LIFTS OF CONVEX SETS AND CONE FACTORIZATIONS

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ABSTRACT. In this paper we address the basic geometric question of when a given convex set is the image under a linear map of an affine slice of a given closed convex cone. Such a representation or "lift" of the convex set is especially useful if the cone admits an efficient algorithm for linear optimization over its affine slices. We show that the existence of a lift of a convex set to a cone is equivalent to the existence of a factorization of an operator associated to the set and its polar via elements in the cone and its dual. This generalizes a theorem of Yannakakis that established a connection between polyhedral lifts of a polytope and nonnegative factorizations of its slack matrix. Symmetric lifts of convex sets can also be characterized similarly. When the cones live in a family, our results lead to the definition of the rank of a convex set with respect to this family. We present results about this rank in the context of cones of positive semidefinite matrices. Our methods provide new tools for understanding cone lifts of convex sets.

1. Introduction

Linear optimization over convex sets plays a central role in optimization. In many instances, a convex set $C \subset \mathbb{R}^n$ may come with a complicated representation that cannot be altered if one is restricted in the number of variables and type of representation that can be used. For instance, the n-dimensional cross-polytope

$$C_n := \{x \in \mathbb{R}^n \,:\, \pm x_1 \pm x_2 \cdots \pm x_n \leq 1\}$$

requires the above 2^n constraints in any representation of it by linear inequalities in n variables. However, C_n is the projection onto the x-coordinates of the polytope

$$Q_n := \{(x, y) \in \mathbb{R}^{2n} \,:\, \sum_{i=1}^{n} y_i = 1, -y_i \leq x_i \leq y_i \,\forall\, i = 1, \ldots, n\}$$

which is described by $2n + 1$ linear constraints and $2n$ variables, and one can optimize a linear function $\langle c, x\rangle$ over C_n by instead optimizing it over Q_n. Since the running time of linear programming algorithms depends on the number of linear constraints of the feasible region, the latter representation allows rapid optimization over C_n. More generally, if a convex set $C \subset \mathbb{R}^n$ can be written as the image under a linear map of an affine slice of a cone that admits efficient algorithms for linear optimization, then one can optimize a linear function efficiently over C as well. For instance, linear optimization over affine slices of the k-dimensional nonnegative orthant $\mathbb{R}^k_+$ is linear programming, and over the cone of $k \times k$ real symmetric positive semidefinite matrices $\mathcal{S}^k_+$ is semidefinite programming, both of which admit efficient algorithms. Motivated by this fact, we ask the following basic geometric questions about a given convex set $C \subset \mathbb{R}^n$:

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(1) *Given a full-dimensional closed convex cone $K \subset \mathbb{R}^m$, when does there exist an affine subspace $L \subset \mathbb{R}^m$ and a linear map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ such that $C = \pi(K \cap L)$?*
(2) *If the cone K comes from a family (K_k) (e.g. $(\mathbb{R}^k_+)$ or $(\mathcal{S}^k_+)$), then what is the least k for which $C = \pi(K_k \cap L)$ for some π and L?*

If $C = \pi(K \cap L)$, then $K \cap L$ is called a *K-lift* of C. In [30], Yannakakis points out a remarkable connection between the smallest k for which a polytope has a $\mathbb{R}^k_+$-lift and the *nonnegative rank* of its *slack matrix*. The main result of our paper is an extension of Yannakakis' result to the general scenario of K being any closed convex cone and C any convex set, answering Question (1) above. The main tool is a generalization of nonnegative factorizations of nonnegative matrices to *cone factorizations* of *slack operators* of convex sets.

This paper is organized as follows. In Section 2 we present our main result (Theorem 2.4) characterizing the existence of a K-lift of a convex set $C \subset \mathbb{R}^n$, when K is a full-dimensional closed convex cone in $\mathbb{R}^m$. A K-lift of C is *symmetric* if it respects the symmetries of C. In Theorem 2.12, we characterize the existence of a *symmetric K-lift* of C. Although symmetric lifts are quite special, they have received much attention. The main result in [30] was that a symmetric $\mathbb{R}^k_+$-lift of the *matching polytope* of the complete graph on n vertices requires k to be at least exponential in n. Results in [17] and [24] have shown that symmetry imposes strong restrictions on the minimum size of polyhedral lifts. Proposition 2.8 describes geometric operations on convex sets that preserve the existence of cone lifts.

In Section 3 we focus on polytopes. As a corollary of Theorem 2.4 we obtain Theorem 3.3 which generalizes Yannakakis' result for polytopes [30, Theorem 3] to arbitrary closed convex cones K. We illustrate Theorems 3.3 and 2.12 using polygons in the plane.

Section 4 tackles Question (2) and considers ordered families of cones, $\mathcal{K} = (K_k)$, that can be used to lift a given $C \subset \mathbb{R}^n$, or more simply, to factorize a nonnegative matrix M. When all faces of all cones in $\mathcal{K}$ are again in $\mathcal{K}$, we define $\text{rank}_{\mathcal{K}}(C)$ (respectively, $\text{rank}_{\mathcal{K}}(M)$) to be the smallest k such that C has a K_k-lift (respectively, M has a K_k-factorization). We focus on the case of $\mathcal{K} = (\mathbb{R}^k_+)$ when $\text{rank}_{\mathcal{K}}(\cdot)$ is called nonnegative rank, and $\mathcal{K} = (\mathcal{S}^k_+)$ when $\text{rank}_{\mathcal{K}}(\cdot)$ is called psd rank. Section 4.1 gives the basic definitions and properties of cone ranks. We find (different) families of nonnegative matrices that show that the gap between any pair among: rank, psd rank and nonnegative rank, can become arbitrarily large. In Section 4.2 we derive lower bounds on nonnegative and psd ranks of polytopes. We note that the nonnegative rank of a polytope is also called the *extension complexity* of the polytope by some authors in reference to this invariant being the smallest k for which the polytope admits a $\mathbb{R}^k_+$-lift. Corollary 4.13 shows a lower bound for the nonnegative rank of a polytope in terms of the size of an antichain of its face lattice. Corollary 4.18 gives an upper bound on the number of facets of a polytope with psd rank k. This subsection also finds families of polytopes whose slack matrices exhibit arbitrarily large gaps between rank and nonnegative rank, as well as rank and psd rank.

In Section 5 we give two applications of our methods. When $C = \text{STAB}(G)$ is the *stable set polytope* of a graph G with n vertices, Lovász constructed a convex approximation of C called the *theta body* of G. This body is the projection of an affine slice of $\mathcal{S}^{n+1}_+$, and when G is a *perfect graph*, it coincides with $\text{STAB}(G)$. Our methods show that this construction is optimal in the sense that for any G, $\text{STAB}(G)$ cannot admit a $\mathcal{S}^k_+$-lift for any $k \leq n$. A result of Burer shows that every $\text{STAB}(G)$ has a $\mathcal{C}^*_{n+1}$-lift where $\mathcal{C}^*_{n+1}$ is the cone of *completely positive matrices* of size $(n+1) \times (n+1)$. We illustrate Burer's result in terms of Theorem 2.4 on a cycle of length five. The second part of Section 5 interprets Theorem 2.4 in the context of *rational lifts* of convex hulls of algebraic sets. We show in Theorem 5.6 that in this case, the positive semidefinite factorizations required by Theorem 2.4 can be interpreted in terms of sums of squares polynomials and rational maps.

In the last few decades, several *lift-and-project* methods have been proposed in the optimization literature that aim to provide tractable descriptions of convex sets. These methods construct a series of nested convex approximations to $C \subset \mathbb{R}^n$ that arise as projections of higher dimensional convex sets. Examples can be found in [1, 28, 20, 19, 23, 15, 18] and [7]. In these methods, C is either a 0/1-polytope or more generally, the convex hull of a semialgebraic set, and the cones that are used in the lifts are either nonnegative orthants or the cones of positive semidefinite matrices. The success of a lift-and-project method relies on whether a lift of C is obtained at some step of the procedure. Questions (1) and (2), and our answers to them, address this convergence question and offer a uniform framework within which to study all lift-and-project methods for convex sets using closed convex cones.

There have been several recent developments that were motivated by the results of Yannakakis in [30]. As mentioned earlier, Kaibel, Pashkovich and Theis proved that symmetry can impose severe restrictions on the minimum size of a polyhedral lift of a polytope. An exciting new result of Fiorini, Massar, Pokutta, Tiwary and de Wolf shows that there are *cut*, *stable set* and *traveling salesman* polytopes for which there can be no polyhedral lift of size polynomial in the number of vertices of the associated graphs. Their paper [12] also gives an interpretation of positive semidefinite rank of a nonnegative matrix in terms of quantum communication complexity extending the connection between nonnegative rank and classical communication complexity established in [30].

2. Cone lifts of convex bodies

A convex set is called a *convex body* if it is compact and contains the origin in its interior. To simplify notation, we will assume throughout the paper that the convex sets $C \subset \mathbb{R}^n$ for which we wish to study cone lifts are all convex bodies, even though our results hold for all convex sets. Recall that the *polar* of a convex set $C \subset \mathbb{R}^n$ is the set

$$C^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1, \quad \forall x \in C\}.$$

Let $\mathrm{ext}(C)$ denote the set of *extreme points* of C, namely, all points $p \in C$ such that if $p = (p_1 + p_2)/2$, with $p_1, p_2 \in C$, then $p = p_1 = p_2$. Since C is compact with the origin in its interior, both C and C° are convex hulls of their respective extreme points. Consider the operator $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $S(x, y) = 1 - \langle x, y\rangle$. We define the *slack operator* S_C, of the convex set C, to be the restriction of S to $\mathrm{ext}(C) \times \mathrm{ext}(C^\circ)$.

Definition 2.1. Let $K \subset \mathbb{R}^m$ be a full-dimensional closed convex cone and $C \subset \mathbb{R}^n$ a full-dimensional convex body. A *K-lift* of C is a set $Q = K \cap L$, where $L \subset \mathbb{R}^m$ is an affine subspace, and $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map such that $C = \pi(Q)$. If L intersects the interior of K we say that Q is a *proper K-lift* of C.

We will see that the existence of a K-lift of C is intimately connected to properties of the slack operator S_C. Recall that the *dual* of a closed convex cone $K \subset \mathbb{R}^m$ is

$$K^* = \{y \in \mathbb{R}^m : \langle x, y\rangle \geq 0, \quad \forall x \in K\}.$$

A cone K is *self-dual* if $K^* = K$. In particular, the cones $\mathbb{R}^n_+$ and $\mathcal{S}^k_+$ are self-dual.

Definition 2.2. Let C and K be as in Definition 2.1. We say that the slack operator S_C is *K-factorizable* if there exist maps (not necessarily linear)

$$A : \text{ext}(C) \to K \quad \text{and} \quad B : \text{ext}(C^\circ) \to K^*$$

such that $S_C(x, y) = \langle A(x), B(y)\rangle$ for all $(x, y) \in \text{ext}(C) \times \text{ext}(C^\circ)$.

Remark 2.3. The maps A and B may be defined over all of C and C° by picking a representation of each $x \in C$ (similarly, $y \in C^\circ$) as a convex combination of extreme points of C (respectively, C°) and extending A and B linearly. Such extensions are not unique.

With the above set up, we can now characterize the existence of a K-lift of C.

Theorem 2.4. *If C has a proper K-lift then S_C is K-factorizable. Conversely, if S_C is K-factorizable then C has a K-lift.*

Proof: Suppose C has a proper K-lift. Then there exists an affine space $L = w_0 + L_0$ in $\mathbb{R}^m$ (L_0 is a linear subspace) and a linear map $\pi : \mathbb{R}^m \to \mathbb{R}^n$ such that $C = \pi(K \cap L)$ and $w_0 \in \text{int}(K)$. Equivalently,

$$C = \{x \in \mathbb{R}^n : x = \pi(w), \quad w \in K \cap (w_0 + L_0)\}.$$

We need to construct the maps $A : \text{ext}(C) \to K$ and $B : \text{ext}(C^\circ) \to K^*$ that factorize the slack operator S_C, from the K-lift of C. For $x_i \in \text{ext}(C)$, define $A(x_i) := w_i$, where w_i is any point in the non-empty convex set $\pi^{-1}(x_i) \cap K \cap L$.

Let c be an extreme point of C°. Then $\max\{\langle c, x\rangle : x \in C\} = 1$ since $\langle c, x\rangle \leq 1$ for all $x \in C$, and if the maximum was smaller than one, then c would not be an extreme point of C°. Let M be a full row rank matrix such that $\ker M = L_0$. Then the following hold:

$$1 = \max_{x \in C} \langle c, x\rangle = \max_{w \in K \cap (w_0 + L_0)} \langle c, \pi(w)\rangle = \max_{\substack{Mw = Mw_0 \\ w \in K}} \langle \pi^*(c), w\rangle$$

Since w_0 lies in the interior of K, by Slater's condition we have strong duality, and we get

$$1 = \min \langle Mw_0, y\rangle \; : \; M^T y - \pi^*(c) \in K^*$$

with the minimum being attained. Further, setting $z = M^T y$ we have that

$$1 = \min \langle w_0, z\rangle \; : \; z - \pi^*(c) \in K^*, \, z \in L_0^\perp$$

with the minimum being attained. Now define $B : \text{ext}(C^\circ) \to K^*$ as the map that sends $y_i \in \text{ext}(C^\circ)$ to $B(y_i) := z - \pi^*(y_i)$, where z is any point in the nonempty convex set $L_0^\perp \cap (K^* + \pi^*(y_i))$ that satisfies $\langle w_0, z\rangle = 1$. Note that for such a z, $\langle w_i, z\rangle = 1$ for all $w_i \in L$. Then $B(y_i) \in K^*$, and for an $x_i \in \text{ext}(C)$,

$$\begin{aligned} \langle x_i, y_i\rangle = \langle \pi(w_i), y_i\rangle = \langle w_i, \pi^*(y_i)\rangle &= \langle w_i, z - B(y_i)\rangle \\ &= 1 - \langle w_i, B(y_i)\rangle = 1 - \langle A(x_i), B(y_i)\rangle. \end{aligned}$$

Therefore, $S_C(x_i, y_i) = 1 - \langle x_i, y_i\rangle = \langle A(x_i), B(y_i)\rangle$ for all $x_i \in \text{ext}(C)$ and $y_i \in \text{ext}(C^\circ)$.

Suppose now S_C is K-factorizable, i.e., there exist maps $A : \text{ext}(C) \to K$ and $B : \text{ext}(C^\circ) \to K^*$ such that $S_C(x, y) = \langle A(x), B(y)\rangle$ for all $(x, y) \in \text{ext}(C) \times \text{ext}(C^\circ)$. Consider the affine space

$$L = \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m : 1 - \langle x, y\rangle = \langle z, B(y)\rangle \, , \; \forall \, y \in \text{ext}(C^\circ)\},$$

and let L_K be its coordinate projection into $\mathbb{R}^m$. Note that $0 \notin L_K$ since otherwise, there exists $x \in \mathbb{R}^n$ such that $1 - \langle x, y\rangle = 0$ for all $y \in \text{ext}(C^\circ)$ which implies that C° lies in the affine hyperplane $\langle x, y\rangle = 1$. This is a contradiction since C° contains the origin. Also, $K \cap L_K \neq \emptyset$ since for each $x \in \text{ext}(C)$, $A(x) \in K \cap L_K$ by assumption.

Let x be some point in $\mathbb{R}^n$ such that there exists some $z \in K$ for which (x, z) is in L. Then, for all extreme points y of C° we will have that $1 - \langle x, y\rangle$ is nonnegative. This implies, using convexity, that $1 - \langle x, y\rangle$ is nonnegative for all y in C°, hence $x \in (C^\circ)^\circ = C$.

We now argue that this implies that for each $z \in K \cap L_K$ there exists a unique $x_z \in \mathbb{R}^n$ such that $(x_z, z) \in L$. That there is one, comes immediately from the definition of L_K. Suppose now that there is another such point x'_z. Then $(tx_z + (1-t)x'_z, z) \in L$ for all reals t which would imply that the line through x_z and x'_z would be contained in C, contradicting our assumption that C is compact.

The map that sends z to x_z is therefore well-defined in $K \cap L_K$, and can be easily checked to be affine. Since the origin is not in L_K, we can extend it to a linear map $\pi : \mathbb{R}^m \to \mathbb{R}^n$. To finish the proof it is enough to show $C = \pi(K \cap L_K)$. We have already seen that $\pi(K \cap L_K) \subseteq C$ so we just have to show the reverse inclusion. For all extreme points x of C, $A(x)$ belongs to $K \cap L_K$, and therefore, $x = \pi(A(x)) \in \pi(K \cap L_K)$. Since $C = \text{conv}(\text{ext}(C))$ and $\pi(K \cap L_K)$ is convex, $C \subseteq \pi(K \cap L_K)$. □

The restriction to proper lifts in Theorem 2.4 is not important if the cone K has a well-understood facial structure as in the case of nonnegative orthants and cones of positive semidefinite matrices. If there exists a K-lift that is not proper, then there is a proper lift to a face of K and we could pass to this face to obtain a cone factorization. Since our proof uses strong duality, it is not obvious how to remove the properness assumption for a general closed convex cone. However, there is a situation under which properness can be dropped.

Definition 2.5. [8] A cone K is *nice* if $K^* + F^\perp$ is closed for all faces F of K.

Corollary 2.6. *If K is a nice cone, then whenever C has a K-lift (not necessarily proper), S_C has a K-factorization.*

Proof: In [25] Pataki notes that K is nice if and only if $F^* = K^* + F^\perp$ for all faces F of K. Let $A : \text{ext}(C) \to F$ and $B : \text{ext}(C^\circ) \to F^*$ be the F-factorization of S_C from the proper lift of C to a face F of K. Then A is also a map from $\text{ext}(C)$ to K. Define $B' : \text{ext}(C^\circ) \to K^*$ as $B'(y) = z \in K^*$ such that $B(y) - z \in F^\perp$. Then $\langle A(x), B(y)\rangle = \langle A(x), B'(y)\rangle$ for all $(x, y) \in \text{ext}(C) \times \text{ext}(C^\circ)$ and we obtain a K-factorization of S_C. □

Polyhedral cones, second order cones and the cones of real symmetric psd matrices $\mathcal{S}^k_+$ are all nice. In [25] Pataki shows that if a cone is nice then all its faces are exposed and he conjectures that the converse is also true.

We now present a simple illustration of Theorem 2.4 using $K = \mathcal{S}^2_+$.

Example 2.7. Let C be the unit disk in $\mathbb{R}^2$ which can be written as

$$C = \left\{ (x, y) \in \mathbb{R}^2 : \begin{pmatrix} 1+x & y \\ y & 1-x \end{pmatrix} \succeq 0 \right\}.$$

This means that S_C must have a $\mathcal{S}^2_+$ factorization. Since $C^\circ = C$, $\text{ext}(C) = \text{ext}(C^\circ) = \partial C$, and so we have to find maps $A, B : \text{ext}(C) \to \mathcal{S}^2_+$ such that for all $(x_1, y_1), (x_2, y_2) \in \text{ext}(C)$,

$$\langle A(x_1, y_1), B(x_2, y_2)\rangle = 1 - x_1 x_2 - y_1 y_2.$$

But this is accomplished by the maps

$$A(x_1, y_1) = \begin{pmatrix} 1+x_1 & y_1 \\ y_1 & 1-x_1 \end{pmatrix}$$

and

$$B(x_2, y_2) = \frac{1}{2}\begin{pmatrix} 1-x_2 & -y_2 \\ -y_2 & 1+x_2 \end{pmatrix}$$

which factorize S_C and can easily be checked to be positive semidefinite in their domains.

The lifts of convex bodies are preserved by many common geometric operators.

Proposition 2.8. *If C_1 and C_2 are convex bodies, and K_1 and K_2 are closed convex cones such that C_1 has a K_1-lift and C_2 has a K_2-lift, then the following are true:*

1. *If π is any linear map, then $\pi(C_1)$ has a K_1-lift;*
2. *C_1° has a K_1^*-lift;*
3. *Every exposed face of C_1 has a K_1-lift;*
4. *The cartesian product $C_1 \times C_2$ has a $K_1 \times K_2$-lift;*
5. *The Minkowski sum $C_1 + C_2$ has a $K_1 \times K_2$-lift;*
6. *The convex hull $\mathrm{conv}(C_1 \cup C_2)$ has a $K_1 \times K_2$-lift.*

Proof: The first property follows immediately from the definition of a K_1-lift. The second is an immediate consequence of Theorem 2.4. For the third property, if a face F of C_1 is exposed, then $F = C_1 \cap H$ where H is a hyperplane in $\mathbb{R}^n$. If $K_1 \cap L$ is a K_1-lift of C, then $K_1 \cap L'$ is a K_1-lift of F where L' is the affine space obtained by adding the equation of H to the equations defining L. The fourth property is again easy to derive from the definition since, if $C_1 = \pi_1(K_1 \cap L_1)$ and $C_2 = \pi_2(K_2 \cap L_2)$, then $C_1 \times C_2 = (\pi_1 \times \pi_2)(K_1 \times K_2 \cap L_1 \times L_2)$. The fifth one follows from (1) and the fact that the Minkowski sum $C_1 + C_2$ is a linear image of the cartesian product $C_1 \times C_2$.

For the sixth, we use the fact that $\mathrm{conv}(C_1 \cup C_2)^\circ = C_1^\circ \cap C_2^\circ$. Given factorizations A_1, B_1 of S_{C_1} and A_2, B_2 of S_{C_2}, we have seen that we can extend A_i to all of C_i, and B_i to all of C_i°, and get that $1 - \langle x, y\rangle = \langle A_i(x), B_i(y)\rangle$ for all $(x, y) \in C_i \times C_i^\circ$. Furthermore, extend A_1 to $\mathrm{conv}(C_1 \cup C_2)$ by defining it to be zero outside C_1 and set A_2 to be zero outside $C_2 \setminus C_1$. Then, since $\mathrm{ext}(\mathrm{conv}(C_1 \cup C_2)) \subseteq \mathrm{ext}(C_1) \cup \mathrm{ext}(C_2)$ and $\mathrm{ext}(C_1^\circ \cap C_2^\circ)$ is contained in both C_1° and C_2°, the maps, $(A_1, A_2) : \mathrm{ext}(\mathrm{conv}(C_1 \cup C_2)) \to K_1 \times K_2$ and $(B_1, B_2) : \mathrm{ext}(\mathrm{conv}(C_1 \cup C_2)^\circ) \to K_1^* \times K_2^*$ give a $K_1 \times K_2$ factorization of $S_{\mathrm{conv}(C_1 \cup C_2)}$. $\square$

Explicit constructions of the lifts guaranteed in Proposition 2.8 can be found in the work of Ben-Tal, Nesterov and Nemirovski; see e.g. [5, 22]. They were especially interested in the case of lifts into the cones of positive semidefinite matrices. Of significant interest is the relationship between lifts and duality, particularly when considering a self-dual cone K. When K is self dual, Theorem 2.4 shows that the existence of a K-lift is a property of both the convex body and its polar making the theory invariant under duality. We now examine the behavior of cone lifts under projective transformations.

Proposition 2.9. *Let $C \subset \mathbb{R}^n$ be a convex body with a K-lift where $K \subset \mathbb{R}^m$ is a closed convex cone. If Π is a projective transformation with $\Pi(C)$ compact, then $\Pi(C)$ has a K-lift.*

Proof: Without loss of generality we may assume the lift to be proper by passing to the smallest face of K containing the lift of C. Then, by Theorem 2.4, there exists maps

$A : \mathrm{ext}(C) \to K$ and $B : \mathrm{ext}(C^\circ) \to K^*$ factorizing S_C, and we can extend their domains to C and C° as noted in Remark 2.3. Recall that a real projective transformation Π in $\mathbb{R}^n$ is a map sending x to $Px/(1 + \langle c, x\rangle)$ where P is some $n \times n$ (invertible) real matrix, and c a vector in $\mathbb{R}^n$. The compactness of $\Pi(C)$ is equivalent to $1 + \langle c, x\rangle$ not vanishing on C and so we may assume without loss of generality that $1 + \langle c, x\rangle$ is positive on C.

Since for $y \in \Pi(C)^\circ$ and $x \in C$, $0 \leq 1-\langle y, \Pi(x)\rangle = 1-\frac{y^T Px}{1+\langle c,x\rangle} = \frac{1+\langle c,x\rangle - y^T Px}{1+\langle c,x\rangle}$, we have that $\left\langle P^T y - c, x\right\rangle \leq 1$, and therefore, $z_y := P^T y - c \in C^\circ$. Consider the maps $A' : \Pi(C) \to K$ and $B' : \Pi(C)^\circ \to K^*$ given by $A'(x) = A(\Pi^{-1}(x))/(1 + \langle c, \Pi^{-1}(x)\rangle)$ and $B'(y) = B(z_y)$. These maps form a K-factorization of $S_{\Pi(C)}$ and hence, $\Pi(C)$ has a K-lift by Theorem 2.4. The case of affine transformations is trivial, but can be seen as a particular case of the projective case we just proved. $\square$

A restricted class of lifts that has received much attention is that of *symmetric lifts*. The idea there is to demand that the lift not only exists, but also preserves the symmetries of the object being lifted. Several definitions of symmetry have been studied in the context of lifts to nonnegative orthants in papers such as [30], [17] and [24]. Theorem 2.4 can be extended to symmetric lifts.

Let G be a subgroup of GL_n acting on $\mathrm{ext}(C)$. A simple example of such a group would be $\mathrm{Aut}(C)$, the group of all rigid linear transformations φ of $\mathbb{R}^n$ such that $\varphi(C) = C$, restricted to $\mathrm{ext}(C)$. Any such group G is compact, hence has a unique measure μ_G, its Haar measure, such that $\mu_G(G) = 1$ and μ_G is invariant under multiplication, i.e., $\mu_G(gU) = \mu_G(U)$ for all $g \in G$ and all $U \subseteq G$. Note that allowing affine transformations instead of linear ones, would not be essentially different, as any group of affine transformations acting on a compact set has a common fixed point, so after a translation of C it would be simply a subgroup of GL_n.

Definition 2.10. Let K be a closed convex cone and C a convex body, such that $C = \pi(K \cap L)$ for some affine subspace L and linear map π. Furthermore, let $G \subseteq \mathrm{GL}_n$ be a group acting on $\mathrm{ext}(C)$ and $H \subseteq \mathrm{GL}_m$ a group acting on K. We say that the lift $K \cap L$ of C is (G, H)-*symmetric* if there exists a group homomorphism from G to H sending $\varphi \in G$ to $f_\varphi \in H$ such that $f_\varphi(K\cap L) = K\cap L$ and $\pi\circ f_\varphi = \varphi\circ\pi$, when restricted to $K\cap L\cap\pi^{-1}(\mathrm{ext}(C))$. We will say the lift is *symmetric* if it is $(\mathrm{Aut}(C), \mathrm{Aut}(K))$-symmetric.

The lifts obtained from the traditional lift-and-project methods mentioned in the Introduction are often symmetric in the sense of Definition 2.10, so it makes sense to study such lifts. In order to get a symmetric version of Theorem 2.4, we have to introduce a notion of *symmetric factorization* of S_C.

Definition 2.11. Let C, K, G and H be as in Definition 2.10, and $A : \mathrm{ext}(C) \to K$ and $B : \mathrm{ext}(C^\circ) \to K^*$ a K-factorization of S_C. We say that the factorization is (G, H)-*symmetric* if there exists a group homomorphism from G to H sending $\varphi \in G$ to $f_\varphi \in H$ such that $A \circ \varphi = f_\varphi \circ A$. Call the factorization *symmetric* if it is $(\mathrm{Aut}(C), \mathrm{Aut}(K))$-symmetric.

Note that any action of $G \subseteq \mathrm{GL}_n$ on C defines trivially an action of G on C°, and similarly any action of $H \subseteq \mathrm{GL}_m$ on K defines an action on K^*. With these actions we can see that if a K-factorization is (G, H)-symmetric in the sense of the previous definition, the group homomorphism f would also verify $B \circ \varphi = f_\varphi \circ B$. Hence, Definition 2.11 is actually invariant with respect to polarity, even if it seems to only depend on the map A. This observation would still be true if we had considered G and H to be subgroups of projective transformations of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, but general linear groups are enough to cover all interesting examples we know. We can now establish the symmetric version of Theorem 2.4.

Theorem 2.12. *If C has a proper (G,H)-symmetric K-lift then S_C has a (G,H)-symmetric K-factorization. Conversely, if S_C has a (G,H)-symmetric K-factorization then C has a (G,H)-symmetric K-lift.*

Proof: First suppose that C has a proper (G,H)-symmetric K-lift with $C = \pi(K \cap L)$. For each orbit of the action of the group G on $\text{ext}(C)$, pick a representative x_0, and let $A'(x_0)$ be any point in $K \cap L$ such that $\pi(A'(x_0)) = x_0$. Let $G_{x_0} \subseteq G$ be the subgroup of all automorphisms that fix x_0. Then we can define

$$A(x_0) := \int_{\varphi \in G_{x_0}} f_\varphi(A'(x_0)) d\mu_{G_{x_0}}$$

which generalizes the construction in [30, Step 2, pp 449]. For a finite group, this is just the usual average of all images of $A'(x_0)$ under the action of G_{x_0}. For any other point x' in the same orbit as x_0, pick any ψ such that $\psi(x_0) = x'$ and define $A(x') := f_\psi(A(x_0))$. The point $A(x')$ in $K \cap L$ does not actually depend on the choice of ψ. To see this it is enough to note that $f_\mu \circ A(x_0) = A(x_0)$ for all $\mu \in G_{x_0}$ and if ψ_1 and ψ_2 both send x_0 to x', then $f_{\psi_1}^{-1} \circ f_{\psi_2} = f_{\psi_1^{-1}\psi_2}$ and $\psi_1^{-1}\psi_2$ is in G_{x_0}.

Since $K \cap L$ is a proper lift of C, we know we have a K-factorization of S_C by Theorem 2.4. If we follow the proof of that result, we see that it is actually constructive, in the sense that we can pick as a map from $\text{ext}(C) \to K$ any section of the projection π. In particular, we can pick the map A we just defined, since we have $\pi(A(x)) = x$ for every $x \in \text{ext}(C)$. This means that such a map A can be extended to a K-factorization A, B of S_C. For any $\mu \in G$ and $x \in \text{ext}(C)$, we have $A \circ \mu(x) = A \circ \mu \circ \psi(x_0)$, for some ψ and x_0 in the orbit of x and so, by the above considerations,

$$A \circ \mu(x) = f_{\mu \circ \psi} \circ A(x_0) = f_\mu \circ f_\psi \circ A(x_0) = f_\mu \circ A(\psi x_0) = f_\mu \circ A(x),$$

and hence, we have a (G,H)-symmetric K-factorization of S_C.

Suppose now we have a (G,H)-symmetric K-factorization of S_C. Since it is in particular a K-factorization of S_C, we have a K-lift $K \cap L$ of C by Theorem 2.4. From the proof of that theorem we know that $A(x)$ is in $K \cap L$ for all $x \in \text{ext}(C)$. Let L' be the affine subspace of L spanned by all such points $A(x)$. It is clear from the definition that L' is f_φ invariant for all $\varphi \in G$. Furthermore, given any $y \in L'$ we can write it as an affine combination $\sum_i \alpha_i A(x_i)$ for some x_i in $\text{ext}(C)$, and so for all $\varphi \in G$, we have

$$\pi(f_\varphi(y)) = \sum_i \alpha_i \pi(f_\varphi(A(x_i))) = \sum_i \alpha_i \pi(A(\varphi x_i)) = \sum_i \alpha_i \varphi x_i,$$

which is simply the image of $\pi(y)$ under φ. Hence, $K \cap L'$ is a (G,H)-symmetric lift of C. $\square$

3. Cone lifts of polytopes

The results developed in the previous section for general convex bodies specialize nicely to polytopes, providing a more general version of the original result of Yannakakis relating polyhedral lifts of polytopes and nonnegative factorizations of their slack matrices. We first introduce the necessary definitions.

For a full-dimensional polytope P in $\mathbb{R}^n$, let $V_P = \{p_1, \ldots, p_v\}$ be its set of vertices, F_P its set of facets, and $f := |F_P|$. Recall that each facet F_i in F_P corresponds to a unique (up to multiplication by nonnegative scalars) linear inequality $h_i(x) \geq 0$ that is valid on P such that $F_i = \{x \in P : h_i(x) = 0\}$. These form (again up to multiplication by nonnegative scalars) the unique irredundant representation of P as

$$P = \{x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_f(x) \geq 0\}.$$

Since we are assuming that the origin is in the interior of P, $h_i(0) > 0$ for each $i = 1, \ldots, f$. Therefore, we can make the facet description of P unique by normalizing each h_i to verify $h_i(0) = 1$. We will call this the *canonical inequality representation of* P.

Definition 3.1. Let P be a full-dimensional polytope in $\mathbb{R}^n$ with vertex set $V_P = \{p_1, \ldots, p_v\}$ and with an inequality representation

$$P = \{x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_f(x) \geq 0\}.$$

Then the nonnegative matrix in $\mathbb{R}^{v \times f}$ whose (i, j)-entry is $h_j(p_i)$ is called a *slack matrix of* P. If the h_i form the canonical inequality representation of P, we call the corresponding slack matrix the *canonical slack matrix of* P.

In the case of a polytope P, $\mathrm{ext}(P)$ is just V_P, and the elements of $\mathrm{ext}(P^\circ)$ are in bijection with the facets of P. This means that the operator S_P is actually a finite map from $V_P \times F_P$ to $\mathbb{R}_+$ that sends a pair (p_i, F_j) to $h_j(p_i)$, where h_j is the canonical inequality corresponding to the facet F_j. Hence,we may identify the slack operator of P with the canonical slack matrix of P and use S_P to also denote this matrix. We now need a definition about factorizations of non-negative matrices.

Definition 3.2. Let $M = (M_{ij}) \in \mathbb{R}_+^{p \times q}$ be a nonnegative matrix and K a closed convex cone. Then a K-*factorization* of M is a pair of ordered sets $a^1, \ldots, a^p \in K$ and $b^1, \ldots, b^q \in K^*$ such that $\langle a^i, b^j \rangle = M_{ij}$.

Note that $M \in \mathbb{R}_+^{p \times q}$ has a $\mathbb{R}_+^k$-factorization if and only if there exist a $p \times k$ nonnegative matrix A and a $k \times q$ nonnegative matrix B such that $M = AB$. Therefore, Definition 3.2 generalizes nonnegative factorizations of nonnegative matrices to arbitrary closed convex cones. Since any slack matrix of P can be obtained from the canonical one by multiplication by a diagonal nonnegative matrix, it is K-factorizable if and only if S_P is K-factorizable. We can now state Theorem 2.4 for polytopes.

Theorem 3.3. *If a full-dimensional polytope P has a proper K-lift then every slack matrix of P admits a K-factorization. Conversely, if some slack matrix of P has a K-factorization then P has a K-lift.*

Theorem 3.3 is a direct translation of Theorem 2.4 using the identification between the slack operator of P and the canonical slack matrix of P. The original theorem of Yannakakis [30, Theorem 3] proved this result in the case where K was some nonnegative orthant $\mathbb{R}_+^l$.

Example 3.4. To illustrate Theorem 3.3 consider the regular hexagon in the plane with canonical inequality description

$$H = \left\{ (x_1, x_2) \in \mathbb{R}^2 \ : \ \begin{pmatrix} 1 & \sqrt{3}/3 \\ 0 & 2\sqrt{3}/3 \\ -1 & \sqrt{3}/3 \\ -1 & -\sqrt{3}/3 \\ 0 & -2\sqrt{3}/3 \\ 1 & -\sqrt{3}/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

We will denote the coefficient matrix by F and the right hand side vector by d. It is easy to check that H cannot be the projection of an affine slice of $\mathbb{R}^k_+$ for $k < 5$. Therefore, we ask whether it can be the linear image of an affine slice of $\mathbb{R}^5_+$, which turns out to be surprisingly non-trivial. Using Theorem 3.3 this is equivalent to asking if the canonical slack matrix of the hexagon,

$$S_H := \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{pmatrix},$$

has a $\mathbb{R}^5_+$-factorization. Check that

$$S_H = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where we call the first matrix A and the second matrix B. We may take the rows of A as elements of $\mathbb{R}^5_+$, and the columns of B as elements of $\mathbb{R}^5_+ = (\mathbb{R}^5_+)^*$, and they provide us a $\mathbb{R}^5_+$-factorization of the slack matrix S_H, proving that this hexagon has a $\mathbb{R}^5_+$-lift while the trivial polyhedral lift would have been to $\mathbb{R}^6_+$.

We can construct the lift explicitly using the proof of the Theorem 2.4. Note that

$$H = \{(x_1, x_2) \in \mathbb{R}^2 : \exists \ y \in \mathbb{R}^5_+ \text{ s.t. } Fx + B^T y = d\}.$$

Hence, the exact slice of $\mathbb{R}^5_+$ that is mapped to the hexagon is simply

$$\{y \in \mathbb{R}^5_+ : \exists \ x \in \mathbb{R}^2 \text{ s.t. } B^T y = d - Fx\}.$$

By eliminating the x variables in the system we get

$$\{y \in \mathbb{R}^5_+ \ : \ y_1 + y_2 + y_3 + y_5 = 2, y_3 + y_4 + y_5 = 1\},$$

and so we have a three dimensional slice of $\mathbb{R}^5_+$ projecting down to H. This projection is visualized in Figure 1.

The hexagon is a good example to see that the existence of lifts depends on more than the combinatorics of the facial structure of the polytope. If instead of a regular hexagon we

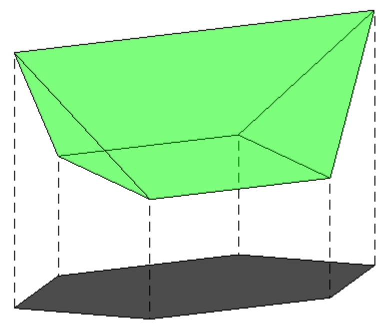

FIGURE 1. Lift of the regular hexagon.

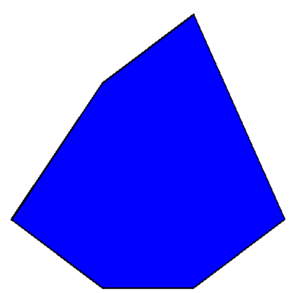

FIGURE 2. Irregular hexagon with no $\mathbb{R}^5_+$-lift.

take the hexagon with vertices $(0,-1)$, $(1,-1)$, $(2,0)$, $(1,3)$, $(0,2)$ and $(-1,0)$, as seen in Figure 2, a valid slack matrix would be

$$S := \begin{pmatrix} 0 & 0 & 1 & 4 & 3 & 1 \\ 1 & 0 & 0 & 4 & 4 & 3 \\ 7 & 4 & 0 & 0 & 4 & 9 \\ 3 & 4 & 4 & 0 & 0 & 1 \\ 3 & 5 & 6 & 1 & 0 & 0 \\ 0 & 1 & 3 & 5 & 3 & 0 \end{pmatrix}.$$

One can check that if a 6×6 matrix with the zero pattern of a slack matrix of a hexagon has a $\mathbb{R}^5_+$-factorization, then it has a factorization with either the same zero pattern as the matrices A and B obtained before, or the patterns given by applying a cyclic permutation to the rows of A and the columns of B. A simple algebraic computation then shows that the slack matrix S above has no such decomposition hence this irregular hexagon has no $\mathbb{R}^5_+$-lift.

Symmetric lifts of polytopes are especially interesting to study since the automorphism group of a polytope is finite. We now show that there are polygons with n sides for which a symmetric $\mathbb{R}^k_+$-lift requires k to be at least n.

Proposition 3.5. *A regular polygon with n sides where n is either a prime number or a power of a prime number cannot admit a symmetric $\mathbb{R}^k_+$-lift where $k < n$.*

Proof: A symmetric $\mathbb{R}^k_+$-lift of a polytope P implies the existence of an injective group homomorphism from $\text{Aut}(P)$ to $\text{Aut}(\mathbb{R}^k_+)$. Since the rigid transformations of $\mathbb{R}^k_+$ are the permutations of coordinates, $\text{Aut}(\mathbb{R}^k_+)$ is the symmetric group S_k. This implies that the cardinality of $\text{Aut}(P)$ must divide $k!$.

Let P be a regular p-gon where p is prime. Since $\text{Aut}(P)$ has $2p$ elements, and the smallest k such that $2p$ divides $k!$ is p (since $p > 2$), we can never do better than a symmetric $\mathbb{R}^p_+$-lift for P. If P is a p^t-gon, then the homomorphism from $\text{Aut}(P)$ to S_k must send an element of order p^t to an element whose order is a multiple of p^t. The smallest symmetric group with an element of order p^t is S_{p^t} and hence, P cannot have a symmetric $\mathbb{R}^k_+$-lift with $k < p^t$. $\square$

In Example 3.4 we saw a $\mathbb{R}^5_+$-lift of a regular hexagon, but notice that the accompanying factorization is not symmetric.

Remark 3.6. Ben-Tal and Nemirovski have shown in [6] that a regular n-gon admits a $\mathbb{R}^k_+$-lift where $k = O(\log n)$. Combining their result with Proposition 3.5 provides a simple family of polytopes where there is an exponential gap between the sizes of the smallest possible symmetric and non-symmetric lift into nonnegative orthants. This provides a simple illustration of the impact of symmetry on the size of lifts, a phenomenon that was investigated in detail by Kaibel, Pashkovich and Theis in [17].

4. Cone ranks of convex bodies

In Section 2 we established necessary and sufficient conditions for the existence of a K-lift of a given convex body $C \subset \mathbb{R}^n$ for a fixed cone K. In many instances, the cone K belongs to a family such as $(\mathbb{R}^i_+)_i$ or $(\mathcal{S}^i_+)_i$. In such cases, it becomes interesting to determine the smallest cone in the family that admits a lift of C. In this section, we study this scenario and develop the notion of *cone rank* of a convex body.

4.1. **Definitions and basics.**

Definition 4.1. A *cone family* $\mathcal{K} = (K_i)_{i\in\mathbb{N}}$ is a sequence of closed convex cones K_i indexed by $i \in \mathbb{N}$. The family $\mathcal{K}$ is said to be *closed* if for every $i \in \mathbb{N}$ and every face F of K_i there exists $j \leq i$ such that F is isomorphic to K_j.

Example 4.2.

(1) The set of nonnegative orthants $(\mathbb{R}^i_+,\ i \in \mathbb{N})$ form a closed cone family.
(2) The family $(\mathcal{S}^i_+, i \in \mathbb{N})$ where $\mathcal{S}^i_+$ is the set of all $i \times i$ real symmetric positive semidefinite matrices is closed since every face of $\mathcal{S}^i_+$ is isomorphic to a $\mathcal{S}^j_+$ for $j \leq i$ [3, Chapter II.12].
(3) Recall that a $i \times i$ symmetric matrix A is *copositive* if $x^T A x \geq 0$ for all $x \in \mathbb{R}^i_+$. Let the cone of $i \times i$ symmetric copositive matrices be denoted as C_i. This family is not closed — the set of all $i \times i$ matrices with zeroes on the diagonal and nonnegative off-diagonal entries form a face of C_i that is isomorphic to the nonnegative orthant of dimension $\binom{i}{2}$.
(4) The dual of C_i is the cone C_i^* of all *completely positive* matrices which are exactly those symmetric $i \times i$ matrices that factorize as BB^T for some $B \in \mathbb{R}^{i\times k}_+$. The family $(C_i^*,\ i \in \mathbb{N})$ is also not closed since $\dim C_i^* = \binom{i}{2}$ while C_i^* has facets (faces of dimension $\binom{i}{2} - 1$) which therefore, cannot belong to the family.

Recall the definition of a cone factorization of a nonnegative matrix in Definition 3.2.

Definition 4.3. Let $\mathcal{K} = (K_i)_{i\in\mathbb{N}}$ be a closed cone family.

(1) The $\mathcal{K}$-rank of a nonnegative matrix M, denoted as $\operatorname{rank}_{\mathcal{K}}(M)$, is the smallest i such that M has a K_i-factorization. If no such i exists, we say that $\operatorname{rank}_{\mathcal{K}}(M) = +\infty$.
(2) The $\mathcal{K}$-rank of a convex body $C \subset \mathbb{R}^n$, denoted as $\operatorname{rank}_{\mathcal{K}}(C)$, is the smallest i such that the slack operator S_C has a K_i-factorization. If such an i does not exist, we say that $\operatorname{rank}_{\mathcal{K}}(C) = +\infty$.

In this paper, we will be particularly interested in the families $\mathcal{K} = (\mathbb{R}^i_+)$ and $\mathcal{K} = (\mathcal{S}^i_+)$. In the former case, we set $\operatorname{rank}_+(\cdot) := \operatorname{rank}_{\mathcal{K}}(\cdot)$ and call it *nonnegative rank*, and in the latter case we set $\operatorname{rank}_{\mathrm{psd}}(\cdot) := \operatorname{rank}_{\mathcal{K}}(\cdot)$ and call it *psd rank*. Our interest in cone ranks comes from their connection to the existence of cone lifts. The following is immediate from Theorem 2.4.

Theorem 4.4. *Let $\mathcal{K} = (K_i)_{i\geq 0}$ be a closed cone family and $C \subset \mathbb{R}^n$ a convex body. Then $\operatorname{rank}_{\mathcal{K}}(C)$ is the smallest i such that C has a K_i-lift.*

Proof: If $i = \operatorname{rank}_{\mathcal{K}}(C)$, then we have a K_i-factorization of the slack operator S_C, and therefore, by Theorem 2.4, C has a K_i-lift. Take the smallest j for which C has a K_j-lift and suppose $j < i$. If the lift was proper, we would get a K_j factorization of S_C for $j < i$, which contradicts that $i = \operatorname{rank}_{\mathcal{K}}(C)$. Therefore, the K_j-lift of C is not proper, and C has a lift to a proper face of K_j. Since $\mathcal{K}$ is closed, this would imply a K_l-lift of C for $l < j$ contradicting the definition of j. $\square$

In practice one might want to consider lifts to products of cones in a family. This could be dealt with by defining rank as the tuple of indices of the factors in such a product, minimal under some order. In this paper we are mostly working with the families $(\mathbb{R}^i_+)$ and $(\mathcal{S}^i_+)$, and in the first case, $\mathbb{R}^n_+ \times \mathbb{R}^m_+ = \mathbb{R}^{n+m}_+$, and in the second case, $\mathcal{S}^n_+ \times \mathcal{S}^m_+ = \mathcal{S}^{m+n}_+ \cap L$ where L is a linear space. Therefore, in these situations, there is no incentive to consider lifts to products of cones. However, if one wants to study lifts to the family of second order cones, considering products of cones makes sense.

Having defined $\operatorname{rank}_{\mathcal{K}}(M)$ for a nonnegative matrix M, it is natural to ask how it compares with the usual rank of M. We now look at this relationship for the nonnegative and psd ranks of a nonnegative matrix.

The nonnegative rank of a nonnegative matrix arises in several contexts and has wide applications [10]. As mentioned earlier, its relation to $\mathbb{R}^k_+$-lifts of a polytope was studied by Yannakakis [30]. Determining the nonnegative rank of a matrix is NP-hard in general [29], but there are obvious upper and lower bounds on it.

Lemma 4.5. *For any $M \in \mathbb{R}^{p\times q}_+$, $\operatorname{rank}(M) \leq \operatorname{rank}_+(M) \leq \min\{p, q\}$.*

Further, it is not possible in general, to bound $\operatorname{rank}_+(M)$ by a function of $\operatorname{rank}(M)$.

Example 4.6. Consider the $n\times n$ matrix M_n whose (i, j)-entry is $(i-j)^2$. Then $\operatorname{rank}(M_n) = 3$ for all n since $M_n = A_nB_n$ where row i of A_n is $(i^2, -2i, 1)$ for $i = 1, \ldots, n$ and column j of B_n is $(1, j, j^2)^T$ for $j = 1, \ldots, n$. If M_n has a $\mathbb{R}^k_+$-factorization, then there exists $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}^k_+$ such that $\langle a_i, b_j\rangle \neq 0$ for all $i \neq j$. Notice that for $i \neq j$ if $\operatorname{supp}(b_j) \subseteq \operatorname{supp}(b_i)$ then $\langle a_i, b_i\rangle = 0$ implies $\langle a_i, b_j\rangle = 0$, and hence, all the b_i's (and also all the a_i's) must have supports that are pairwise incomparable. By Sperner's lemma, the largest antichain in the Boolean lattice of subsets of $[k]$ has cardinality $\binom{k}{\lfloor \frac{k}{2}\rfloor}$, and thus we get that $n \leq \binom{k}{\lfloor \frac{k}{2}\rfloor}$. Therefore, $\operatorname{rank}_+(M_n)$ is bounded below by the smallest integer k such that $n \le \binom{k}{\lfloor \frac{k}{2} \rfloor}$. For large k, we have $\binom{k}{\lfloor \frac{k}{2} \rfloor} \approx \sqrt{\frac{2}{\pi k}} \cdot 2^k$, and the easy bound $\binom{k}{\lfloor \frac{k}{2} \rfloor} \le 2^k$ yields $\mathrm{rank}_+(M_n) \ge \log_2 n$.

The psd rank of a nonnegative matrix is connected to rank and rank_+ as follows.

Proposition 4.7. *For any nonnegative matrix M*

$$\frac{1}{2}\sqrt{1+8\,\mathrm{rank}(M)} - \frac{1}{2} \le \mathrm{rank}_{psd}(M) \le \mathrm{rank}_+(M).$$

Proof: Suppose $a_1, \dots, a_p, b_1, \dots, b_q$ give a $\mathbb{R}^r_+$-factorization of $M \in \mathbb{R}^{p\times q}_+$. Then the diagonal matrices $A_i := \mathrm{diag}(a_i)$ and $B_j := \mathrm{diag}(b_j)$ give a $\mathcal{S}^r_+$-factorization of M, and we obtain the second inequality.

Now suppose $A_1, \dots A_p, B_1, \dots, B_q$ give a $\mathcal{S}^r_+$-factorization of M. Consider the vectors

$$a_i = (A_{11}, \dots, A_{rr}, 2A_{12}, \dots, 2A_{1r}, 2A_{23}, \dots, 2A_{(r-1)r})$$

and

$$b_j = (B_{11}, \dots, B_{rr}, B_{12}, \dots, B_{1r}, B_{23}, \dots, B_{(r-1)r})$$

in $\mathbb{R}^{\binom{r+1}{2}}$ where $A = A_i$ and $B = B_j$. Then $\langle a_i, b_j\rangle = \langle A_i, B_j\rangle = M_{ij}$ so M has rank at most $\binom{r+1}{2}$. By solving for r we get the desired inequality. $\square$

There is a simple, yet important situation where $\mathrm{rank}(M)$ is an upper bound on $\mathrm{rank}_{\mathrm{psd}}(M)$.

Proposition 4.8. *Take $M \in \mathbb{R}^{p\times q}$ and let M' be the nonnegative matrix obtained from M by squaring each entry of M. Then* $\mathrm{rank}_{psd}(M') \le \mathrm{rank}(M)$. *In particular, if M is a* $0/1$ *matrix,* $\mathrm{rank}_{psd}(M) \le \mathrm{rank}(M)$.

Proof: Let $\mathrm{rank}(M) = r$ and $v_1, \dots, v_p, w_1, \dots, w_q \in \mathbb{R}^r$ be such that $\langle v_i, w_j\rangle = M_{ij}$. Consider the matrices $A_i = v_i v_i^T$, $i = 1, \dots, p$ and $B_j = w_j w_j^T$, $j = 1, \dots, q$ in $\mathcal{S}^r_+$. Then, since $\langle A_i, B_j\rangle = \langle v_i, w_j\rangle^2 = M'_{ij}$, the matrix M' has a $\mathcal{S}^r_+$-factorization. $\square$

Barvinok has generalized the above result in a recent preprint [2] to show that when the number of distinct entries in a nonnegative matrix M does not exceed k, then the psd rank of M is bounded above by $\binom{k-1+\mathrm{rank}(M)}{k-1}$. We now see that the gap between the nonnegative and psd rank of a nonnegative matrix can become arbitrarily large.

Example 4.9. Let E_n be the $n \times n$ matrix, $n \ge 2$, whose (i,j)-entry is $i - j$. Then $\mathrm{rank}(E_n) = 2$ since the vectors $a_i := (i, -1)$, $i = 1, \dots, n$ and $b_j = (1, j)$, $j = 1, \dots, n$ have the property that $\langle a_i, b_j\rangle = i - j$. Therefore, by Proposition 4.8, the matrix M_n with (i,j)-entry equal to $(i-j)^2$ has psd rank two and an explicit $\mathcal{S}^2_+$-factorization of M_n is given by the psd matrices

$$A_i := \begin{pmatrix} i^2 & -i \\ -i & 1 \end{pmatrix}, \quad i = 1, \dots, n \text{ and } B_j := \begin{pmatrix} 1 & j \\ j & j^2 \end{pmatrix}, \quad j = 1, \dots, n.$$

However, we saw in Example 4.6 that $\mathrm{rank}_+(M_n)$ grows with n. A family of $n \times n$ matrices for which psd rank is $\mathrm{O}(\log n)$ and nonnegative rank at least n^{constant} is given in [12]. For the family $\{M_n\}$, the gap between rank and psd rank can become arbitrary large.

Thus, so far we have seen that the gap between $\mathrm{rank}(M)$ and $\mathrm{rank}_+(M)$ as well as the gap between $\mathrm{rank}_{\mathrm{psd}}(M)$ and $\mathrm{rank}_+(M)$ can be made arbitrarily large for nonnegative matrices M. Results in the next subsection will imply that there are nonnegative matrices for which the gap between $\mathrm{rank}(M)$ and $\mathrm{rank}_{\mathrm{psd}}(M)$ can also become arbitrarily large.

4.2. **Lower bounds on the nonnegative and psd ranks of polytopes.** A well-known lower bound to the nonnegative rank of a matrix is the Boolean rank of the support of the matrix. The *support* of a matrix $M \in \mathbb{R}_+^{p\times q}$, is the Boolean matrix $\text{supp}(M)$ obtained by turning every non-zero entry in M to a one. The rank of $\text{supp}(M)$ in Boolean arithmetic (where $1+1=1$ and all other additions and multiplications among 0 and 1 are as for the integers) is called the *Boolean rank* of $\text{supp}(M)$ (and also of M). In terms of factorizations, Boolean rank can be defined as follows.

Definition 4.10. The *Boolean rank* of a matrix $T \in \{0,1\}^{p\times q}$ is the least integer r for which there exists $A \in \{0,1\}^{p\times r}$ and $B \in \{0,1\}^{r\times q}$ such that $T = AB$ where all additions and multiplications are in Boolean arithmetic.

We will denote the Boolean rank of $\text{supp}(M)$ as $\text{rank}_B(M)$. It is easy to see that $\text{rank}_B(M) \leq \text{rank}_+(M)$. However, it is NP-hard to compute Boolean rank and most lower bounds to $\text{rank}_+(M)$ are, in fact, lower bounds to $\text{rank}_B(M)$.

The ideas in Example 4.6 provide an elegant way of thinking about lower bounds for the nonnegative rank of a polytope. Let C be a polytope and let $L(C)$ be its face lattice. If C has a lift as $C = \pi(\mathbb{R}_+^k \cap L)$, then the map π^{-1} sends faces of C to faces of $\mathbb{R}_+^k \cap L$. Since each face of $\mathbb{R}_+^k \cap L$ is the intersection of a face of $\mathbb{R}_+^k$ with L, the map π^{-1} is an injection from $L(C)$ to the faces of $\mathbb{R}_+^k$. The faces of $\mathbb{R}_+^k$ can be identified with subsets of $[k]$ as they are of the form $F_J = \{x \in \mathbb{R}_+^k : \text{supp}(x) \subseteq J\}$ for $J \subseteq [k]$. So the map π^{-1} determines an embedding of the lattice $L(C)$ into $2^{[k]}$, the Boolean lattice of subsets of $[k]$.

Theorem 4.11. *For a polytope C, there is a Boolean factorization of* $\text{supp}(S_C)$ *of intermediate dimension k if and only if there is a lattice embedding of $L(C)$ into $2^{[k]}$.*

Proof: In this proof it is convenient to identify a subset U of $[k]$ with its incidence vector in $\{0,1\}^k$ defined as having 1 in position i if and only if $i \in U$. Given an embedding ϕ of $L(C)$ into $2^{[k]}$, a Boolean factorization AB of $\text{supp}(S_C)$ is gotten by taking the row of A indexed by vertex v of C to be $\phi(v)$, and the column of B indexed by facet F of C to be $[k]\backslash\phi(F)$. Then the (v,F) entry of $\text{supp}(M)$ is zero if and only if $\phi(v) \subseteq \phi(F)$ if and only if $v \in F$.

Suppose now we have a Boolean factorization AB of $\text{supp}(S_P)$ of intermediate dimension k. For every face F of P define

$$\phi(F) := \bigcup_{v\in F} A(v)$$

where $A(v)$ denotes the row of A indexed by vertex v. Clearly $H \subseteq F$ implies $\phi(H) \subseteq \phi(F)$. To see the reverse inclusion, suppose $H \not\subseteq F$. Pick a vertex $w \in H \setminus F$ and a facet $\tilde{F}$ containing F but not w. Let $B(\tilde{F})$ denote the column of B indexed by facet $\tilde{F}$. Since $A(w) \cap B(\tilde{F}) \neq \emptyset$, we have $\phi(H) \cap B(\tilde{F}) \neq \emptyset$. On the other hand, for all $v \in F$, we have $v \in \tilde{F}$ which implies that $A(v) \cap B(\tilde{F}) = \emptyset$ and so, $\phi(F) \cap B(\tilde{F}) = \emptyset$. Therefore, $\phi(H) \not\subseteq \phi(F)$, completing the proof. ☐

Theorem 4.11 immediately yields a lower bound on the nonnegative rank of a polytope based solely on the facial structure of the polytope.

Corollary 4.12. *Let $C \subset \mathbb{R}^n$ be a polytope and k the smallest integer such that there exists an embedding of the face lattice $L(C)$ into the Boolean lattice $2^{[k]}$. Then* $\text{rank}_+(C) \geq k$.

The Boolean rank of a 0/1 matrix is also called its *rectangle covering number.* Theorem 2.9 in [11] phrases a version of the above results in terms of rectangle covering number.

Corollary 4.13. *If $C \subset \mathbb{R}^n$ is a polytope, then the following hold:*

(1) *Let p be the size of a largest* antichain *of faces of C (i.e., a largest set of faces such that no one is contained in another). Then* $\text{rank}_+(C)$ *is bounded below by the smallest k such that $p \leq \binom{k}{\lfloor \frac{k}{2} \rfloor}$;*
(2) *(Goemans [14]) Let n_C be the number of faces of C, then* $\text{rank}_+(C) \geq \log_2(n_C)$.

Proof: The first bound follows from Corollary 4.12 since lattice embeddings preserve antichains, and the size of the largest antichain of the Boolean lattice $2^{[k]}$ is $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ (Sperner's lemma). The second bound follows from the easy fact that any embedding of $L(C)$ into $2^{[k]}$ requires $\#L(C) \leq 2^k$. □

Note that a (weaker) version of the first bound can be found in [13, Corollary 4] with the size of the largest antichain replaced by the number of vertices. As mentioned, the second lower bound essentially appears in [14]. Further lower bounds for the nonnegative rank of a polytope are overviewed in [11]. The two bounds in Corollary 4.13 are in general different. For instance, if C is a square in the plane, the Goemans bound says that $\text{rank}_+(C) \geq \log_2(10) \sim 3.32$ while the antichain bound says that $\text{rank}_+(C) \geq 4$, and thus both give the same value after rounding up. For C a three-dimensional cube, $\log_2(28) = 4.807355$ while the maximum size of an antichain of faces is 12 (take the 12 edges) and hence, the antichain lower bound is 6. Although the antichain bound can be better than Goemans' (as this example shows), asymptotically they are roughly equivalent. To see this, we notice that if $p \approx \binom{k}{\lfloor \frac{k}{2} \rfloor}$, then an asymptotic expansion yields $k \approx C_1 + \log_2 p + \frac{1}{2}\log_2(C_2 + 2\log p)$, for some small explicit constants C_1 and C_2. Since p (antichain size) is always less than or equal to the number of faces n_C, we have $\log_2 p \leq \log_2 n_C$, and thus the antichain bound is at most an additive logarithmic term greater than the Goemans bound.

We close the study of nonnegative ranks with a family of polytopes for which all slack matrices have constant rank while their nonnegative ranks can grow arbitrarily high.

Example 4.14. Let S_n be the slack matrix of a regular n-gon in the plane. Then $\text{rank}(S_n) = 3$ for all n, while, by Corollary 4.13, $\text{rank}_+(S_n) \geq \log_2(n)$.

The above lower bound is of optimal order since a regular n-gon has a $\mathbb{R}^k_+$-lift where $k = O(\log_2(n))$ by the results in [6].

The psd rank of a nonnegative matrix or convex body seems to be even harder to study than nonnegative rank and no techniques are known for finding upper or lower bounds for it in general. Here we will derive some coarse complexity bounds by providing bounds for algebraic degrees. To derive our results, we begin with a rephrasing of part of [26, Theorem 1.1] about quantifier elimination.

Theorem 4.15. *Given a formula of the form*

$$\exists\, y \in \mathbb{R}^{m-n} \,:\, g_i(x,y) \geq 0 \quad \forall\, i = 1, \ldots, s$$

where $x \in \mathbb{R}^n$ and $g_i \in \mathbb{R}[x,y]$ are polynomials of degree at most d, there exists a quantifier elimination method that produces a quantifier free formula of the form

$$\bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} (h_{ij}(x)\, \Delta_{ij}\, 0) \tag{1}$$

where $h_{ij} \in \mathbb{R}[x]$, $\Delta_{ij} \in \{>, \geq, =, \neq, \leq, <\}$ such that

$$I \leq (sd)^{Kn(m-n)},\ J_i \leq (sd)^{K(m-n)}$$

and the degree of h_{ij} is at most $(sd)^{K(m-n)}$, where K is a constant.

The following result of Renegar on *hyperbolic programs* offers a semialgebraic description by k polynomial inequalities of degree at most k, of an affine slice of a $\mathcal{S}_+^k$ (a spectrahedron) that contains a positive definite matrix.

Theorem 4.16. [27] *Let $Q = \{z \in \mathbb{R}^m : C + \sum z_i A_i \succeq 0\}$ be a spectrahedron with $E := C + \sum z'_i A_i \succ 0$ for some $z' \in Q$, and C, A_i are symmetric matrices of size $k \times k$. Then Q is a semialgebraic set described by $g^{(i)}(z) \geq 0$ for $i = 1, \ldots, k$ where $g^{(0)}(z) := \det(C + \sum z_i A_i)$ and $g^{(i)}(z)$ is the i-th Renegar derivative of $g^{(0)}(z)$ in direction E.*

With these two results, we can give a lower bound on the psd rank of a full-dimensional, convex, semi-algebraic set C. The *Zariski closure* of the boundary of C is a hypersurface in $\mathbb{R}^n$ since the boundary of C has codimension one. We define the *degree of C* to be the degree of a minimal degree (nonzero) polynomial whose zero set is the Zariski closure of the boundary of C. By construction, this polynomial vanishes on the boundary of C.

Proposition 4.17. *If $C \subseteq \mathbb{R}^n$ is a full-dimensional convex semialgebraic set with a $\mathcal{S}_+^k$-lift, then the degree of C is at most $k^{O(k^2 n)}$.*

Proof: We may assume that C has a proper $\mathcal{S}_+^k$-lift since otherwise we can restrict to a face of $\mathcal{S}_+^k$ and obtain a $\mathcal{S}_+^r$-lift with $r < k$. Hence there is an affine subspace L that intersects the interior of $\mathcal{S}_+^k$ such that $C = \pi(\mathcal{S}_+^k \cap L)$. This implies that there exist $k \times k$ symmetric matrices $A_1, \ldots, A_n, B_{n+1}, \ldots, B_m$ and a positive definite matrix A_0 such that

$$L = \left\{ A_0 + \sum x_i A_i + \sum y_j B_j, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^{m-n} \right\},$$

and $\pi(A_0 + \sum x_i A_i + \sum y_j B_j) = (x_1, \ldots, x_n)$. Let

$$Q = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^{m-n} \ :\ A_0 + \sum x_i A_i + \sum y_j B_j \succeq 0 \right\}.$$

Then by Theorem 4.16, Q is a basic semialgebraic set cut out by the k Renegar derivatives, $g_i(x, y) \geq 0$, of $\det(A_0 + \sum x_i A_i + \sum y_j B_j)$, with the degree of each g_i at most k.

Since C is the projection of Q, by the Tarski-Seidenberg transfer principle [21], C is again semialgebraic and has a quantifier free formula of the type (1). Hence the boundary of C is described by at most $(k^2)^{K(m-n)(n+1)}$ polynomials of degree at most $(k^2)^{K(m-n)}$ where K is a constant. Since $m < \binom{k+1}{2} \leq k^2$, by multiplying all those polynomials together we get a polynomial vanishing on the boundary of C of degree at most $(k)^{2K(k^2-n)(n+2)} = k^{O(k^2 n)}$. $\square$

The above result provides bounds on the psd ranks of polytopes.

Corollary 4.18. *If $C \subset \mathbb{R}^n$ is a full-dimensional polytope whose slack matrix has psd rank k, then C has at most $k^{O(k^2 n)}$ facets.*

Proof: If the psd rank of the slack matrix of C is k then C has a $\mathcal{S}_+^k$-lift. By Proposition 4.17 the degree of C is then at most $k^{O(k^2 n)}$. Since the minimal degree polynomial that vanishes on the boundary of a polytope is the product of the linear polynomials that vanish on each of its facets, the degree of C is the number of facets of C. $\square$

This shows that even for slack matrices of polytopes there is no function of rank that bounds psd rank.

Example 4.19. As in 4.14, let S_n be the slack matrix of a regular n-gon in the plane. Then by Proposition 4.17, $\text{rank}_{psd}(S_n)$ grows to infinity as n increases. But as we have seen before, $\text{rank}(S_n) = 3$ for all n.

In this section, we have shown that the gap between all pairs of ranks: rank, rank_+ and rank_{psd} can become arbitrarily large for nonnegative matrices. For slack matrices of polytopes we have given examples where the gaps between rank and rank_+, and rank and rank_{psd}, can also grow arbitrarily large. However, no family of slack matrices are known for which rank_+ can become arbitrarily bigger than rank_{psd} or at least exponentially bigger. Such a family would provide the first concrete proof that semidefinite programming can provide smaller representations of polytopes than linear programming.

5. Applications

5.1. **Stable set polytopes.** An interesting example of polytopes that arise from combinatorial optimization is that of stable set polytopes. Let G be a graph with vertices $V = \{1, \ldots, n\}$ and edge set E. A subset $S \subseteq V$ is *stable* if there are no edges between elements in S. To each stable set S we can associate a vector $\chi_S \in \{0,1\}^n$ where $(\chi_S)_i = 1$ if $i \in S$ and $(\chi_S)_i = 0$ otherwise. The *stable set polytope* of the graph G is the polytope

$$\text{STAB}(G) = \text{conv}\{\chi_S : S \text{ is a stable set of } G\}.$$

Finding the largest stable set in a (possibly vertex-weighted) graph is a classic NP-hard problem in combinatorial optimization that can be formulated as linear optimization over $\text{STAB}(G)$. The polytopes $\text{STAB}(G)$ give rise to one of the most celebrated results in semidefinite lifts of polytopes. Recall that a graph is *perfect* if the chromatic number of every induced subgraph equals the size of its largest clique.

Theorem 5.1. [20] *Let G be a perfect graph with n vertices, then* $\text{STAB}(G)$ *has a* $\mathcal{S}_+^{n+1}$*-lift.*

The proof is by explicit construction. Suppose $X \in \mathcal{S}_+^{n+1}$ has rows and columns indexed by $0, 1, \ldots, n$. Lovász showed that when G is perfect, the cone $\mathcal{S}_+^{n+1}$ sliced by the planes given by

$$X_{0,0} = 1, \quad X_{i,i} = X_{0,i} \;\; \forall \;\; i, \quad X_{i,j} = 0 \;\; \forall \;\; (i,j) \in E,$$

and projected onto the coordinates $X_{i,i}$ for $i = 1, \ldots, n$, is exactly $\text{STAB}(G)$. If G is not perfect this construction offers a convex relaxation of $\text{STAB}(G)$ called the *theta body* of G. In [30], Yannakakis showed that if G is perfect, $\text{STAB}(G)$ has a $\mathbb{R}_+^k$-lift where $k = n^{O(\log n)}$. It is an open problem as to whether $\text{STAB}(G)$, when G is perfect, admits a polyhedral lift of size polynomial in the number of vertices of G. Such a result is plausible since one can find a maximum weight stable set in a perfect graph in polynomial time by semidefinite programming over the above lift. On the other hand, it would also be interesting if $\text{STAB}(G)$ does not admit a polyhedral lift of size polynomial in n when G is a perfect graph. Such a result would provide the first example of a family of discrete optimization problems where semidefinite lifts are appreciably smaller than polyhedral lifts. In fact, until recently no explicit family of graphs was known for which $\text{STAB}(G)$ does not admit a polyhedral lift of size polynomial in the number of vertices of G. In [12], the authors construct non-perfect graphs G with n vertices for which $\text{rank}_+(\text{STAB}(G))$ is $2^{\Omega(n^{1/2})}$.

In the context of Theorem 5.1, a natural question is whether there could exist a positive semidefinite lift of the stable set polytope of a perfect graph to some $\mathcal{S}_+^k$ where $k < n+1$. The next theorem settles this question.

Theorem 5.2. *Let G be any graph with n vertices. Then* $\mathrm{STAB}(G)$ *does not admit a $\mathcal{S}_+^n$-lift.*

Proof: Using Theorem 3.3 it is enough to show that the slack matrix of $\mathrm{STAB}(G)$ has no $\mathcal{S}_+^n$-factorization. Furthermore we may restrict ourselves to a submatrix of the slack matrix. Consider the subset V' of vertices of $\mathrm{STAB}(G)$ consisting of the origin and all the standard basis vectors $e_1,\ldots,e_n$. The set V' is in the vertex set of every stable set polytope since the empty set and all singleton vertices are stable in any graph. Consider also a set of facets F' containing some facet that does not touch the origin, and all n facets given by the nonnegativities $x_i \geq 0$. The submatrix of the slack matrix whose rows are indexed by V' and columns by F' has the block structure

$$S' = \begin{pmatrix} 1 & 0_n \\ *_n & I_n \end{pmatrix}$$

where $*_n$ is some unknown $n \times 1$ vector, 0_n the zero vector of size $1 \times n$ and I_n the $n \times n$ identity matrix. Suppose S' has a $\mathcal{S}_+^n$ factorization with $A_0, \ldots, A_n \in \mathcal{S}_+^n$ associated to rows and $B_0, \ldots, B_n \in \mathcal{S}_+^n$ associated to columns. By looking at the first row of S' we see $\langle A_0, B_i \rangle = 0$ for all $i \geq 1$ which implies $A_0 B_i = 0$ for all $i \geq 1$ since all matrices are psd. Therefore, the columns of each B_i are in the kernel of A_0 for $i \geq 1$. Since A_0 is a nonzero $n \times n$ matrix, its kernel has dimension at most $n-1$, and contains all columns of B_i for $i = 1, \ldots, n$. By a dimension count we get that all the columns of one of the B_i, say B_k, are in the span of the columns of B_i, $i \geq 1$ and $i \neq k$. Consider now A_k. Again, $A_k B_i = 0$ for all $i \geq 1$ and $i \neq k$, which implies that all columns of those B_i are in the kernel of A_k. But this implies that so are the columns of B_k. Therefore, $\langle B_k, A_k \rangle = 0$ which contradicts the structure of S'. $\square$

Remark 5.3.
(1) In fact, the above proof shows that any polytope in $\mathbb{R}^n$ that has a vertex that locally looks like a nonnegative orthant has no $\mathcal{S}_+^n$-lift. Recently, it has been shown [16] that the psd rank of a n-dimensional polytope in $\mathbb{R}^n$ is at least $n+1$.
(2) The result in Theorem 5.2 is simple, and yet remarkable in a couple of ways. First, it is an illustration of the usefulness of the factorization theorem (Theorem 2.4) to prove the optimality of a lift. Secondly, it is impressive that the simple and natural semidefinite lift proposed by Lovász is optimal in this sense.
(3) Theorem 4.2 in [15] implies that any n-dimensional polytope with a 0/1-slack matrix admits a $\mathcal{S}_+^{n+1}$-lift. A simple proof of this fact follows from Proposition 4.8 since the rank of a slack matrix of a polytope in $\mathbb{R}^n$ is at most $n+1$.

We close this subsection with an interesting class of lifts of stable set polytopes to completely positive cones. Recall that $\mathcal{C}_n^*$ is the cone of $n \times n$ completely positive matrices.

Theorem 5.4. [9] *For any graph G with n vertices, the polytope* $\mathrm{STAB}(G)$ *has a $\mathcal{C}_{n+1}^*$-lift.*

Proof: This is an immediate consequence of Proposition 3.2 in [9] applied to this problem. $\square$

The $\mathcal{C}_{n+1}^*$-lift of $\mathrm{STAB}(G)$ is given by the same linear constraints on $X \in \mathcal{C}_{n+1}^*$ that were used to construct the $\mathcal{S}_+^{n+1}$-lift. These lifts are of very small size and work for all graphs, but have limited interest in practical computations since copositive/completely positive programming is not known to have any efficient algorithms. We illustrate the copositive/completely positive factorization that is expected for this lift in the case of a 5-cycle.

Example 5.5. From Theorem 5.4 we know that the stable set polytope of a 5-cycle has a C_6^*-lift, and hence by Theorem 2.4, its slack matrix must have a C_6^*-factorization. This polytope has 11 vertices: the origin, the five standard basis vectors $e_1, \ldots, e_5$ and the five sums $e_1 + e_3, e_2 + e_4, e_3 + e_5, e_4 + e_1, e_5 + e_2$ corresponding to the five stable sets of the 5-cycle with two elements. We will denote these last five vertices by $s_1, \ldots, s_5$, respectively. Furthermore, there are 11 facets for this stable set polytope given by the inequalities:

$$x_i \geq 0, \quad x_i + x_{i+1} \leq 1, \ \ \forall \ i = 1, \ldots, 5, \quad \text{and} \quad \sum_{j=1}^{5} x_j \leq 2$$

where we identify x_6 with x_1.

Since we know a $\mathcal{C}_6^*$-lift, the A map that takes vertices of $\mathrm{STAB}(G)$ to $\mathcal{C}_6^*$ is easy to get. Send each vertex $v \in \mathbb{R}^5$ to $A(v) = (1, v)^T(1, v) \in \mathbb{R}^{6\times 6}$, and since all coordinates are nonnegative, $A(v)$ is completely positive. For the copositive lifts of the facets, we go case by case. For $x_i \geq 0$ take the matrix $(0, e_i)^T(0, e_i)$, while for $1 - x_i - x_{i+1} \geq 0$ take $(1, -e_i - e_{i+1})^T(1, -e_i - e_{i+1})$. All these matrices are positive semidefinite and hence also copositive. It is also easy to check that they satisfy the factorization requirements.

It remains to find a copositive matrix for the odd-cycle inequality $2 - \sum_{j=1}^{5} x_j \geq 0$. This is non-trivial, but it can be checked that the following matrix works for the factorization:

$$\begin{pmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

To see that it is copositive, by Theorem 2 in [9], we just have to show that

$$2\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

is copositive, and this is a well known Horn form, that is copositive. For a proof, see for instance, Lemma 2.1 in [4].

5.2. **Rational lifts of algebraic sets.** Our last application is an interpretation of Theorem 2.4 for an important class of positive semidefinite lifts, called *rational lifts*, of zero sets of polynomial equations. Suppose we have a system of polynomial equations

$$p_1(x) = p_2(x) = \cdots = p_m(x) = 0 \tag{2}$$

where the p_i's have real coefficients and n variables, and I is the *ideal* they generate in the polynomial ring $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$. The set of zeros of (2), denoted by $\mathcal{V}_{\mathbb{R}}(I)$, is the *real variety* of the ideal I, and we consider positive semidefinite lifts of $C = \mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))$. Since different polynomial systems can generate the same convex hull, we define the *convex radical ideal* of I to be the ideal $\sqrt[\text{conv}]{I}$ of polynomials vanishing on $\mathcal{V}_{\mathbb{R}}(I) \cap \text{ext}(C)$. Replacing I by $\sqrt[\text{conv}]{I}$ does not change C and so we will, to simplify arguments, assume that $I = \sqrt[\text{conv}]{I}$.

We consider special kinds of $\mathcal{S}_+^k$-factorizations of the slack operator S_C, namely, those where the map $A : \text{ext}(C) \to \mathcal{S}_+^k$ is of the form $A(x) = v(x)v(x)^T$, where $v(x)$ is a vector of rational functions $v(x) = (v_1(x), \dots, v_n(x))$. By factoring out the common denominators, we can rewrite such a map as $A(x) = \frac{1}{p(x)^2} w(x)w(x)^T$ where $w(x)$ is a vector of polynomials. We say that A is a *rational map*, and if $p(x) = 1$ we say that A is a *polynomial map*. A $\mathcal{S}_+^k$-factorization of S_C is called a rational (respectively, polynomial) factorization if the map A used in the factorization is a rational (respectively, polynomial) map.

These lifts turn out to be related to the sums of squares techniques for lift-and-project methods. Given a polynomial $q(x) \in \mathbb{R}[x]$, we say that it is a *sum of squares (sos) modulo* I, if there exist polynomials $h_1(x), \dots, h_s(x) \in \mathbb{R}[x]$ such that $q(x) - \sum h_i(x)^2 \in I$. If the degrees of all the h_i are bounded above by k we say that q is *k-sos modulo I*. This is a sufficient condition for nonnegativity over a real variety that has been used to construct sequences of semidefinite relaxations of the convex hull of the variety. One such hierarchy is given by the *theta bodies* of I, introduced in [15]. They are defined geometrically by taking the k-th theta body relaxation of $\text{conv}(\mathcal{V}_{\mathbb{R}}(I))$, denoted as $\text{TH}_k(I)$, to be the intersection of all half-spaces $\{x : \ell(x) \geq 0\}$ where $\ell(x)$ is a linear polynomial that is k-sos modulo I.

Theorem 5.6. *Let I be a convex radical ideal and $Z = \mathcal{V}_{\mathbb{R}}(I)$ its zero set such that* $\text{conv}(Z)$ *is compact and contains the origin. Then,*

(1) *the slack operator of* $\text{conv}(Z)$ *has a rational factorization with $A(x) = \frac{1}{p(x)^2} w(x)w(x)^T$ in $\mathcal{S}_+^k$ for all $x \in \text{ext}(\text{conv}(Z))$ if and only if, for every linear polynomial $\ell(x)$ nonnegative over Z, $p(x)^2\ell(x)$ is a sum of squares modulo I, with all the polynomials in the sum of squares being linear combinations of the entries of $w(x)$.*

(2) *The slack operator of* $\text{conv}(Z)$ *has a polynomial factorization with $A(x) = w(x)w(x)^T$ where the degree of each entry in w at most k if and only if* $\text{TH}_k(I) = \text{conv}(Z)$.

Proof: For the first part note that since any linear polynomial $\ell(x)$ nonnegative over Z is a convex combination of extreme points of the polar of $\text{conv}(Z)$, there exists a matrix $B_\ell \in \mathcal{S}_+^k$ such that $\ell(x) = \langle B_\ell, A(x) \rangle$ for all $x \in \text{ext}(\text{conv}(Z)))$. Since I is convex radical this actually implies $\ell(x) = \langle B_\ell, A(x) \rangle$ modulo I, and by rewriting the right hand side we have $p(x)^2\ell(x) = w(x)^T B_\ell w(x)$ modulo I, which is a sum of squares modulo the ideal with the conditions we want. Since all steps in the proof are actually equivalences, this gives us a proof of the first statement.

For the second statement just note that from [15], I is TH_k-exact if and only if all linear polynomials non-negative over $\mathcal{V}_{\mathbb{R}}(I)$ are k-sos modulo I (since I is in particular real radical). Now use the first statement to conclude the proof. $\square$

A rational factorization of the slack matrix of $C := \text{conv}(\mathcal{V}_{\mathbb{R}}(I))$ consists of two maps A and B that assign psd matrices to extreme points of C and C°. On the *primal* side, every extreme point (and hence every point) of C is being lifted to a psd matrix via the map A. On the *dual* side, B is assigning a psd Gram matrix to every linear functional that is nonnegative on $\mathcal{V}_{\mathbb{R}}(I)$ certifying its sum of squares property with respect to this variety.

Several further remarks are in order. The requirements that $\text{conv}(Z)$ is compact and contains the origin in its interior are not essential and are assumed for the sake of simplicity and to keep the discussion in the same setting as in our main theorems. A similar idea could be applied to convex hulls of sets defined by polynomial inequalities, but there the usual lift is not to a positive semidefinite cone but to a product of such cones, making the notation more cumbersome. Finally, the condition that the ideal I is convex radical can be avoided if we use a stronger notion of a polynomial lift that implies factorization over the entire variety and not just over the extreme points of the convex hull of the variety.

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