicitly solve such a reflection problem in §5.1. Section 5.2 describes the fluid networks associated to these special Skorokhod problems.

Let P be a nonnegative matrix with spectral radius strictly smaller than one. To a given càdlàg function Y with values in $\mathbb{R}^n$ such that $Y(0) = 0$, one can associate a càdlàg pair (W, L) with the following properties ($w \in \mathbb{R}_+^n$):

S1 $W(t) = w + Y(t) + (I - P')L(t)$, $t \geq 0$,
S2 $W(t) \geq 0$, $t \geq 0$ and $W(0) = w$,
S3 $L(0) = 0$ and L is nondecreasing, and
S4 $\sum_{j=1}^n \int_0^\infty W_j(t)\, dL_j(t) = 0$.

It is known that such a pair exists and that it is unique; see Harrison and Reiman [15] for the continuous case, Robert [35] or Whitt [37, Thm. 14.2.3] for the càdlàg case, and Kella [24] for a more general result.

It is said that (W, L) is the solution to the Skorokhod problem of Y in $\mathbb{R}_+^n$ with reflection matrix $I - P'$ and initial condition w.

In general, the pair (W, L) cannot be expressed explicitly in terms of the driving process Y, with the notable exception of the one-dimensional case. However, if the Skorokhod problem has a special structure, this property carries over to a multidimensional setting.

5.1. A special Skorokhod problem. It is the aim of this subsection to solve the Skorokhod problem for the pair (W, L) under the following assumptions:

N1 P is strictly upper triangular,
N2 the jth column of P contains exactly one strictly positive element for $j = 2, \ldots, n$, and
N3 Y_j is nondecreasing for $j = 2, \ldots, n$.

In §5.2, we show that these assumptions impose a "tree" structure on fluid networks.

THEOREM 5.1. *Under* **N1–N3**, *the solution to the Skorokhod problem of Y in $\mathbb{R}^n_+$ is given by*

$$L(t) = 0 \vee \sup_{0\le s\le t} [-(I-P')^{-1}Y(s) - (I-P')^{-1}w],$$
$$W(t) = w + Y(t) + (I-P')L(t),$$

where the supremum should be interpreted componentwise.

PROOF. As W is determined by L and **S1**, we only have to prove the expression for L. By Theorem D.3 of Robert [35], we know that L_i satisfies the fixed-point equation

$$L_i(t) = 0 \vee \sup_{0\le s\le t} [(P'L)_i(s) - w_i - Y_i(s)] \tag{5}$$

for $i = 1, \ldots, n$ and $t \ge 0$.

As a consequence of **N1**, we have $(I-P')^{-1} = I + P' + \cdots + P'^{n-1}$, and the jth row of $(I-P')^{-1}$ is the jth row of $I + P' + P'^2 + \cdots + P'^{j-1}$. Therefore, the theorem asserts that

$$L_i(t) = 0 \vee \sup_{0\le s\le t} \left[-\sum_{k=0}^{i-1} [P'^k Y(s) + P'^k w]_i \right]. \tag{6}$$

The proof goes by induction. For $i = 1$, (6) is the same equation as (5). Let us now suppose that we know that (6) holds for $i = 1, \ldots, j-1$, where $j = 2, \ldots, n$. Furthermore, let $j^* < j$ be such that $p_{j^*j} > 0$; it is unique by **N2**. Equation (5) shows that

$$\begin{aligned}
L_j(t) &= 0 \vee \sup_{0\le s\le t} [p_{j^*j} L_{j^*}(s) - w_j - Y_j(s)] \\
&= 0 \vee \sup_{0\le s\le t} \left[\left(0 \vee \sup_{0\le u\le s} - \sum_{k=0}^{j^*-1} p_{j^*j} [P'^k Y(u) + P'^k w]_{j^*} \right) - w_j - Y_j(s) \right] \\
&= 0 \vee \sup_{0\le s\le t} \left[\sup_{0\le u\le s} - \sum_{k=0}^{j^*-1} p_{j^*j} [P'^k Y(u) + P'^k w]_{j^*} - w_j - Y_j(s) \right] \qquad (7) \\
&= 0 \vee \sup_{0\le u\le t} \sup_{u\le s\le t} \left[-\sum_{k=0}^{j^*-1} [P'^{k+1} Y(u) + P'^{k+1} w]_j - w_j - Y_j(s) \right] \\
&= 0 \vee \sup_{0\le u\le t} \left[-\sum_{k=0}^{j^*} [P'^k Y(u) + P'^k w]_j \right], \qquad (8)
\end{aligned}$$

where we have used **N3** for the equalities (7) and (8).

The proof is completed after noting that the jth row of P'^k only contains zeroes for $k = j^*+1, \ldots, j-1$. □

Instead of working directly with W, it is often convenient to work with a transformed version, $\widetilde{W} := (I-P')^{-1}W$. The process $\widetilde{W}$ lies in a cone $\mathscr{C}$, which is a polyhedron and a proper subset of the orthant $\mathbb{R}^n_+$. Under the present assumptions, at least one edge of $\mathscr{C}$ is in the interior of $\mathbb{R}^n_+$ and at least one is an axis. Below, we give an interpretation of $\widetilde{W}$.

We next establish a correspondence between the event that $W_j(t) = 0$ and $\widetilde{W}_j(t) = 0$ under an additional condition.

PROPOSITION 5.1. *Suppose that* **N1–N3** *hold, but with "nondecreasing" replaced by "strictly increasing" in* **N3**. *Then, we have $W_j(t) = 0$ if and only if $\widetilde{W}_j(t) = 0$, for any $j = 1, \ldots, n$ and $t \ge 0$.*

PROOF. For $j = 1$ we have $W_j(t) = \widetilde{W}_j(t)$, so the stated is satisfied; suppose therefore that $j > 1$. Since the matrix $(I-P')^{-1}$ is lower triangular and nonnegative, we straightforwardly get that $\widetilde{W}_j(t) = 0$ implies $W_j(t) = 0$.

For the converse, observe that under **N1–N2** (see the proof of Theorem 5.1; we use the same notation)

$$\widetilde{W}_j(t) = \sum_{k=0}^{j-1} [P'^k W]_j(t) = \sum_{k=0}^{j^*} [P'^k W]_j(t).$$

An induction argument shows that it suffices to prove that $W_j(t) = 0$ implies $W_{j^*}(t) = 0$. To see that this holds, we observe that by **S1** and (5), $W_j(t) = 0$ is equivalent to

$$p_{j^*j} L_{j^*}(t) - w_j - Y_j(t) = 0 \vee \sup_{0\le s\le t} [p_{j^*j} L_{j^*}(s) - w_j - Y_j(s)].$$

The right-hand side of this equality is clearly nondecreasing. Therefore, since Y_j is strictly increasing by assumption, we conclude that $dL_{j^*}(t) > 0$, which immediately yields $W_{j^*}(t) = 0$ by **S4**. This completes the proof. □

5.2. Lévy-driven tree fluid networks. In this subsection, we define a class of Lévy-driven fluid networks, which we call *tree fluid networks*. We are interested in the steady-state behavior of such networks.

Consider n (infinite-buffer) fluid queues, with external input to queue j in the time interval $[0, t]$ given by $J_j(t)$. We assume that $J = \{J(t)\colon t \ge 0\} = \{(J_1(t), \ldots, J_n(t))'\colon t \ge 0\}$ is a càdlàg Lévy process starting in $J(0) = 0 \in \mathbb{R}^n_+$. The buffers are continuously drained at a constant rate as long as there is content in the buffer. These drain rates are given by a vector r; for buffer j, the rate is $r_j > 0$.

The interaction between the queues is modeled as follows. A fraction p_{ij} of the output of station i is immediately transferred to station j, while a fraction $1 - \sum_{j\ne i} p_{ij}$ leaves the system. We set $p_{ii} = 0$ for all i and suppose that $\sum_j p_{ij} \le 1$. The matrix $P = \{p_{ij}\colon i, j = 1, \ldots, n\}$ is called the *routing matrix*. We assume that for any station i, there is at most one station feeding buffer i, and that $p_{ij} = 0$ for $j < i$. The resulting network can be represented by a (directed) tree. Indeed, the stations then correspond to nodes, and there is a vertex from station i and j if $p_{ij} > 0$. We therefore use the name "tree fluid networks." We represent such a fluid network by the triplet (J, r, P). Note that P satisfies **N1–N2** by definition of a tree fluid network.

The *buffer content* process W and *regulator* L associated to the fluid network (J, r, P) are defined as the solution of the Skorokhod problem of

$$Y(t) := J(t) - (I - P')rt$$

with reflection matrix $I - P'$. The buffer content is sometimes called the *workload*, explaining the notation W. Importantly, the dynamics of the network are given by **S1–S4**, as the reader may verify. The process L_j can be interpreted as the cumulative unused capacity in station j.

Associated to the processes W and L, one can also define the process of the *age of the busy period*: for $j = 1, \ldots, n$, we set

$$B_j(t) := t - \sup\{s \le t\colon W_j(s) = 0\}, \tag{9}$$

and let $B(t) = (B_1(t), \ldots, B_n(t))'$. Hence, if there is work in queue j at time t (that is, $W_j(t) > 0$), $B_j(t)$ is the time that elapsed after the last time that the jth queue was empty. If there is no work in queue i at time t, then $B_i(t) = 0$. Similarly, one can also define the *age of the idle period* for $j = 1, \ldots, n$:

$$I_j(t) := t - \sup\{s \le t\colon W_j(s) \ne 0\},$$

and the corresponding vector $I(t)$. As a result of these definitions, $I_j(t) > 0$ implies $B_j(t) = 0$ and $B_j(t) > 0$ implies $I_j(t) = 0$ for $j = 1, \ldots, n$. The quantities $\widetilde{B}_j(t)$ and $\tilde{I}_j(t)$ are defined similarly, but with W_j replaced by the jth element of $\widetilde{W} = (I - P')^{-1}W$.

The random variables $\widetilde{W}_j$, $\widetilde{B}_j$, and $\tilde{I}_j$ have a natural interpretation. Indeed, let us consider all stations on a path from the root of the tree to station j. The *total* content of the buffers along this path is then given by $\widetilde{W}_j$. Consequently, $\widetilde{B}_j$ and $\tilde{I}_j$ correspond to the ages of the busy and idle periods of this aggregate buffer.

In the rest of the paper, we assume that the tree fluid network has the following additional properties:

T1 If $p_{ij} > 0$, then $p_{ij} > r_j/r_i$,
T2 $J_j(t)$ are nondecreasing for $j = 2, \ldots, n$,
T3 J is an n-dimensional Lévy process, and
T4 J is integrable and $(I - P')^{-1}\,\mathbb{E}\,J(1) < r$.

An important consequence of **T1** and **T2** is that Y is componentwise nondecreasing, except for Y_1. Consequently, if **T1** and **T2** hold for a tree fluid network, then **N1–N3** are automatically satisfied for the associated Skorokhod problem. Hence, Theorem 5.1 gives an explicit description of the buffer contents in the network. Note that **T4** ensures *stability* of the network.

Let us now define the process

$$X(t) := (I-P')^{-1}Y(t) = (I-P')^{-1}J(t) - rt.$$

In view of assumption **T1**, the down-stream buffer contents always grow when one of the up-stream buffers is nonempty. Moreover, under **T1**, $\widetilde{W}$ is itself a reflected process, that is $(\widetilde{W}, \widetilde{L})$ is the solution to the Skorokhod problem for X with reflection matrix I and initial condition $(I-P')^{-1}w$. Therefore, each coordinate of $\widetilde{W}$ is a one-dimensional reflected process. A similar assumption facilitates the analysis in Kella [20], Kella [22, Thm. 4.1 and Lem. 4.2], and Kella and Whitt [25].

In the next proposition, we find the steady-state behavior of the buffer content and the age of the busy (and idle) period for the Lévy-driven tree fluid network (J, r, P). We also consider the case where the inequality $p_{ij} > r_j/r_i$ in **T1** holds only weakly (i.e. $p_{ij} \geq r_j/r_i$), as this plays a role in priority fluid systems (see §6.3 below).

Recall the definitions of $G = G^X$ and $H = H^X$ in §§2.1 and 2.2, respectively.

Proposition 5.2. *Suppose that* **T1–T4** *hold for the tree fluid network* (J, r, P).

(i) *For any initial condition* $W(0) = w$, *the triplet of vectors* $(W(t), B(t), I(t))$ *converges in distribution to* $((I-P')\bar{X}, G^X, H^X)$ *as* $t \to \infty$.

(ii) *If the second inequality in* **T1** *holds only weakly, then for any initial condition* $W(0) = w$, *the triplet of vectors* $(W(t), \widetilde{B}(t), \widetilde{I}(t))$ *converges in distribution to* $((I-P')\bar{X}, G^X, H^X)$ *as* $t \to \infty$.

Proof. Throughout this proof, a system of equations like (9) is abbreviated by $B(t) = t - \sup\{s \leq t\colon W(s) = 0\}$.

We start with the proof of (ii). By Theorem 5.1, we have for any $t > 0$

$$\widetilde{W}(t) = [x + X(t)] \vee \sup_{0\leq s\leq t} [X(t) - X(s)],$$

where $x = (I-P')^{-1}w$. Moreover, as a consequence of Proposition 5.1, we have

$$\begin{aligned}\widetilde{B}(t) &= t - \sup\{s \leq t\colon \widetilde{W}(s) = 0\}\\ &= t - \sup\Big\{s \leq t\colon x + X(s) = 0 \wedge \inf_{0\leq u\leq s} [x + X(u)]\Big\}\\ &= t - \sup\Big\{s \leq t\colon x + X(s) = 0 \wedge \inf_{0\leq u\leq t} [x + X(u)]\Big\},\end{aligned}$$

where the last equality is best understood by sketching a sample path of X. The supremum over an empty set should be interpreted as zero.

This reasoning carries over to idle periods:

$$\widetilde{I}(t) = t - \sup\Big\{s \leq t\colon x + X(s) \neq 0 \wedge \inf_{0\leq u\leq s} [x + X(u)]\Big\}.$$

Due to the stationarity of the increments of $\{X(t),\ t \geq 0\}$ (**T3**), we may extend X to the two-sided process $\{X(t),\ t \in \mathbb{R}\}$. This leads to

$$\begin{pmatrix}\widetilde{W}(t)\\ \widetilde{B}(t)\\ \widetilde{I}(t)\end{pmatrix} =_{\mathrm{d}} \begin{pmatrix}[x - X(-t)] \vee \sup_{-t\leq s\leq 0}[-X(s)]\\ -\sup\{s\colon\ -t \leq s \leq 0, -X(s) = [x - X(-t)] \vee \sup_{-t\leq u\leq 0}[-X(u)]\}\\ -\sup\{s\colon\ -t \leq s \leq 0, -X(s) \neq [x - X(-t)] \vee \sup_{-t\leq u\leq s}[-X(u)]\}\end{pmatrix}.$$

Since $x - X(-t) \to -\infty$ almost surely by **T4**, this tends to

$$\begin{pmatrix}\sup_{s\leq 0}[-X(s)]\\ -\sup\{s \leq 0\colon\ -X(s) = \sup_{u\leq 0}[-X(u)]\}\\ -\sup\{s \leq 0\colon\ -X(s) \neq \sup_{u\leq s}[-X(u)]\}\end{pmatrix},$$

a vector that is almost surely finite, again by **T4**. By time-reversibility (see Lemma II.2 of Bertoin [4]), the latter vector is equal in distribution to $(\bar{X}, G^X, H^X)$.

The first claim follows from (ii) after noting that $B(t) = \widetilde{B}(t)$ and $I(t) = \widetilde{I}(t)$ by Proposition 5.1. □

We remark that the above proof does not use **T3** to the fullest. Indeed, for the proposition to hold, it suffices that J has stationary increments and that it is time-reversible.

Let us now suppose that the initial buffer content w is random. Proposition 5.2 shows, after a standard argument, that $\{W(t)\}$ is a stationary process if $W(0) = w$ is distributed as μ^*, where μ^* is the distribution of $(I - P')\bar{X}$. We now show that this stationary distribution is unique.

Corollary 5.1. *Suppose that* **T1–T4** *hold for the tree fluid network* (J, r, P). *Then* μ^* *is the only stationary distribution.*

Proof. Suppose there exists another stationary distribution $\mu_0^* \neq \mu^*$. Let W_0^* be the corresponding stationary process. For any Borel set B in $\mathbb{R}_+^n$ and any $t \geq 0$, we then have $\mathbb{P}(W_0^*(0) \in B) = \mathbb{P}(W_0^*(t) \in B)$. Therefore,

$$\begin{aligned}\mathbb{P}(W_0^*(0) \in B) &= \lim_{t\to\infty} \mathbb{P}(W_0^*(t) \in B)\\ &= \lim_{t\to\infty} \int_0^\infty \mathbb{P}(W_0^*(t) \in B \mid W_0^*(0) = w)\,\mathbb{P}(W_0^*(0) \in dw)\\ &= \int_0^\infty \lim_{t\to\infty} \mathbb{P}(W_0^*(t) \in B \mid W_0^*(0) = w)\,\mathbb{P}(W_0^*(0) \in dw)\\ &= \int_0^\infty \mathbb{P}((I-P')\bar{X} \in B)\,\mathbb{P}(W_0^*(0) \in dw) = \mathbb{P}((I-P')\bar{X} \in B),\end{aligned}$$

where the second last equation is due to Proposition 5.2. This is clearly a contradiction. □

Corollary 5.1 answers, for the special case of tree fluid networks, a question from the paper of Konstantopolous et al. [28] on the uniqueness of the stationary distribution. Note that for the queueing problem related to (J, r, P), the uniqueness of the stationary distribution was discussed in Kella [22]. In contrast to the setting in Kella [22], we allow for the first component of $J(t)$ to be a general Lévy process.

In the next section, we combine Proposition 5.2 with the results given in §§3 and 4 to study particular networks.

6. Tandem networks and priority systems. In this section, we analyze n fluid queues in tandem, which is a tree fluid network with a special structure. We also analyze a closely related priority system.

The tandem structure is specified by the form of the routing matrix: we suppose that P is such that $p_{i,i+1} > 0$ for $i = 1, \ldots, n-1$, and $p_{ij} = 0$ otherwise. Observe that we allow $p_{i,i+1} > 1$, and that it is not really a restriction to exclude $p_{i,i+1} = 0$; otherwise the queueing system splits into independent tandem networks.

In all of our results, we suppose that the tandem system (J, r, P) satisfies **T1–T4**. We rule out the degenerate case where the first $j \geq 1$ components of J are deterministic drifts, since an equivalent problem can then be studied with the first j stations removed. We also impose the following assumptions on the input Lévy process J:

T5 J has mutually independent components, and

T6 The Lévy measure of J_1 is supported on $\mathbb{R}_+$.

Observe that under **T2–T3**, **T5** implies that $J_2, \ldots, J_n$ are independent nonnegative subordinators.

This section consists of three parts. In §6.1, we are interested in the joint (steady-state) distribution of the buffer contents and the ages of the busy periods for fluid tandem networks, i.e., in the distribution of $(W(\infty), B(\infty))$. Section 6.2 considers the situation of a single compound Poisson input to the system. For that system, we are also interested in the ages of the idle periods, i.e., in the vector $I(\infty)$. In §6.3, we analyze buffer contents and busy periods in a priority system.

6.1. Generalities. To find the joint distribution of $W(\infty)$ and $B(\infty)$, throughout this section denoted by W and B respectively, we rely on Proposition 5.2. This motivates the analysis of $X(t) = (I - P')^{-1}J(t) - rt$. For $i = 2, \ldots, n$, we define the cumulant of $J_i(t)$ by $\theta_i^J(\beta) := -\log \mathbb{E}\, e^{-\beta J_i(1)}$, $\beta \geq 0$. As in §4, we write ψ_i (defined by $\psi_i(\beta) = \log \mathbb{E}\, e^{-\beta X_i(1)}$) for the Laplace exponent of $-X_i$. Its inverse is again denoted by Φ_i.

Under **T2** and **T6**, the Lévy measure of X is supported on $\mathbb{R}_+^n$. Moreover, as we ruled out trivial queues in the network, each of the components of $\bar{X}$ has a nondegenerate distribution. Therefore, let us recall that the following holds (see, e.g., Theorem VII.4 in Bertoin [4]): for $\alpha, \beta \geq 0$, $(\alpha, \beta) \neq (0, 0)$, $\beta \neq \Phi_i(\alpha)$, $i = 1, \ldots, n$, we have

$$\mathbb{E}\, e^{-\alpha G_i - \beta \bar{X}_i} = -\mathbb{E}\, X_i(1) \frac{\Phi_i(\alpha) - \beta}{\alpha - \psi_i(\beta)}. \tag{10}$$

This identity plays a crucial role in the results of this section. For notational convenience, we shall write that (10) holds for any $\alpha, \beta \geq 0$, without the requirements $(\alpha, \beta) \neq (0, 0)$ and $\beta \neq \Phi_i(\alpha)$.

Now we can formulate the main result of this subsection. We remark that the first formula also holds if J_1 is not necessarily spectrally positive. For instance, it allows for phase-type downward jumps; see Dieker [7] for the joint transform of $\bar{X}_j$ and G_j in that case.

THEOREM 6.1. *Consider a tandem fluid network (J, r, P) for which* **T1–T6** *holds. Then for $\omega, \beta \in \mathbb{R}_+^n$, the transform $\mathbb{E}\,e^{-\langle\omega, W\rangle - \langle\beta, B\rangle}$ equals*

$$\prod_{j=1}^{n-1} \frac{\mathbb{E}\,e^{-\left[\sum_{\ell=j+1}^n \theta_\ell^J(\omega_\ell)+\sum_{\ell=j+1}^n (p_{\ell-1\ell}r_{\ell-1}-r_\ell)\omega_\ell+\sum_{\ell=j}^n \beta_\ell\right]G_j-\omega_j\bar{X}_j}}{\mathbb{E}\,e^{-\left[\sum_{\ell=j+1}^n \theta_\ell^J(\omega_\ell)+\sum_{\ell=j+1}^n (p_{\ell-1\ell}r_{\ell-1}-r_\ell)\omega_\ell+\sum_{\ell=j+1}^n \beta_\ell\right]G_j-p_{j,j+1}\omega_{j+1}\bar{X}_j}} \times \mathbb{E}\,e^{-\beta_n G_n-\omega_n\bar{X}_n}.$$

Consequently, we have for $\omega, \beta \in \mathbb{R}_+^n$,

$$\begin{aligned}\mathbb{E}\,e^{-\langle\omega, W\rangle-\langle\beta, B\rangle} &= -\mathbb{E}\,X_n(1)\frac{\Phi_n(\beta_n)-\omega_n}{\beta_n-\psi_n(\omega_n)}\\ &\quad\times\prod_{j=1}^{n-1}\frac{\Phi_j(\sum_{\ell=j+1}^n \theta_\ell^J(\omega_\ell)+\sum_{\ell=j+1}^n(p_{\ell-1,\ell}r_{\ell-1}-r_\ell)\omega_\ell+\sum_{\ell=j}^n\beta_\ell)-\omega_j}{\Phi_j(\sum_{\ell=j+1}^n \theta_\ell^J(\omega_\ell)+\sum_{\ell=j+1}^n(p_{\ell-1,\ell}r_{\ell-1}-r_\ell)\omega_\ell+\sum_{\ell=j+1}^n\beta_\ell)-p_{j,j+1}\omega_{j+1}}\\ &\quad\times\prod_{j=1}^{n-1}\frac{\sum_{\ell=j+1}^n \theta_\ell^J(\omega_\ell)+\sum_{\ell=j+1}^n(p_{\ell-1,\ell}r_{\ell-1}-r_\ell)\omega_\ell+\sum_{\ell=j+1}^n\beta_\ell-\psi_j(p_{j,j+1}\omega_{j+1})}{\sum_{\ell=j+1}^n \theta_\ell^J(\omega_\ell)+\sum_{\ell=j+1}^n(p_{\ell-1,\ell}r_{\ell-1}-r_\ell)\omega_\ell+\sum_{\ell=j}^n\beta_\ell-\psi_j(\omega_j)}.\end{aligned}$$

PROOF. By Proposition 5.2(i), $(W, B) =_{\mathrm{d}} ((I-P')\bar{X}, G^X)$. Hence we have

$$\mathbb{E}\,e^{-\langle\omega, W\rangle-\langle\beta, B\rangle} = \mathbb{E}\,e^{-\langle(I-P)\omega, (I-P')^{-1}W\rangle-\langle\beta, B\rangle} = \mathbb{E}\,e^{-\langle\beta, G\rangle-\langle(I-P)\omega, \bar{X}\rangle}. \tag{11}$$

Now note that the stability condition **T4** for (J, r, P) implies **D** for X by the law of large numbers. Thus, to apply Theorem 3.1 for (11), it is enough to check that **G** holds. Standard algebraic manipulations give

$$X_1(t) = J_1(t) - r_1 t$$

and

$$X_{i+1}(t) = p_{i,i+1}X_i(t) + J_{i+1}(t) + (p_{i,i+1}r_i - r_{i+1})t$$

for $i = 1, \ldots, n-1$. Hence, **G** holds with $K_i = p_{i-1,i}$ and $\Upsilon_i(t) = J_i(t) + (p_{i-1,i}r_{i-1} - r_i)t$.

As a result, we know that from Theorem 3.1,

$$\begin{aligned}\mathbb{E}\,e^{-\langle\beta, G\rangle-\langle(I-P)\omega, \bar{X}\rangle} &= \mathbb{E}\,e^{-\langle\beta, G\rangle-\langle\tilde{\omega}, \bar{X}\rangle}\\ &= \prod_{j=1}^{n-1}\frac{\mathbb{E}\,e^{-[\sum_{\ell=j+1}^n \theta_\ell^\Upsilon(\sum_{k=\ell}^n K_\ell^k\tilde{\omega}_k)+\sum_{\ell=j}^n\beta_\ell]G_j-(\sum_{k=j}^n K_j^k\tilde{\omega}_k)\bar{X}_j}}{\mathbb{E}\,e^{-[\sum_{\ell=j+1}^n \theta_\ell^\Upsilon(\sum_{k=\ell}^n K_\ell^k\tilde{\omega}_k)+\sum_{\ell=j+1}^n\beta_\ell]G_j-(\sum_{k=j+1}^n K_j^k\tilde{\omega}_k)\bar{X}_j}} \times \mathbb{E}\,e^{-\beta_n G_n-\tilde{\omega}_n\bar{X}_n},\end{aligned}$$

where we have set $\tilde{\omega} = (I-P)\omega$ for notational convenience.

The reader may check that $\sum_{k=j}^n K_j^k\tilde{\omega}_k = \omega_j$ and $\sum_{k=j+1}^n K_j^k\tilde{\omega}_k = p_{j,j+1}\omega_{j+1}$, leading to the first claim. The second assertion is a consequence of the first and (10). □

Theorem 6.1 extends several results from the literature on the steady-state distribution of the buffer content for tandem Lévy networks. In particular, if $J(t) = (J_1(t), 0)'$, $P = (p_{ij})$, with $p_{12} = 1$ and zeroes elsewhere, if one chooses $\beta_1 = \beta_2 = 0$ and $\omega_1 = 0$ in Theorem 6.1, then one obtains Theorem 3.2 of Dębicki et al. [6]. Additionally, if one chooses $\beta_1 = \beta_2 = 0$ and supposes that J_1 is a subordinator, we recover the results of Kella [20].

Even if the Laplace transform of $(G_j, \bar{X}_j)$ can be inverted, it is generally not straightforward to invert the Laplace transform of (W, B) given in Theorem 6.1. Some progress has been recently made in case $n = 2$; for a Brownian fluid system, Lieshout and Mandjes [30] calculate the distribution of W. Avram et al. [2] study a compound Poisson setting with exponential jumps. A different type of explicit solution can be found in the work of Harrison [14]; he gives an example closely related to the framework of the present paper.

For use in §6.3, we point out that the expression in Theorem 6.1 is $\mathbb{E}\,e^{-\langle\omega, W\rangle-\langle\beta, \tilde{B}\rangle}$ if the second inequality in **T1** is weak, cf. Proposition 5.2(ii).

The lengths of the busy periods. Besides the Laplace transforms of the *ages* B of the busy periods, Theorem 6.1 also enables us to find the Laplace transforms of the *length* V of the steady-state running busy periods. Indeed, let D_i, $i = 1, \ldots, n$ denote the steady-state remaining lengths of the running busy period, so that $V_i = B_i + D_i$. We know that D_i and B_i are equal in distribution. In fact, following for instance (Asmussen [1, Sec. V.3]), we have

$$(B_i, D_i) =_{\mathrm{d}} (U_i V_i, (1-U_i)V_i), \tag{12}$$

where U_i are i.i.d. and uniform on $[0, 1]$.

For the Brownian (single-station) fluid queue, the following result is Corollary 3.8 of Salminen and Norros [36].

COROLLARY 6.1. *Consider a tandem fluid network (J, r, P) for which* **T1–T6** *holds. Then for $\alpha, \beta \geq 0$, $\alpha \neq \beta$,*

$$\mathbb{E}\, e^{-\alpha B_i - \beta D_i} = -\mathbb{E}\, X_i(1) \frac{\Phi_i(\alpha) - \Phi_i(\beta)}{\alpha - \beta}.$$

Moreover, we have for $\alpha \geq 0$,

$$\mathbb{E}\, e^{-\alpha V_i} = -\mathbb{E}\, X_i(1) \frac{d\Phi_i(\alpha)}{d\alpha}.$$

PROOF. Since the second claim follows straightforwardly from the first, we only prove the first expression. Following (12), we have for $\alpha \neq \beta$,

$$\begin{aligned}(\alpha - \beta)\, \mathbb{E}\, e^{-\alpha B_i - \beta D_i} &= (\alpha - \beta)\, \mathbb{E}\, e^{-(\alpha-\beta) U_i V_i - \beta V_i} = (\alpha - \beta)\, \mathbb{E} \int_0^1 e^{-(\alpha-\beta) u V_i - \beta V_i}\, du \\ &= \mathbb{E} \int_\beta^\alpha e^{-u V_i}\, du = \mathbb{E} \int_0^\alpha e^{-u V_i}\, du - \mathbb{E} \int_0^\beta e^{-u V_i}\, du.\end{aligned}$$

The two identities that result upon setting $\beta = 0$ and $\alpha = 0$ can be used to express the first and second expectation in terms of the Laplace transform of B_i and D_i respectively; this yields for $\alpha \neq \beta$

$$\mathbb{E}\, e^{-\alpha B_i - \beta D_i} = \frac{1}{\alpha - \beta} [\alpha\, \mathbb{E}\, e^{-\alpha B_i} - \beta\, \mathbb{E}\, e^{-\beta B_i}],$$

where we have used the equality in distribution of B_i and D_i. Application of (10) completes the proof. □

6.2. A single compound Poisson input. In this subsection, we examine a tandem fluid network with a single compound Poisson input (Kella and Whitt [25]). The following assumption formalizes our framework.

T7 $p_{i,i+1} = 1$ for $i = 1, \ldots, n-1$, while $p_{ij} = 0$ otherwise, and

T8 J_1 is a compound Poisson process with positive drift $\mathtt{d}$ and intensity λ, and $J_j \equiv 0$ for $j = 2, \ldots, n$. Moreover, r_j decreases strictly in j and $\mathbb{E}\, J(1) < r_n$.

An important consequence of **T7** and **T8** is that

$$(r_j - r_k)\omega = \psi_j(\omega) - \psi_k(\omega), \tag{13}$$

which simplifies the resulting expressions in view of fact that we often deal with ratios of the fluctuation identity (10). Interestingly, it is also possible to study (joint distributions of) idle periods under these assumptions.

The following corollary collects some results that follow from **T7** and **T8** and Theorem 6.1. Many interesting formulas can be derived, but we have selected two examples for which the formulas are especially appealing.

COROLLARY 6.2. *Consider a tandem fluid network (J, r, P) for which* **T7–T8** *holds.*

(i) *For $i = 1, \ldots, n$, and $\omega, \beta \geq 0$, we have*

$$\mathbb{E}\, e^{-\omega W_i - \beta B_i} = -\mathbb{E}\, X_i(1) \frac{\Phi_i(\beta) - \omega}{\beta + (r_{i-1} - r_i)\omega} \times \frac{\Phi_{i-1}((r_{i-1} - r_i)\omega + \beta)}{\Phi_{i-1}((r_{i-1} - r_i)\omega + \beta) - \omega}.$$

Moreover, $\mathbb{P}(W_i = 0) = \mathbb{P}(B_i = 0) = \mathbb{E}\, X_i(1)/(\mathtt{d} - r_i)$.

(ii) *For $i = 2, \ldots, n$ and $\omega, \beta \geq 0'$, in analogy with (i)*

$$\mathbb{E}[e^{-\omega W_i - \beta B_i}; W_{i-1} = 0] = -\frac{\mathbb{E}\, X_i(1)}{\mathtt{d} - r_{i-1}} \frac{\Phi_i(\beta) - \omega}{\Phi_{i-1}((r_{i-1} - r_i)\omega + \beta) - \omega}.$$

PROOF. To prove (i), apply Theorem 6.1 to obtain for $i = 1, \ldots, n$,

$$\mathbb{E}\, e^{-\omega W_i - \beta B_i} = \frac{\mathbb{E}\, e^{-[(r_{i-1} - r_i)\omega + \beta] G_{i-1}}}{\mathbb{E}\, e^{-[(r_{i-1} - r_i)\omega + \beta] G_{i-1} - \omega \bar{X}_{i-1}}}\, \mathbb{E}\, e^{-\beta G_i - \omega \bar{X}_i}.$$

With (10), this leads immediately to the given formula after invoking (13).

We find $\mathbb{P}(W_i = 0)$ upon choosing $\omega = 0$ and noting that

$$\mathbb{P}(W_i = 0) = \mathbb{P}(B_i = 0) = \lim_{\beta \to \infty} \mathbb{E}\, e^{-\beta G_i} = -\mathbb{E}\, X_i(1) \lim_{\beta \to \infty} \frac{\Phi_i(\beta)}{\beta} = \frac{\mathbb{E}\, X_i(1)}{\mathtt{d} - r_i},$$

where the last equality follows from Proposition I.2 in Bertoin [4].

The second claim uses a similar argument; it follows from Theorem 6.1 that for $i = 2, \ldots, n$

$$\mathbb{E}\, e^{-\omega_i W_i - \beta_{i-1} B_{i-1} - \beta_i B_i} = \frac{\mathbb{E}\, e^{-[(r_{i-1} - r_i)\omega_i + \beta_{i-1} + \beta_i] G_{i-1}}}{\mathbb{E}\, e^{-[(r_{i-1} - r_i)\omega_i + \beta_i] G_{i-1} - \omega_i \bar{X}_{i-1}}}\, \mathbb{E}\, e^{-\beta_i G_i - \omega_i \bar{X}_i},$$

and the numerator of the fraction tends to $\mathbb{P}(W_{i-1} = 0)$ as $\beta_{i-1} \to \infty$. Now apply (10) and (13). □

We end this subsection with an application of the theory in §4, which enables us to study the idle periods in a tandem fluid network satisfying **T7–T8**. For $\gamma \in \mathbb{R}_+^{k-1}$, we set

$$\mathscr{D}_j^k(\gamma) := c_j \sum_{\ell=j}^{k-1} \left(\frac{1}{c_{\ell+1}} - \frac{1}{c_\ell} \right) \left(\lambda + \sum_{p=1}^{\ell} \gamma_p \right),$$

which is similar to the definition of $\mathscr{C}_j^k$ in §4.

Proposition 6.1. *Consider a tandem fluid network (J, r, P) for which* **T7–T8** *holds. For $\gamma \in \mathbb{R}_+^n$, we have*

$$\mathbb{E}\,\mathrm{e}^{-\langle \gamma, I\rangle} = 1 - \sum_{k=1}^{n} \mathbb{P}(W_k = 0)\, \mathbb{E}_k^{\downarrow}\big[\mathrm{e}^{-\sum_{\ell=1}^{k-1}\gamma_\ell H_\ell}(1 - \mathrm{e}^{-\gamma_k H_k})\big],$$

where $\mathbb{P}(W_j = 0)$ is given in Corollary 6.2(i), *and*

$$\mathbb{E}_k^{\downarrow}\,\mathrm{e}^{-\sum_{\ell=1}^{k}\gamma_\ell H_\ell} = \frac{\lambda + \sum_{\ell=1}^{k-1}\gamma_\ell(1 - c_k/c_\ell) - c_k \Phi_1(\mathscr{D}_1^k(\gamma))}{\lambda + \sum_{\ell=1}^{k}\gamma_\ell} \times \prod_{j=1}^{k-1} \frac{\lambda + \sum_{\ell=1}^{k-1}\gamma_\ell - \sum_{\ell=j+1}^{k-1}(c_k/c_\ell)\gamma_\ell - c_k \Phi_{j+1}(\mathscr{D}_{j+1}^k(\gamma))}{\lambda + \sum_{\ell=1}^{k-1}\gamma_\ell - \sum_{\ell=j+1}^{k}(c_k/c_\ell)\gamma_\ell - c_k \Phi_j(\mathscr{D}_j^k(\gamma))}. \tag{14}$$

Proof. Note that **T7** and **T8** imply **H**. The first claim follows from Proposition 5.2 and the facts that for $k = 2, \ldots, n$,

$$\mathbb{E}\,\mathrm{e}^{-\sum_{\ell=1}^{k}\gamma_\ell H_\ell} = \mathbb{E}\,\mathrm{e}^{-\sum_{\ell=1}^{k-1}\gamma_\ell H_\ell} + \mathbb{E}_k^{\downarrow}\big[\mathrm{e}^{-\sum_{\ell=1}^{k-1}\gamma_\ell H_\ell}(1 - \gamma_k H_k)\big]\,\mathbb{P}(\bar{X}_k = 0),$$

and $\mathbb{E}\,\mathrm{e}^{\gamma_1 H_1} = 1 - \mathbb{E}_1^{\downarrow}[1 - \mathrm{e}^{-\gamma_1 H_1}]\,\mathbb{P}(\bar{X}_1 = 0)$. These identities follow after observing that H_k vanishes on the event $\{\bar{X}_k = 0\}$, and that $\{\bar{X}_k = 0\}$ is the complement of $\{\bar{X}_k > 0\}$.

Let us now prove the expression for the $\mathbb{P}_k^{\downarrow}$-distribution of $(H_1, \ldots, H_k)'$. From Proposition 4.1 and Proposition 4.2, we know that

$$\mathbb{E}_k^{\downarrow}\,\mathrm{e}^{-\sum_{\ell=1}^{k}\gamma_\ell H_\ell} = \frac{\lambda\,\mathbb{E}\,\mathrm{e}^{-\mathscr{D}_1^k(\gamma)\left(\rho_1^{(1)} - \sigma_1^{(1)}\right)}}{\lambda + \sum_{\ell=1}^{k}\gamma_\ell} \prod_{j=1}^{k-1} \frac{\sum_{\ell=1}^{j}\gamma_\ell + \lambda\,\mathbb{E}\,\mathrm{e}^{-\mathscr{D}_{j+1}^k(\gamma)\left(\rho_{j+1}^{(j+1)} - \sigma_{j+1}^{(j+1)}\right)}}{\sum_{\ell=1}^{j}\gamma_\ell + \lambda\,\mathbb{E}\,\mathrm{e}^{-\mathscr{D}_j^k(\gamma)\left(\rho_j^{(j)} - \sigma_j^{(j)}\right)}}.$$

The proof is finished after invoking (4) and noting that for $j = 1, \ldots, k-1$,

$$\frac{c_k}{c_j}\left[\lambda + \sum_{\ell=1}^{j}\gamma_\ell + \mathscr{D}_j^k(\gamma)\right] = \frac{c_k}{c_{j+1}}\left[\lambda + \sum_{\ell=1}^{j}\gamma_\ell + \mathscr{D}_{j+1}^k(\gamma)\right] = \lambda + \sum_{\ell=1}^{k-1}\gamma_\ell - \sum_{\ell=j+1}^{k-1}\frac{c_k}{c_\ell}\gamma_\ell,$$

and

$$\frac{c_k}{c_1}[\lambda + \mathscr{D}_1^k(\gamma)] = \lambda + \sum_{\ell=1}^{k-1}\gamma_\ell - \sum_{\ell=1}^{k-1}\frac{c_k}{c_\ell}\gamma_\ell,$$

as the reader readily verifies. □

6.3. A priority fluid system. In this subsection, we analyze a single station which is drained at a constant rate $\mathtt{r} > 0$. It is fed by n external inputs ("traffic classes") $J_1(t), \ldots, J_n(t)$, each equipped with its own (infinite-capacity) buffer. The queue discipline is (preemptive resume) *priority*, meaning that for each $i = 1, \ldots, n$, the ith buffer is continuously drained only if first $i-1$ buffers do not require the full capacity $\mathtt{r}$. We call such a system a *priority fluid system*.

The aim of this section is to find the Laplace transform of (W, E), where $W_j = W_j(\infty)$ is the stationary buffer content of class-j input traffic, and $E_j = E_j(\infty)$ is the stationary age of the busy period for class j. We impose the following assumptions.

P1 J is an n-dimensional Lévy process with mutually independent components, and its Lévy measure is supported on $\mathbb{R}_+^n$, $J(0) = 0$,

P2 $J_j(t)$ are nondecreasing for $j = 2, \ldots, n$, and

P3 J is integrable and $\sum_{i=1}^{n}\mathbb{E}\,J_i(1) < \mathtt{r}$.

The central idea is that W evolves in the same manner as the solution to the Skorokhod problem that corresponds to a tandem fluid network (J, r, P), with $r = (\mathtt{r}, \ldots, \mathtt{r})'$ and $P = (p_{ij})$ such that $p_{i,i+1} = 1$ for $i =$

$1, \dots, n-1$ and $p_{ij} = 0$ otherwise. This equivalence has been noticed, for instance, by Elwalid and Mitra [12]. It allows us to use the notation of §6.1.

It is important to observe that **P1–P3** for the priority system implies **T1–T6** for the corresponding tandem fluid network, except that the second inequality in **T1** only holds as a weak inequality. However, as remarked in §6.1, the Laplace transform of the distribution of $(W, \widetilde{B})$ is then still given in Theorem 6.1.

The steady-state ages of the busy periods E can also be expressed in terms of the solution (W, L) to this Skorokhod problem, but it does *not always* equal $\widetilde{B}$ as in §6.1. To see this, notice that if class-1 traffic (highest priority) arrives to an empty system at time t, we have $W_2(t) = 0$, while $\widetilde{W}_2(t) > 0$ so that $\widetilde{B}_2(t) > 0$. However, it must hold that $E_2(t) = 0$.

Still, the following theorem shows that it is possible to express the distribution of (W, E) in terms of $(W, \widetilde{B})$.

Theorem 6.2. *Consider a priority fluid network for which* **P1–P3** *holds. Then for $\omega, \beta \in \mathbb{R}_+^n$, the transform $\mathbb{E}\, e^{-\langle \omega, W\rangle - \langle \beta, E\rangle}$ equals*

$$\mathbb{E}\, e^{-\langle \omega, W\rangle - \langle \beta, \widetilde{B}\rangle} + \sum_{j=2}^{n} \mathbb{E}\big[e^{-\sum_{\ell=1}^{j-1}\omega_\ell W_\ell - \sum_{\ell=1}^{j-1}\beta_\ell \widetilde{B}_\ell}(1 - e^{-\beta_j \widetilde{B}_j});\, W_j = \cdots = W_n = 0\big].$$

Proof. In principle, E_j equals $\widetilde{B}_j$, except when $W_j = 0$. In fact, it follows from the above reasoning that

$$\mathbb{E}\, e^{-\langle \omega, W\rangle - \langle \beta, E\rangle} = \mathbb{E}\big[e^{-\omega_1 W_1 - \beta_1 \widetilde{B}_1};\, W_2 = \cdots = W_n = 0\big] + \sum_{j=2}^{n} \mathbb{E}\big[e^{-\sum_{\ell=1}^{j}\omega_\ell W_\ell - \sum_{\ell=1}^{j}\beta_\ell \widetilde{B}_\ell};\, W_j > 0,\, W_{j+1} = \cdots = W_n = 0\big].$$

Now, use the fact that $\{W_j > 0\}$ is the complement of $\{W_j = 0\}$ and rearrange terms. $\square$

If the $J_2, \dots, J_n$ are *strictly* increasing, it can be seen (for instance with Theorem 6.1) that

$$\mathbb{E}\big[e^{-\sum_{\ell=1}^{j-1}\omega_\ell W_\ell - \sum_{\ell=1}^{j-1}\beta_\ell \widetilde{B}_\ell}(1 - e^{-\beta_j \widetilde{B}_j});\, W_j = \cdots = W_n = 0\big] = 0.$$

Therefore, in that case, we have the equality in distribution $(W, E) =_{\mathrm{d}} (W, \widetilde{B})$.

Another important special case is when $J_1, \dots, J_n$ are compound Poisson processes, say with intensities $\lambda_1, \dots, \lambda_n$ respectively. Much is known about the resulting priority system, see for instance Jaiswal [19] for this and related models. To our knowledge, the distribution of (W, E) has not been investigated. However, it is given by Theorem 6.2 and Theorem 6.1 upon noting that $\theta_\ell^J(\omega) \to \lambda_\ell$ as $\omega \to \infty$. Since it is not so instructive to write out the resulting formulas, we leave this to the reader.

Appendix A. Some calculations for a compound Poisson process with negative drift. In this appendix, we study a compound Poisson process Z with negative drift, and derive some results on the excursions of $Z - \underline{Z}$ from 0, just before its entrance to 0. These results are applied in §4.

Let us first fix the notation. Throughout this appendix, Z is a Lévy process on $(\Omega, \mathscr{F}, \mathbb{P})$ with Laplace exponent

$$\psi_{-Z}(\beta) := \log \mathbb{E}\, e^{-\beta Z(1)} = c\beta - \lambda \int_{\mathbb{R}_+} (1 - e^{-\beta z})\, F(dz),$$

where $c > 0$, $\lambda \in (0, \infty)$, and F is a probability distribution on $(0, \infty)$. That is, Z is a compound Poisson process under $\mathbb{P}$ with rate λ and negative drift $-c$, and its (positive) jumps are governed by F. We suppose that $\mathbb{E}\, Z(1) < 0$, so that Z drifts to $-\infty$. In analogy to §4, the inverse of ψ_{-Z} is denoted by Φ_{-Z}; it is uniquely defined since ψ_{-Z} is increasing. Observe that $\Phi_{-Z}(0) = 0$.

Set $T_0 = 0$, and let T_i denote the epoch of the ith jump of Z. To the ith jump of Z, we associate a vector of *marks*, denoted by $M_i \in \mathbb{R}_+^m$ (for some $m \in \mathbb{Z}_+$). We suppose that M_i is independent of the process $T \equiv \{T_n\colon n \geq 1\}$, and that it is also independent of $(Z(T_j) - Z(T_j-), M_j)$ for $j \neq i$. However, we allow for a dependency between M_i and $Z(T_i) - Z(T_i-)$. In fact, an interesting choice for M_i is $M_i = Z(T_i) - Z(T_i-)$ (so that $m = 1$).

Define τ_- as the first hitting time of zero, and N_- as the index of the last jump before τ_-, i.e.,

$$\tau_- := \inf\{t \geq 0\colon Z(t) = 0\}, \qquad N_- = \inf\{n \geq 0\colon Z(T_{n+1}-) \leq 0\}.$$

Write $\mathbb{P}_\xi$ for the law of $Z + \xi$ under $\mathbb{P}$ with initial mark $M_0 = M$. We suppose that the initial condition (ξ, M) is independent of Z, and has the same distribution as $(Z(T_1) - Z(T_1-), M_1)$. Observe that both τ_- and N_- are $\mathbb{P}_\xi$-almost surely finite, and that (by the Markov property) the "overshoot of the first excursion" $T_{N_-+1} - \tau_-$ has an exponential distribution with parameter λ.

In this appendix, it is our aim to characterize the $\mathbb{P}_\xi$-distribution of τ_- (excursion length), $\tau_- - T_{N_-}$ (excursion "undershoot"), and M_{N_-} (mark of the last jump). Overshoots and undershoots have been studied extensively in the literature. However, as opposed to what we have here, these results are all related to the situation that a Lévy process can cross a boundary by jumping over it (strictly speaking, this is the only case where the terms "overshoot" and "undershoot" seem to be appropriate). See Doney and Kyprianou [10] for a recent contribution and for references.

In view of the results of Dufresne and Gerber [11], it is tempting to believe that $\tau_- - T_{N_-}$ has an exponential distribution. However, it turns out that this "undershoot" has a completely different distribution.

Proposition A.1. *We have for $\beta, \gamma \ge 0$ and $\kappa \in \mathbb{R}^m_+$,*

$$\begin{aligned}\mathbb{E}_\xi \,\mathrm{e}^{-\beta(\tau_- - T_{N_-})-\gamma\tau_- -\langle\kappa, M_{N_-}\rangle} &= \frac{[\beta+\gamma-c\Phi_{-Z}(\gamma)+\lambda]\,\mathbb{E}\,\mathrm{e}^{-(\beta+\gamma+\lambda)\xi/c-\langle\kappa, M\rangle}}{\beta+\lambda\,\mathbb{E}\,\mathrm{e}^{-(\beta+\gamma+\lambda)\xi/c}}\\ &= \frac{[\beta+\lambda\,\mathbb{E}_\xi\,\mathrm{e}^{-\gamma\tau_-}]\,\mathbb{E}\,\mathrm{e}^{-(\beta+\gamma+\lambda)\xi/c-\langle\kappa, M\rangle}}{\beta+\lambda\,\mathbb{E}\,\mathrm{e}^{-(\beta+\gamma+\lambda)\xi/c}}.\end{aligned}$$

To prove this proposition, we need an auxiliary result on Poisson processes. Consider a Poisson point process $N(t)$ with parameter μ, and let ζ be a positive random variable, independent of N. Let $A(t)$ be the backward recurrence time process defined by N, that is the time from ζ to the nearest point to the left. The following lemma characterizes the joint distribution of $N(\zeta)$, $A(\zeta)$, and ζ.

Lemma A.1. *We have for $\beta, \gamma \ge 0$ and $0 \le s \le 1$,*

$$\mathbb{E}\, s^{N(\zeta)}\mathrm{e}^{-\beta A(\zeta)-\gamma\zeta} = \frac{\beta}{\beta+s\mu}\,\mathbb{E}\,\mathrm{e}^{-(\beta+\gamma+\mu)\zeta} + \frac{s\mu}{\beta+s\mu}\,\mathbb{E}\,\mathrm{e}^{-[\gamma+(1-s)\mu]\zeta}.$$

Proof. We only prove the claim for $\gamma = 0$; the general case follows by replacing the distribution of ζ by the (defective) distribution of $\tilde\zeta$ given by $\mathbb{E}\,\mathrm{e}^{-\beta\tilde\zeta} = \mathbb{E}\,\mathrm{e}^{-(\beta+\gamma)\zeta}$. Let $U_0 = 0$ and $U_1, U_2, \ldots$ be the location of consecutive points of N. Observe that

$$\begin{aligned}\mathbb{E}\, s^{N(\zeta)}\mathrm{e}^{-\beta A(\zeta)} &= \sum_{n=0}^\infty s^n\,\mathbb{E}[\mathrm{e}^{-\beta(\zeta-U_n)}; 0\le \zeta-U_n \le U_{n+1}-U_n]\\ &= \sum_{n=0}^\infty s^n \int_0^\infty\int_0^t \mathrm{e}^{-(\beta+\mu)(t-x)}\,\mathbb{P}_{U_n}(dx)\,\mathbb{P}_\zeta(dt) = \sum_{n=0}^\infty s^n \phi_n(\mu+\beta),\end{aligned} \tag{15}$$

where

$$\phi_n(\beta) := \mathbb{E}[\mathrm{e}^{-\beta(\zeta-U_n)}; \zeta \ge U_n].$$

Clearly, $\phi_0(\beta) = \mathbb{E}\,\mathrm{e}^{-\beta\zeta}$. If we let B be the forward recurrence time process, we have for $n \ge 1$,

$$\begin{aligned}\phi_n(\beta) &= \mathbb{E}[\mathrm{e}^{-\beta(\zeta-U_n)}; \zeta\ge U_{n-1}] - \mathbb{E}[\mathrm{e}^{-\beta(\zeta-U_n)}; U_{n-1}\le\zeta<U_n]\\ &= \mathbb{E}[\mathrm{e}^{-\beta(\zeta-U_{n-1})+\beta(U_n-U_{n-1})}; \zeta\ge U_{n-1}] - \mathbb{E}[\mathrm{e}^{-\beta(\zeta-U_n)}; U_{n-1}\le\zeta<U_n]\\ &= \mathbb{E}[\mathrm{e}^{\beta(U_n-U_{n-1})}]\,\mathbb{E}[\mathrm{e}^{-\beta(\zeta-U_{n-1})}; \zeta\ge U_{n-1}] - \mathbb{E}[\mathrm{e}^{\beta B(\zeta)} \mid N(\zeta)=n-1]\,\mathbb{P}(N(\zeta)=n-1)\\ &= \frac{\mu}{\mu-\beta}[\phi_{n-1}(\beta) - \mathbb{P}(N(\zeta)=n-1)],\end{aligned}$$

where we used the lack-of-memory property of the exponential distribution for the last equality. After iteration, we obtain

$$\phi_n(\beta) = \left(\frac{\mu}{\mu-\beta}\right)^n \mathbb{E}\,\mathrm{e}^{-\beta\zeta} - \sum_{i=0}^{n-1}\left(\frac{\mu}{\mu-\beta}\right)^{n-i}\mathbb{P}(N(\zeta)=i).$$

Therefore, taking $0 < s < \beta/\mu$ (later we may use an analytic-continuation argument), we deduce from (15) that

$$\mathbb{E}[s^{N(\zeta)}\mathrm{e}^{-\beta A(\zeta)}] = \mathbb{E}\,\mathrm{e}^{-(\beta+\mu)\zeta}\sum_{n=0}^\infty\left(-\frac{s\mu}{\beta}\right)^n - \sum_{n=1}^\infty s^n\sum_{i=0}^{n-1}\left(-\frac{\mu}{\beta}\right)^{n-i}\mathbb{P}(N(\zeta)=i).$$

The double sum in this expression can be rewritten as

$$-\frac{s\mu}{\beta+s\mu}\sum_{i=0}^\infty s^i\,\mathbb{P}(N(\zeta)=i) = -\frac{s\mu}{\beta+s\mu}\,\mathbb{E}\,\mathrm{e}^{-(1-s)\mu\zeta},$$

and the claim follows. □

Lemma A.1 is the main ingredient to prove Proposition A.1.

PROOF OF PROPOSITION A.1. The crucial yet simple observation is that

$$\begin{aligned}\mathbb{E}_\xi e^{-\beta(\tau_- - T_{N_-})-\gamma\tau_- - \langle\kappa, M_{N_-}\rangle} &= \mathbb{E}_\xi[e^{-\beta(\tau_- - T_{N_-})-\gamma\tau_- - \langle\kappa, M_{N_-}\rangle}; N_- = 0] + \mathbb{E}_\xi[e^{-\beta(\tau_- - T_{N_-})-\gamma\tau_- - \langle\kappa, M_{N_-}\rangle}; N_- \geq 1]\\ &= \mathbb{E}\, e^{-(\lambda+\beta+\gamma)\xi/c - \langle\kappa, M\rangle} + \mathbb{E}_\xi[e^{-\beta(\tau_- - T_{N_-})-\gamma\tau_- - \langle\kappa, M_{N_-}\rangle}; N_- \geq 1].\end{aligned} \tag{A1}$$

To analyze the second term, we exploit the fact that there are several excursions of $Z - \underline{Z}$ from 0. Therefore, we set

$$C(t) := \inf\{s \geq 0 \colon Z(s) - Z(0) = -t\},$$

where an infimum over an empty set should be interpreted as infinity.

It is obvious that C is a subordinator with drift $1/c$, and that it jumps at rate λ/c with jumps distributed as τ_- under $\mathbb{P}_\xi$. This observation implies with Theorem VII.1 of Bertoin [4] that

$$\Phi_{-Z}(\gamma) = \frac{\gamma}{c} + \frac{\lambda}{c}(1 - \mathbb{E}_\xi e^{-\gamma\tau_-}). \tag{A2}$$

Lemma A.1 can be applied to the Poisson process N constituted by the jump epochs of C, $\mu = \lambda/c$, and $\zeta = \xi$. Each jump of C corresponds to an excursion of $Z - \underline{Z}$ from 0, for which the "excursion overshoot," the excursion length, and the marks of the last jump are of interest. Observe that these quantities have the same distribution as $\tau_- - T_{N_-}$, τ_-, and M_{N_-}, respectively. Using the notation of Lemma A.1, this yields

$$\mathbb{E}_\xi[e^{-\beta(\tau_- - T_{N_-})-\gamma\tau_- - \langle\kappa, M_{N_-}\rangle}; N_- \geq 1] = \mathbb{E}[(\mathbb{E}_\xi e^{-\gamma\tau_-})^{N(\xi)-1} e^{-\beta A(\xi)/c - \gamma\xi/c}; N(\xi) \geq 1]\, \mathbb{E}_\xi e^{-\beta(\tau_- - T_{N_-})-\gamma\tau_- - \langle\kappa, M_{N_-}\rangle}. \tag{A3}$$

Therefore, Lemma A.1 yields

$$\begin{aligned}\mathbb{E}[s^{N(\xi)-1} e^{-\beta A(\xi)/c - \gamma\xi/c}; N(\xi) \geq 1] &= \frac{\mathbb{E}[s^{N(\xi)} e^{-\beta A(\xi)/c - \gamma\xi/c}] - \mathbb{E}\, e^{-(\lambda+\beta+\gamma)\xi/c}}{s}\\ &= \frac{\lambda}{\lambda s + \beta}[\mathbb{E}\, e^{-((1-s)\lambda+\gamma)\xi/c} - \mathbb{E}\, e^{-(\lambda+\beta+\gamma)\xi/c}].\end{aligned}$$

Upon combining this with (A1) and (A3), we arrive at

$$\mathbb{E}_\xi e^{-\beta(\tau_- - T_{N_-})-\gamma\tau_- - \langle\kappa, M_{N_-}\rangle} = \frac{[\beta + \lambda\,\mathbb{E}_\xi e^{-\gamma\tau_-}]\,\mathbb{E}\, e^{-(\lambda+\beta+\gamma)\xi/c - \langle\kappa, M\rangle}}{\lambda\,\mathbb{E}_\xi e^{-\gamma\tau_-} + \beta - \lambda\,\mathbb{E}\, e^{-(\lambda(1-\mathbb{E}_\xi e^{-\gamma\tau_-})+\gamma)\xi/c} + \lambda\,\mathbb{E}\, e^{-(\lambda+\beta+\gamma)\xi/c}},$$

which, with the help of (A2), reduces to

$$\frac{[\beta + \gamma - c\Phi_{-Z}(\gamma) + \lambda]\,\mathbb{E}\, e^{-(\beta+\gamma+\lambda)\xi/c - \langle\kappa, M\rangle}}{\beta + \gamma - c\Phi_{-Z}(\gamma) - \lambda(\mathbb{E}\, e^{-\Phi_{-Z}(\gamma)\xi} - 1) + \lambda\,\mathbb{E}\, e^{-(\beta+\gamma+\lambda)\xi/c}}.$$

By definition of Φ_{-Z}, we have

$$\gamma = \psi_{-Z}(\Phi_{-Z}(\gamma)) = c\Phi_{-Z}(\gamma) + \lambda(\mathbb{E}\, e^{-\Phi_{-Z}(\gamma)\xi} - 1),$$

and the claim follows. $\square$

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