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Rothblum's description of the stable marriage polyhedron is TDI

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Rothblum's description of the stable marriage polyhedron is TDI*

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Abstract

Rothblum showed in [7] that the convex hull of the stable matchings of a bipartite preference system can be described by an elegant system of linear inequalities. In this note we show that the description given by Rothblum is totaly dual integral. Our proof is based on the results of Gusfield and Irving on rotations.

1 Introduction

The stable marriage problem was introduced by Gale and Shapley [4], who showed that every bipartite preference system has a stable matching, and gave an algorithm that finds one. Since then, a lot of progress has been made in understanding the problem and its non-bipartite version, the so-called stable roommates problem. Of particular interest are the results of Vande Vate [8] and Rothblum [7], who gave simple systems of linear inequalities that describe the convex hull of stable matchings of a bipartite preference system.

The contribution of the present paper is that the linear system of Rothblum is in fact a totally dual integral system, and integral dual optimal solutions can be derived from the dual solutions of the associated rotation system.

Let $G = (U, V; E)$ be a bipartite graph, and for every $w \in U \cup V$ let $<_w$ be a linear order of the edges incident to w. The set of these linear orders is denoted by $\mathcal{O}$, and the pair $(G, \mathcal{O})$ is called a *bipartite preference system.* The notation $e \leqslant_w f$ is used if $e <_w f$ or $e = f$ (we say that e *dominates* f). An edge e is said to be *better at w*

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than f if $e <_v f$. We use $e <_U f$ to denote that e and f has a common endnode in U and there e is better than f.

Let $E' \subseteq E$ be a set of edges. An edge $e \in E$ *blocks* E' if e is not dominated by any element of E'. A matching M is called *stable* if it is not blocked by any edge of E, i.e. M dominates every edge. In particular, every stable matching is inclusion-wise maximal. For a node w covered by M, let $p_M(w)$ denote the other endnode of the edge of M covering w.

The rest of this section contains some well-known results on stable matchings that are used in the subsequent proofs. Gale and Shapley proved that every bipartite preference system has a stable matching, and they gave an algorithm for finding one. The structural properties of stable matchings were first described in [6]. It is easy to see that any two stable matchings M, N cover the same node set. For $u \in U$, let $\min_u(M, N)$ be the best edge of $M \cup N$ at u, and let $\max_u(M, N)$ be the worst edge of $M \cup N$ at u (if u is not covered by $M \cup N$, then $\min_u(M, N) = \max_u(M, N) = \emptyset$). Let $M \wedge N := \{\min_u(M, N) \mid u \in U\}$ and $M \vee N := \{\max_u(M, N) \mid u \in U\}$. It is easy to see that $M \wedge N$ and $M \vee N$ are stable matchings. Conway proved that the set $\mathcal{M}$ of stable matchings with the operations $\wedge$ and $\vee$ forms a distributive lattice, which has a unique minimal element (the U-optimal stable matching M_U) and a unique maximal element (the V-optimal stable matching M_V). The algorithm of Gale and Shapley finds M_U or M_V.

2 Rotations

In this section we describe the basic properties of so-called rotations, and show how a minimum cost stable matching can be found using the structure of the rotations of the bipartite preference system. These results are taken from the book of Gusfield and Irving [5].

Let $(G, \mathcal{O})$ be a bipartite preference system, M a stable matching, and $u \in U$ a node covered by M. Let $s_M(u)$ denote the node $v \in V$ (if it exists) for which the following hold:

- $(u, v) \in E$ and (u, v) is better at v than $(u, p_M(u))$,
- if $(u, v') \in E$ and $(u, v') <_u (u, v)$, then v' is covered by M, and (u, v') is not better at v' than $(p_M(v'), v')$.

It is easy to see that if an edge (u, v) is between $(u, s_M(u))$ and $(u, p_M(u))$ according to the linear order at u, then (u, v) is not in a stable matching.

A cycle $\rho = (v_1, u_1, v_2, u_2, \ldots v_k, u_k)$ is called a *rotation* if there is a stable matching M such that $(u_i, v_i) \in M$ and $v_{i+1} = s_M(u_i)$ for every $i = 1, 2, \ldots k$ (where $v_{k+1} = v_1$). If these properties hold for a rotation ρ and a stable matching M, then we say that ρ can be *eliminated* from M. Let $M/\rho := M \setminus \{(v_i, u_i) : i = 1, 2, ...k\} \cup \{(u_i, v_{i+1}) : i =$

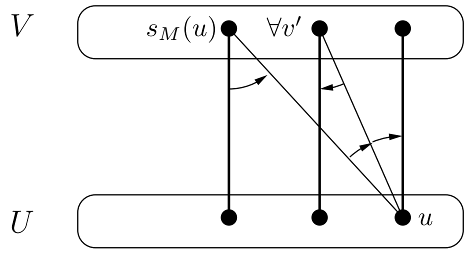

Figure 1: Definition of $s_M(u)$

$1, 2, ...k\}$, this is obtained by *eliminating* ρ from M. We say that the edges of type (v_i, u_i) are *discarded edges* in ρ, and the edges of type (u_i, v_{i+1}) are *promoted edges* in ρ. Thus at u_i the discarded edge is better than the promoted edge, and at v_i the promoted edge is better than the discarded edge.

Claim 2.1 ([5]). *M/ρ is a stable matching, and M/ρ covers M in the lattice $\mathcal{M}$.* □

It turns out that rotations have a very rich structure which completely describes the structure of stable matchings. Let R be the set of rotations of the bipartite preference system. The following are true:

- Let M and N be stable matchings such that $M \wedge N = M$ in the lattice of stable matchings. Then N can be obtained from M by successively eliminating a sequence of rotations. The set of rotations that have to be eliminated is unique (but the sequence is not). In particular, every stable matching can be obtained by eliminating a sequence of rotations from M_U. We say that a rotation ρ is *eliminated* in M if ρ is in the set of rotations that have to be eliminated to obtain M from M_U.
- A partial order can be defined on R: $\rho \preceq \rho'$ if ρ is eliminated in every stable matching where ρ' is eliminated. A set X of rotations can be eliminated from M_U in some order if and only if it is a *closed set* in this partial order ('closed' means that if $\rho_1 \in X$ and $\rho_2 \preceq \rho_1$, then $\rho_2 \in X$).

It follows from the above facts that there is a one-to-one correspondence between the closed sets in the partial order of rotations (including the empty set and the set R) and the stable matchings of the preference system. Let R_M denote the set of rotations corresponding to the stable matching M.

Given a cost function $c : E \to \mathbb{Z}$ on the edges of the graph, we can define a cost function $c' : R \to \mathbb{Z}$ on the rotations the following way: For a rotation $\rho := (v_1, u_1, \ldots, v_k, u_k)$ let

$$c'(\rho) := -c(v_1u_1) + c(u_1v_2) - c(v_2u_2) + c(u_2v_3) - \cdots + c(u_kv_1). \tag{1}$$

Then for every stable matching M, $c(M) = c(M_U) + c'(R_M)$. This means that a minimum cost stable matching corresponds to a minimum cost closed set of rotations.

In the following we define a directed graph $D = (R, A)$ on the set of rotations, with the property that its transitive closure is the partial order $\prec$, i.e. there is a directed path in D from ρ to ρ' if and only if $\rho \preceq \rho'$. The digraph D has two types of edges.

Type 1: $(\rho, \rho') \in A$ if there is an edge $(u, v) \in E$ $(u \in U, v \in V)$ contained in both rotations such that in ρ the other edge at u is better than (u, v), and n ρ' the other edge at u is worse than (u, v).

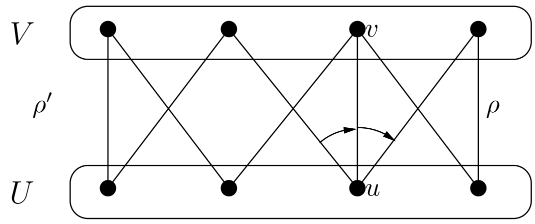

Figure 2: Edge of type 1 in D

Type 2: $(\rho, \rho') \in A$ if there is an edge $(u, v) \in E$ $(u \in U, v \in V)$ which is between the two edges of ρ' incident to u according to $<_u$, and is between the two edges of ρ incident to v according to $<_v$.

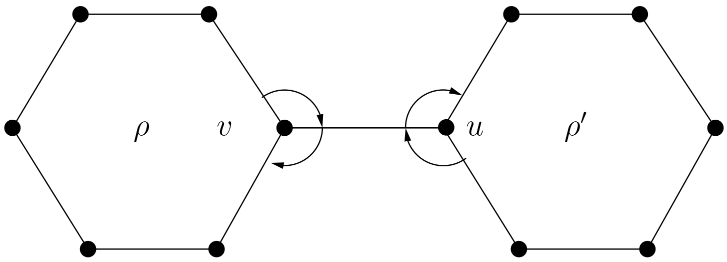

Figure 3: Edge of type 2 in D

It is easy to see that if $(\rho, \rho') \in A$, then $\rho \prec \rho'$.

Theorem 2.2 ([5]). *The transitive closure of the digraph D is the partial order $\preceq$.*

Corollary 2.3 ([5]). *There is a one-to-one correspondence between the stable matchings of the preference system $(G, \mathcal{O})$ and the sets of in-degree 0 of D.*

The sets of in-degree 0 of D can be described as the integer points of the following polyhedron:

$$\begin{aligned} 0 \leqslant x \leqslant 1&, \\ x(\rho) - x(\rho') \geqslant 0&, \quad \text{if } (\rho, \rho') \in A. \end{aligned} \tag{2}$$

The matrix of this system is totally unimodular since it is a network matrix. Given a cost function c, finding a minimum cost stable matching corresponds to the problem of finding an integer solution x of the above system for which $c'x$ is minimal, where c' is the cost function defined in (1). The minimum value of $c'x$ is equal to $c(M_{opt})-c(M_U)$, where M_{opt} is a minimum weight stable matching.

If we consider the dual problem, we obtain the following result.

Corollary 2.4. *Let c' be the cost function defined in* (1). *There exists an integer vector $z \in \mathbb{Z}^{R\cup A}$ such that*

$$\begin{aligned} z_\rho &\geqslant 0 && \text{if } \rho \in R, \\ z_{(\rho,\rho')} &\geqslant 0, && \text{if } (\rho,\rho') \in A, \\ -z_\rho + z(\Delta^+(\rho)) - z(\Delta^-(\rho)) &\leqslant c'(\rho), && \text{if } \rho \in R, \end{aligned} \tag{3}$$

and

$$-\sum_{\rho\in R} z_\rho = c(M_{opt}) - c(M_U),$$

where M_{opt} is a minimum weight stable matching, and $\Delta^+(\rho)$ (resp. $\Delta^-(\rho)$) denotes the set of edges of D leaving (resp. entering) ρ.

3 Polyhedral results

3.1 Variables on stable matching edges

Given a bipartite preference system $(G=(U,V;E),\mathcal{O})$, let E_{st} denote the set of edges in E that belong to some stable matching. We first show a TDI system with variables $x \in \mathbb{R}^{E_{st}}$ that describes the convex hull of stable matchings.

Theorem 3.1. *The following system with variables $x \in \mathbb{R}^{E_{st}}$ is TDI:*

$$\begin{aligned} \min cx \quad & s.t. \\ x &\geqslant 0, \\ x(\varphi_{st}(e)) &\geqslant 1, && \text{if } e \in E \setminus E_{st}, \\ x(\varphi_{st}(e)) &= 1, && \text{if } e \in E_{st}, \end{aligned} \tag{4}$$

where $\varphi_{st}(e)$ is the set of edges in E_{st} that dominate e. Furthermore, the system describes the convex hull of stable matchings.

Proof. It is easy to see that all stable matchings satisfy the inequalities, so they belong to the polyhedron. Let x be an integer element of the polyhedron, and $e=(u,v) \in E_{st}$ such that e is the worst edge at u in E_{st}. We know from the lattice property of stable matchings that e is the best edge at v, so $\varphi_{st}(e) = D_{st}(u)$, where $D_{st}(u)$ is the set of edges in E_{st} incident to u. It follows that $x(D_{st}(u)) = 1$ if u is covered by a stable matching, so x is a matching that covers the nodes covered by every stable matching. The other inequalities imply that x is actually a stable matching.

By the above argument, the TDI property implies that the system describes the convex hull of stable matchings. To prove the TDI property, it is enough to show that for every integer cost function $c \in \mathbb{Z}^{E_{st}}$ there exists an integer dual vector $y \in \mathbb{Z}^E$ that satisfies the following:

$$y(e) \geqslant 0, \qquad \text{if } e \in E \setminus E_{st}, \tag{5}$$

$$y(\psi(e)) \leqslant c(e), \qquad \text{if } e \in E_{st}, \tag{6}$$

$$\sum_{e \in E} y(e) = c(M_{opt}), \tag{7}$$

where $\psi(e)$ is the set of edges dominated by e, and M_{opt} is a minimum cost stable matching.

We will construct a feasible y using the vector $z \in \mathbb{Z}^{R \cup A}$ which exists according to Corollary 2.4 (here R is the set of rotations and $D = (R, A)$ is the acyclic digraph referred to in Theorem 2.2). Let $\rho_1, \dots, \rho_r$ be a topological order of D, i.e. an order of the rotations in which they can be eliminated. We will denote z_{ρ_i} by z_i and $z_{(\rho_i, \rho_j)}$ by z_{ij}.

The construction of y consists of constructing a sequence of vectors $y_0, y_1, \dots, y_r$, such that y_t satisfies the inequalities of type (6) on the edges of M_U and on the edges that appear in rotations $\rho_1, \dots, \rho_t$, and $\sum_{e \in E} y_t(e) = c(M_U) - \sum_{i=1}^t z_i$. This would imply that $y := y_r$ satisfies all inequalities of type (6) and $\sum_{e \in E} y(e) = c(M_{opt})$, since $\sum_{i=1}^r z_i = c(M_U) - c(M_{opt})$. Thus the constructed y would have the required properties.

In order to make this step-by-step construction possible, some additional technical conditions are required for the vectors $y_0, \dots, y_r$. For $i = 1, \dots, r$, let us choose an arbitrary promoted edge e_0^i of the rotation ρ_i. If $(\rho_i, \rho_j) \in A$ and it is an edge of type 2, then we choose an edge $e_{ij} = uv$ that is between the two edges of ρ_i incident to v according to $<_v$, and is between the two edges of ρ_j incident to u according to $<_u$ (such an edge exists because (ρ_i, ρ_j) is an edge of type 2).

For $0 \leqslant t \leqslant r$ and $e \in E_{st}$, let

$$c_t(e) = c(e) - \sum \{ z_{li} \mid l \leqslant t < i, \ e \leqslant_V e_0^l, \ (\rho_l, \rho_i) \in A \}.$$

Note that $c_r = c$. We will define vectors $y_0, y_1, \dots, y_r$ in $\mathbb{Z}^E$ such that the following conditions hold for every $0 \leqslant t \leqslant r$:

(C1) $y_t(e) \geqslant 0$ if $e \in E \setminus E_{st}$,

(C2) $y_t(\psi(e)) \leqslant c_t(e)$ if $e \in M_U \cup \rho_1 \cup \rho_2 \cup \dots \cup \rho_t$,

(C3) $\sum_{e \in E} y_t(e) = c(M_U) - \sum_{i=1}^{t} z_i$, (C4) supp $y_t \subseteq M_U \cup \rho_1 \cup \rho_2 \cup \cdots \cup \rho_t \cup \{e_{ij} : (i, j \leqslant t)\}$.

The condition (C2) means that at some edges the inequality (6) should hold with a surplus that depends on t and the digraph D. This surplus can be used in the construction of subsequent y_i vectors. As we have already mentioned, the vector $y := y_r$ is in the dual polyhedron, and $\sum_{e\in E} y_e = c(M_{opt})$, so it is an optimal dual solution.

Let

$$y_0(e) := \begin{cases} c(e) & \text{if } e \in M_U, \\ 0 & \text{otherwise} \end{cases}$$

Then (C2) holds for every edge of M_U, and $\sum_{e\in E} y_0(e) = c(M_U)$, as required. The other conditions are also satisfied.

The vector y_t is obtained from y_{t-1} by changing it only on the edges of $\rho_1 \cup \cdots \cup \rho_t$ and on the edges e_{lt} (for $l \leqslant t$). Let $w^t_1, w^t_2, \dots w^t_{2k}$ be the nodes of the rotation ρ_t in reverse order, such that $(w^t_{2k}, w^t_1) = e^t_0$. Thus w^t_i is in U if i is odd and it is in V if i is even.

Let e^t_i denote the edge (w^t_i, w^t_{i+1}). Then $e^t_i <_{w^t_i} e^t_{i-1}$. Every edge e^t_{2i+1} is a discarded edge in ρ_t, hence $y_{t-1}(\psi(e^t_{2i+1})) \leqslant c_{t-1}(e^t_{2i+1})$.

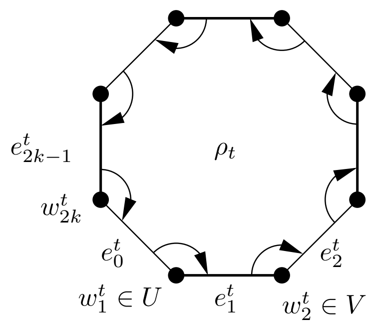

Figure 4: The rotation ρ_t (the thick edges are the discarded edges)

We will use the fact that if en edge e is incident to w^t_{2i+1} and $y_{t-1}(e) \neq 0$, then $e \leqslant_{w^t_{2i+1}} e^t_{2i+1} <_{w^t_{2i+1}} e^t_{2i}$. Analogously, if e is incident to w^t_{2i} and $y_{t-1}(e) \neq 0$, then $e \geqslant_{w^t_{2i}} e^t_{2i-1} >_{w^t_{2i}} e^t_{2i}$.

Before constructing y_t, we will define a vector y'_t that satisfies conditions (C1), (C3) and (C4) for t, and $y'_t(\psi(e)) \leqslant c_{t-1}(e)$ for every $e \in M_U \cup \rho_1 \cup \rho_2 \cup \cdots \cup \rho_t$ except for e^t_0. The vector y'_t is obtained from y_{t-1} by changing the values only on the edges of ρ_t. For $j = 0, 1, \dots k-1$ let

$$y'_t(e^t_{2j+1}) := y_{t-1}(e^t_{2j+1}) + \sum_{i=1}^{2j} (-1)^{i+1} c(e^t_i),$$

$$y'_t(e^t_{2j}) := \sum_{i=1}^{2j} (-1)^i c(e^t_i),$$

and let

$$y'_t(e^t_0) := -z_t.$$

It is easy to check that if $e \in M_U \cup \rho_1 \cup \rho_2 \cup \dots \cup \rho_{t-1}$, then $y'_t(\psi(e)) \leqslant y_{t-1}(\psi(e))$, so (C2) holds with respect to $t-1$.

We also have to check that (C2) for $t-1$ holds on the promoted edges of ρ_t. Observe that $c(e^t_{2j}) - c(e^t_{2j-1}) = c_{t-1}(e^t_{2j}) - c_{t-1}(e^t_{2j-1})$ and $y_{t-1}(\psi(e^t_{2j})) = y_{t-1}(\psi(e^t_{2j-1}))$ for every j. Using these facts,

$$\begin{aligned}
&y'_t(\psi(e^t_{2j}) = y'_t(e^t_{2j}) + y_{t-1}(\psi(e^t_{2j-1})) + y'_t(e^t_{2j-1}) - y_{t-1}(e^t_{2j-1}) = \\
&= \sum_{i=1}^{2j} (-1)^i c(e^t_i) + y_{t-1}(\psi(e^t_{2j-1})) + \sum_{i=1}^{2j-2} (-1)^{i+1} c(e^t_i) = \\
&= y_{t-1}(\psi(e^t_{2j-1})) - c_{t-1}(e^t_{2j-1}) + c_{t-1}(e^t_{2j}) \leqslant \\
&\leqslant c_{t-1}(e^t_{2j}),
\end{aligned}$$

so $y'_t(\psi(e)) \leqslant c_{t-1}(e)$ for every $e \in M_U \cup \rho_1 \cup \rho_2 \cup \dots \cup \rho_t$ except for e^t_0.

Condition (C3) holds because

$$\sum_{e \in E} y'_t(e) = \sum_{e \in E} y_{t-1}(e) - z_t = c(M_U) - \sum_{i=1}^{t} z_i.$$

To obtain y_t from y'_t, we make the following changes for every l for which $(\rho_l, \rho_t) \in A$.

Suppose that $(\rho_l, \rho_t) \in A$ is an edge of type 1. Then there is a common edge in the two rotations that has even index in ρ_l and odd index in ρ_t, say $e^l_{2i} = e^t_{2j+1}$. For every $0 \leqslant p \leqslant i-1$ we increase y'_t by z_{lt} on the edges e^l_{2p}, and decrease by z_{lt} on the edges e^l_{2p+1}. For every $j+1 \leqslant q \leqslant k-1$ we increase y'_t by z_{lt} on the edges e^t_{2q}, and we decrease by z_{lt} on the edges e^t_{2q+1}.

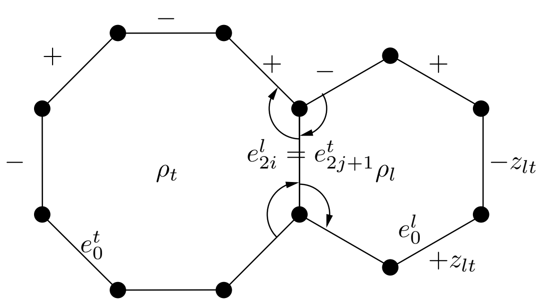

Figure 5: If (ρ_l, ρ_t) is an edge of type 1

If there is an edge of type 2 from ρ_l to ρ_t, then the edge e_{lt} has an endnode of even index in ρ_l, say w^l_{2i}, and it has an endnode of odd index in ρ_t, say w^l_{2j+1}.

For every $0 \leqslant p \leqslant i-1$ we increase y'_t by z_{lt} on the edges e^l_{2p}, and decrease by z_{lt} on the edges e^l_{2p+1}. We increase y'_t by z_{lt} on e_{lt} and on the edges e^t_{2q} for every $j+1 \leqslant q \leqslant k-1$. For every $j \leqslant q \leqslant k-1$ we decrease by z_{lt} on the edges e^t_{2q+1}.

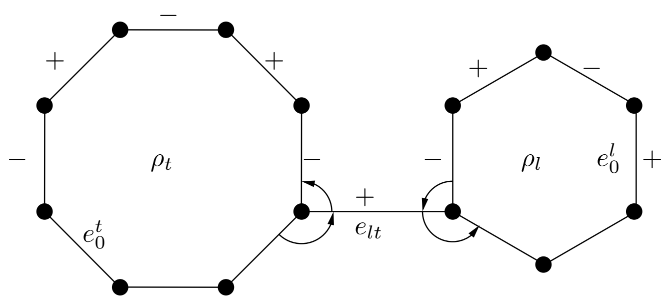

Figure 6: If (ρ_l, ρ_t) is an edge of type 2

After such a modification, condition (C1) holds because y'_t is increased on e_{lt} by a non-negative value.

The number of edges where we increased by z_{lt} is the same as the number of edges where we decreased, thus

$$\sum_{e \in E} y_t(e) = \sum_{e \in E} y'_t(e) = c(M_U) - \sum_{i=1}^{t} z_i.$$

For a fixed l, if $e \in M_U \cup \rho_1 \cup \dots \cup \rho_t \setminus \{e^t_0\}$ and $e \not\leqslant_V e^l_0$ then the number of edges in $\psi(e)$ that were increased by z_{lt} is the same as the number of edges in $\psi(e)$ that were decreased. If $e \in M_U \cup \rho_1 \cup \dots \cup \rho_t \setminus \{e^t_0\}$ and $e \leqslant_V e^l_0$ then $y'_t(\psi(e))$ increases by z_{lt}, which is exactly the allowed amount, since the term z_{lt} appears in $c_{t-1}(e)$ but not in $c_t(e)$.

Since $\{e \in \rho_t : \; e \leqslant_V e^t_0\} = \{e^t_0\}$, the only condition we have yet to verify is that $y_t(\psi(e^t_0)) \leqslant c_t(e^t_0)$. Let us introduce the notation $\alpha := \sum\{z_{lt} \mid l < t, \; e^t_0 \leqslant_V e^l_0, \; (\rho_l, \rho_t) \in A\}$. Then $c_t(e^t_0) = c_{t-1}(e^t_0) - z(\Delta^+(\rho_t)) + \alpha$, so the following holds:

$$\begin{aligned}
&y_t(\psi(e^t_0)) = y'_t(\psi(e^t_0)) - z(\Delta^-(\rho_t)) + \alpha = \\
&= y'_t(e^t_0) + y_{t-1}(\psi(e^t_{2k-1})) + y'_t(e^t_{2k-1}) - y_{t-1}(e^t_{2k-1}) - z(\Delta^-(\rho_t)) + \alpha = \\
&= -z_t + y_{t-1}(\psi(e^t_{2k-1})) + \sum_{i=1}^{2k-2} (-1)^{i+1} c(e^t_i) - z(\Delta^-(\rho_t)) + \alpha \leqslant \\
&\leqslant c'(\rho_t) - z(\Delta^+(\rho_t)) + c_{t-1}(e^t_{2k-1}) - c'(\rho_t) - c(e^t_{2k-1}) + c(e^t_0) + \alpha \leqslant \\
&\leqslant c'(\rho_t) - z(\Delta^+(\rho_t)) + c_{t-1}(e^t_{2k-1}) - c'(\rho_t) - c_{t-1}(e^t_{2k-1}) + c_{t-1}(e^t_0) + \alpha = \\
&= c_{t-1}(e^t_0) - z(\Delta^+(\rho_t)) + \alpha = c_t(e^t_0),
\end{aligned}$$

where we used the fact that $-z_t - z(\Delta^-(\rho_t)) \leqslant c'(\rho_t) - z(\Delta^+(\rho_t))$ by (3). Thus y_t satisfies condition (C2) on all the required edges. □

3.2 Variables on all edges

In [7], Rothblum gave a linear system that describes the convex hull of stable matchings of an arbitrary bipartite preference system. We now prove, using Theorem 3.1, that this system is also TDI.

Theorem 3.2. *The following system, with variables $x \in \mathbb{R}^E$, is totally dual integral:*

$$x \geqslant 0, \tag{8}$$

$$-x(D(w)) \geqslant -1, \qquad \text{if } w \in U \cup V, \tag{9}$$

$$x(\varphi(e)) \geqslant 1, \qquad \text{if } e \in E. \tag{10}$$

Proof. Let $c \in \mathbb{Z}^E$ be an integer cost function. We have to find an integer optimal dual solution for c, i.e. vectors $\pi \in \mathbb{Z}^{U\cup V}$ and $y \in \mathbb{Z}^E$ that satisfy

$$y(e) \geqslant 0, \tag{11}$$

$$\pi(w) \geqslant 0, \tag{12}$$

$$-\pi(u) - \pi(v) + y(\psi(e)) \leqslant c(e), \qquad \text{if } e = (u,v) \in E, \tag{13}$$

and

$$-\sum_{w\in U\cup V} \pi(w) + \sum_{e\in E} y(e) = c(M_{opt}).$$

Let $y_0 \in \mathbb{Z}^E$ be an integral dual optimal solution of the system (4) for the cost function c restricted to E_{st}, which exists by Theorem 3.1, and let π_0 be the all-zero vector on $U \cup V$. Then (y_0, π_0) satisfies (11) if $e \in E \setminus E_{st}$, it satisfies (13) if $e \in E_{st}$, and $-\sum_{w\in U\cup V} \pi_0(w) + \sum_{e\in E} y_0(e) = c(M_{opt})$.

If we increase y by 1 on the edges of a stable matching M and increase π by 1 on every node in U covered by M, then we get a dual vector for which the objective value is the same, and the left side of (13) does not increase for any edge, since $|\psi(e) \cap M| \leqslant 1$, and if $|\psi(e) \cap M| = 1$ then both endnodes of e are covered by M, otherwise e would block M. Moreover, if e is dominated by 2 edges of M then the left side of (13) decreases for e. Let E' be the set of edges that are dominated by 2 edges of some stable matching. By applying modifications of the above type, a dual vector (y_1, π_1) can be constructed which satisfies (11) for every edge, and satisfies (13) for edges in $E_{st} \cup E'$.

Let $e = (u,v) \notin E_{st} \cup E'$. There is no matching M such that $(u, p_M(u)) <_u e <_u (u, s_M(u))$, since then two edges of M would dominate e. It follows that either $e >_u (u, p_{M_V}(u))$ or $e >_v (p_{M_U}(v), v)$. Suppose that the first case holds (the second one can be treated similarly by exchanging U and V). Let $(v = v_1, u_1, v_2, u_2, \ldots v_k, u_k)$ be a maximal sequence such that $(u_i, v_i) \in M_V$ for every $i = 1, \ldots, k$ and $v_{i+1} = s_{M_V}(u_i)$ for every $i = 1, \ldots, k-1$. Since no rotation can be eliminated from M_V, $s_M(u_k)$ must be undefined. This is possible in the following two cases.

Case 1: there is no edge (u_k, v') that is better at v' than $(p_{M_V}(v'), v')$. In this case we increase y by 1 on edges in $M_V \setminus \{(u_1, v_1), \dots, (u_k, v_k)\} \cup \{(u_1, v_2), \dots, (u_{k-1}, v_k)\}$, and increase π by 1 on every node in $U - u_k$ covered by M_V.

Case 2: there is an edge $(u_k, v_{k+1}) \in E$ such that v_{k+1} is not covered by M_V and $(u_k, v_{k+1}) <_{u_k} (u_k, v')$ if (u_k, v') is better at v' than $(p_{M_V}(v'), v')$. In this case we increase y by 1 on the edges in $M_V \setminus \{(u_1, v_1), \dots, (u_k, v_k)\} \cup \{(u_1, v_2), \dots, (u_k, v_{k+1})\}$, and increase π by 1 on every node in U covered by M_V.

It is easy to see that in both cases the objective value remains the same, the left side of (13) does not increase for any edge, and it decreases by 1 for e. So by applying such modifications on every edge where (13) does not hold we can obtain an integer optimal dual solution. □

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