Continuous-time Casino Problems

by

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Abstract

In their fundamental treatise on discrete-time gambling theory, L.E. Dubins and L.J. Savage introduced a natural class of discrete-time stochastic control problems which they called "casinos". A similar class of problems in continuous-time is studied here. Many of the results of Dubins and Savage, including their characterization of "fair casinos", are formulated and proved. A formula is given for the value function of a "continuous-time casino" and the collection of optimal strategies is determined.

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1. Introduction.

Consider the problem of controlling a stochastic process $X = \{X_t,\ t \geq 0\}$ with state space a closed interval $[a,b]$ so as to maximize $P[X \text{ reaches } b]$ where $[X \text{ reaches } b]$ is the event $[X_t = b \text{ for some } t \geq 0]$. Assume the process satisfies a stochastic differential equation

$$(1.1) \qquad X_0 = x, \qquad dX_t = \mu(t)dt + \sigma(t)dW_t$$

where $\{W_t\}$ is a standard Brownian motion process on $(\Omega, \mathcal{F}, P)$ adapted to $\{\mathcal{F}_t\}$, and where each $\mathcal{F}_t$ is independent of $\{W_{t+s} - W_t,\ s \geq 0\}$ and contains all P-null sets. The control processes $\mu(t) = \mu(t,\omega)$ and $\sigma(t) = \sigma(t,\omega)$ are assumed to be real-valued, progressively measurable, and to satisfy the conditions

$$(1.2) \qquad \int_0^t |\mu(s)|ds < \infty \quad \text{a.s.}$$

$$(1.3) \qquad \int_0^t \sigma^2(s)ds < \infty \quad \text{a.s.}$$

for every $t > 0$. In addition, there is, for every $y \in (a,b)$, a <u>control set</u> $C(y)$, a non-empty subset of $\mathbb{R} \times \mathbb{R}^+$ from which the player is required to choose the value of (μ, σ) when the current position is y. More precisely, it is assumed that

$$(\mu(t), \sigma(t)) \in C(X_t)$$

for $X_t \in (a,b)$ and for all t. It is also assumed that $\mu(t) = \sigma(t) = 0$ whenever $X_t = a$ or $X_t = b$. The process X is absorbed at the endpoints a and b.

Let $\Sigma(x)$ be the collection of all such processes $X = \{X_t\}$ starting at state $X_0 = x$ in (a,b), and assume $\Sigma(x)$ is non-empty for every x. Call such a control problem a <u>goal problem</u> on [a,b] and define its <u>value function</u> V by

$$(1.4) \qquad V(x) = \sup_{X \in \Sigma(x)} P[X \text{ reaches } b]$$

for $a<x<b$. Set $V(a) = 0$ and $V(b) = 1$. Goal problems are continuous-time gambling problems in the sense of Pestien and Sudderth [9] and a class of goal problems was studied in [9] with (1.3) replaced there by the more restrictive condition that $E\int_0^t \sigma^2(s)ds < \infty$. However, the results of [9] are not sufficiently general to include the casinos and proportional houses defined below even if the condition on σ were unchanged. (Assumption A of [9] need not hold.)

A goal problem on the unit interval [0,1] is called a <u>continuous-time casino</u> if the control sets C(x) satisfy

$$(1.5) \qquad \begin{aligned} C(x) &\subset C(y) \quad \text{for} \quad 0<x<y<1 \\ pC(x) &\subset C(px) \quad \text{for} \quad 0<x<1, \quad 0<p<1 . \end{aligned}$$

(Here $pC(x) = \{(p\mu, p\sigma): (\mu,\sigma) \in C(x)\}$.) The term "casiño" was used by Dubins and Savage [1] to describe a class of discrete-time stochastic control problems for which conditions corresponding to those of (1.5) are imposed upon the increments of the controlled processes. If the state x is regarded as the player's fortune, then the conditions may be interpreted here as in [1]: "A rich gambler can do whatever a poor one can do" and "A poor gambler can, on a small scale, imitate a rich one". These conditions seem equally natural for investment and portfolio management problems, which are increasingly modeled by stochastic differential equations, as for gambling casinos.

Dubins and Savage devoted a large part of their book [1] on gambling theory to the study of discrete-time casino problems. Many of their results have counterparts in the continuous-time theory. For example, section 3 presents a classification of continuous-time casinos into four types: trivial, subfair, fair, and superfair. The main result, Theorem 3.2, is quite analogous to results in [1], but the proof is much easier in continuous-time. Section 4 gives an exact formula for the value function of many goal problems including a general subfair continuous-time casino. No such result is likely to be found for the more difficult discrete-time problems.

Another natural class of goal problems on [0,1], to which the theory of section 4 is applied in section 5, are the <u>proportional houses</u> for which

$$(1.6) \qquad C(x) = x\hat{C} \qquad \text{for} \quad 0<x<1$$

where $\hat{C}$ is a fixed subset of $\mathbb{R}\times\mathbb{R}^+$. As is the case in many gambling and investment problems, a player's opportunities are proportional to his fortune.

A process $X \in \Sigma(x)$ is <u>optimal at</u> x if $P[X \text{ reaches } b] = V(x)$. Techniques adapted from Dubins and Savage [1] are used in section 6 to characterize optimal strategies for many goal problems.

Three preliminary lemmas are presented in the next section.

2. Basic lemmas.

The major results of the paper rely on two familiar techniques. The first is to use a verification result, Lemma 2.2, to obtain an upper bound on V. The second uses a standard formula, Lemma 2.3, from the theory of diffusion processes to evaluate the probability that a particular process $X \in \Sigma(x)$ reaches b and thereby obtain a lower bound for $V(x)$. We begin with a preliminary result on local martingales.

Lemma 2.1. Let $\{M_t, \mathcal{F}_t\}$ be a local martingale and let Z be an integrable random variable such that $M_t \geq Z$ for all $t \geq 0$. Then $EM_\tau \leq EM_0$ for every a.s. finite stopping time τ.

Proof: Choose stopping times T_j such that $T_j \to \infty$ a.s. and $\{M_{t\wedge T_j}, \mathcal{F}_t\}$ is a uniformly integrable martingale for every j. Set $B_j = [\tau > T_j]$. Then

$$\int_{B_j^c} M_\tau = \int_{B_j^c} M_{\tau\wedge T_j} = E(M_{\tau\wedge T_j}) - \int_{B_j} M_{T_j}$$

$$= EM_0 - \int_{B_j} M_{T_j}$$

$$\leq EM_0 - \int_{B_j} Z \quad .$$

Let $j \to \infty$ to get $EM_\tau \leq EM_0$. ∎

The proof of Lemma 2.1 is due to Steven Orey.

The next lemma is analogous to earlier verification lemmas in [4] and [9]. Let Q be a real-valued function defined on an open set G containing $[a,b]$.

<u>Lemma 2.2</u>. Assume

(a) Q'' exists and is continuous on G,

(b) $\mu Q'(x) + \frac{1}{2}\sigma^2 Q''(x) \leq 0$ for $a<x<b$, $(\mu,\sigma) \in C(x)$,

(c) $Q(x) \geq 0$ for $a \leq x < b$ and $Q(b) \geq 1$.

Then $Q(x) \geq V(x)$ for $a \leq x \leq b$.

<u>Proof</u>: The desired inequality clearly holds for $x = a$ or b. Let $a<x<b$ and let $X \in \Sigma(x)$ satisfy

$$X_t = x + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dW_s$$

where μ and σ are as in the previous section. By Ito's Lemma,

$$Q(X_t) = Q(x) + \int_0^t [\mu(s)Q'(X_s) + \tfrac{1}{2}\sigma^2(X_s)Q''(X_s)]ds$$
$$+ \int_0^t \sigma(s)Q'(X_s)dW_s$$
$$\leq Q(x) + \int_0^t \sigma(s)Q'(X_s)dW_s \quad .$$

Let

$$M_t = \int_0^t \sigma(s)Q'(X_s)dW_s \quad .$$

Then $\{M_t, \mathcal{F}_t\}$ is a local martingale and is bounded below because Q is bounded on $[a,b]$. So by Lemma 2.1 and condition (c),

$$P[X_t = b] \leq EQ(X_t) \leq Q(x) + EM_0 = Q(x) \ .$$

Hence,

$$P[X_t = b \ \text{ for some } t] = \lim_{t\to\infty} P[X_t = b] \leq Q(x) \ .$$

Take the supremum over $X \in \Sigma(x)$ to get $V(x) \leq Q(x)$. ■

For the next lemma, consider a diffusion process X defined by the stochastic differential equation

$$X_0 = x_0, \qquad dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

where $a<x_0<b$, and μ and σ are Borel measurable, real-valued functions on (a,b) which are bounded on closed subsets of (a,b). Assume also that σ has a positive infimum on each closed

subset. For $a<x<b$, let

$$\gamma(x) = \frac{\mu(x)}{\sigma^2(x)}$$

$$\xi(x) = \exp\left(-2\int_{x_0}^{x} \gamma(y)\,dy\right) .$$

Let S be the <u>scale measure</u> on (a,b); i.e.

$$S(B) = \int_B \xi(x)\,dx$$

for Borel sets $B \subset (a,b)$. Also, let

$$\Psi = \int_{x_0}^{b} \frac{S(y,b)}{\xi(y)\sigma^2(y)}\,dy .$$

<u>Lemma 2.3</u>. If $S(a,x_0) < \infty$ and $\Psi < \infty$ then

$$P[X \text{ reaches } b] = \frac{S(a,x_0)}{S(a,b)} ;$$

if $\Psi = \infty$, then

$$P[X \text{ reaches } b] = 0 ;$$

if $S(a,x_0) = \infty$ and $\Psi < \infty$, then

$$P[X \text{ reaches } b] = 1 .$$

<u>Proof</u>: Suppose $S(a,x_0) < \infty$ and $\Psi < \infty$, and let ε and δ satisfy $0<\varepsilon<x_0-a$ and $0<\delta<b-x_0$. It is well-known that if μ

and σ are continuous on $[a+\varepsilon, b-\delta]$, then

$$\text{(2.1)} \qquad P[X \text{ reaches } b-\delta \text{ before } a+\varepsilon] = \frac{S(a+\varepsilon, x_0)}{S(a+\varepsilon, b-\delta)} \quad .$$

Further, if $T_\varepsilon^\delta = \inf\{t: X_t \geq b-\delta \text{ or } X_t \leq a+\varepsilon\}$, then $ET_\varepsilon^\delta < \infty$, and there is an explicit formula for ET_ε^δ in terms of the continuous functions μ and σ (cf. Gihman and Skorohod [3], Corollary 3.15.2). Under our hypotheses, μ and σ are Borel but not necessarily continuous. However, (2.1) still holds, as can be seen by applying the argument in ([3], Theorem 3.15.4) together with Krylov's generalization of Ito's Lemma ([7], Theorem 2.10.1). Following similar reasoning, it is seen that the formula for ET_ε^δ in ([3], Corollary 3.15.2) also still holds. Let δ approach 0 in the formula for ET_ε^δ and use the hypothesis that $\Psi < \infty$ to deduce that $P[X \text{ reaches } a+\varepsilon \text{ or } b] = 1$. This equality, together with the fact that X has continuous paths, implies that

$$P[X \text{ reaches } b] = \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} P[X \text{ reaches } b-\delta \text{ before } a+\varepsilon] \quad .$$

The proof of the first assertion of the lemma can now be completed by letting ε and δ approach 0 in (2.1). The remaining two assertions are immediate consequences of standard arguments in the theory of diffusion processes (cf. Karlin and Taylor [6], Section 15.6). ∎

It is the exact formula of Lemma 2.3 which makes the theory of continuous-time casinos simpler than the discrete-time theory.

3. The four types of casino.

A continuous-time casino is called trivial, fair, superfair, or subfair according as $V(x) = 0$, $V(x) = x$, $V(x) = 1$, or $0 < V(x) < x$ for all $x \in (0,1)$. These terms were introduced by Dubins and Savage [1] who showed that every V corresponding to a discrete-time casino is one of these four types. Their proof was based on a study of the casino inequality:

$$V(px+(1-p)y) \geq V(p)V(x) + (1-V(p))V(y) \quad \text{for } 0 \leq p \leq 1, \quad 0 \leq x \leq y \leq 1 . \tag{3.1}$$

They showed that the value function V of every discrete-time casino satisfies (3.1) and conversely, every V satisfying (3.1) together with $V(0) = 0$, $V(1) = 1$, $0 \leq V(x) \leq 1$ for $0<x<1$, is the value function of some discrete-time casino.

Theorem 3.1. The casino inequality holds for the value function V of any continuous-time casino.

Proof: Essentially the same proof as in [1] will work here also. The only new difficulty is that certain properties of conditional distributions of Ito processes must be checked. (We omit the details because they are straightforward and we will not rely on Theorem 3.1 below.) ∎

In contrast to the situation in discrete-time, not every solution of (3.1) is the value function of a continuous-time casino. As will be seen in section 4, every such value function is twice continuously differentiable on (0,1) whereas the value function of a discrete-time casino can be much more irregular (cf. chapter 5 of [1]).

It follows from Theorem 3.1 and the arguments of Dubins and Savage that every continuous-time casino falls in one of the four classes mentioned above. The characterization of the four classes and, in particular, the characterization of fair casinos is a difficult problem in discrete-time ([1], Theorem 11.3.1). The characterization given here is analogous but relatively easy.

Define

$$C(1) = \bigcup_{0<x<1} C(x) \tag{3.2}$$

$$\rho(1) = \sup\left\{\frac{\mu}{\sigma^2} : (\mu,\sigma) \in C(1)\right\}.$$

In the definition of $\rho(1)$, the convention is made that $\frac{\mu}{0} = -\infty$ if $\mu \le 0$, and $\frac{\mu}{0} = +\infty$ if $\mu > 0$.

Theorem 3.2. A continuous-time casino is trivial, subfair, fair, or superfair according as $\rho(1) = -\infty$, $-\infty < \rho(1) < 0$, $\rho(1) = 0$, or $\rho(1) > 0$. In the subfair case,

$$x^{-2\rho(1)+1} \le V(x) \le \frac{e^{-2\rho(1)x} - 1}{e^{-2\rho(1)} - 1} \tag{3.3}$$

for $0<x<1$.

A useful notion for the proof is that of a proportional strategy. Let $0<x<1$ and $(\mu,\sigma) \in C(1)$. It follows from (1.5) that $x(\mu,\sigma) \in C(x)$. Thus a player can choose the process $X \in \Sigma(x)$ defined by

$$X_0 = x, \qquad dX_t = X_t(\mu\, dt + \sigma\, dW_t) \; .$$

Apply Lemma 2.3 to see that

$$\text{(3.4)} \qquad P[X \text{ reaches } 1] = \begin{cases} x^{-2\mu/\sigma^2+1} & \text{if } \frac{\mu}{\sigma^2} < \frac{1}{2} \\ 1 & \text{if } \frac{\mu}{\sigma^2} \geq \frac{1}{2} \end{cases} .$$

Proof of Theorem 3.2. If $\rho(1) = -\infty$, the $\mu \leq 0$ and $\sigma = 0$ for every $(\mu,\sigma) \in C(1)$ and clearly V is trivial.

If $-\infty < \rho(1) < 0$, use Lemma 2.2 to establish the second inequality in (3.3) and with it the fact that $V(x) < x$. Then in (3.4) take the supremum over $(\mu,\sigma) \in C(1)$ to obtain the first inequality of (3.3).

If $\rho(1) = 0$, use Lemma 2.2 to show $V(x) \leq x$ and take the supremum in (3.4) to prove the opposite inequality.

Finally, suppose $\rho(1) > 0$. It follows that $\rho(1) = \infty$. To see this, choose $(\mu,\sigma) \in C(1)$ with $\mu > 0$ and $\sigma > 0$. By (1.5), the pair (μ_1,σ_1), defined as $\frac{1}{2}(\mu,\sigma)$, is also an element of

$C(1)$ and $\mu_1/\sigma_1^2 = 2\mu/\sigma^2$. Thus $C(1)$ contains elements (μ,σ) for which μ/σ^2 is arbitrarily large. It follows now from (3.4) that $V(x) = 1$ for $0<x<1$. ∎.

The last paragraph of the argument above shows that $\rho(1) = \infty$ for superfair casinos.

Equality can occur in each of the inequalities of (3.3). Here are two examples which demonstrate this. The names are borrowed from Dubins and Savage ([1], section 4.7) who considered similar examples of discrete-time casinos.

Example 3.1. The rich man's casino. The idea here is that a rich man should be able to imitate a poorer one on a large scale. We have already assumed that, in any casino, a poor man can imitate a rich one on a small scale. This suggests the formal definition of a rich man's casino as one for which

$$C(x) = xC(1) , \qquad 0<x<1.$$

If, in addition, $-\infty < \rho(1) < 0$, then (3.3) and Lemma 2.2 can be used to see that $V(x) = x^{-2\rho(1)+1}$.

Example 3.2. The poor man's casino. In this casino a poor man is allowed to do whatever a rich man can. So the control sets $C(x)$

are identically equal to $C(1)$. Thus a player can use a constant control $(\mu,\sigma) \in C(1)$ for which the associated process satisfies

$$X_0 = x, \qquad dX_t = \mu dt + \sigma dW_t$$

and, by Lemma 2.3,

$$P[X \text{ reaches } 1] = \frac{e^{-2\mu x/\sigma^2} - 1}{e^{-2\mu/\sigma^2} - 1} .$$

If $-\infty < \rho(1) < 0$, take the supremum over $(\mu,\sigma) \in C(1)$ to get equality in the second inequality of (3.3).

4. A formula for V.

The value function V of a continuous-time casino is explicitly determined by Theorem 3.2 except for the subfair case. A formula for V in the subfair case will be given in this section.

Consider a goal problem on $[a,b]$ specified by control sets $C(x)$, $a<x<b$. Define

$$\rho(x) = \sup\left\{ \frac{\mu}{\sigma^2} : (\mu,\sigma) \in C(x) \right\}, \tag{4.1}$$

making again the convention that $\frac{\mu}{0} = -\infty$ if $\mu \le 0$. Fix $x_0 \in (a,b)$ and for $a<x<b$, let

$$\xi(x) = \exp\left(-2\int_{x_0}^{x} \rho(y)\,dy\right)$$

$$S(A) = \int_A \xi(y)\,dy \quad ,$$

for Borel sets $A \subset (a,b)$, assuming these integrals are well defined.

<u>Theorem 4.1</u>. The value function V of a subfair, continuous-time casino satisfies

$$V(x) = \frac{S(0,x)}{S(0,1)} \quad .$$

A similar formula was proved in [9], but assumptions were made there which do not hold for subfair casinos. The proof of Theorem 4.1 will be given in three lemmas and will apply to a class of goal problems more general than subfair casinos.

Assumption I. ρ is continuous on (a,b).

Define $Q(a) = 0$, $Q(b) = 1$, and, for $a<x<b$,

(4.2) $$Q(x) = \frac{S(a,x)}{S(a,b)} \quad \text{if} \quad S(a,b) < \infty ,$$

(4.3) $$Q(x) = 0 \quad \text{if} \quad S(x,b) = \infty ,$$

and

(4.4) $$Q(x) = 1 \quad \text{if} \quad S(a,x) = \infty \quad \text{and} \quad S(x,b) < \infty .$$

Notice that if (4.2) holds, then Q is strictly increasing on $[a,b]$ and Q'' is continuous on (a,b).

Lemma 4.1 Under Assumption I, $Q \geq V$.

Proof: This is obvious if $Q = 1$ on $(a,b]$. Next, assume Q satisfies either (4.2) or (4.3). Let $\varepsilon>0$ and $\delta>0$, with $0<\varepsilon+\delta<b-a$ and consider the goal problem on $[a+\varepsilon,b-\delta]$ with control sets $C(x)$. Define

$$Q_\varepsilon^\delta(x) = \frac{S(a+\varepsilon,x)}{S(a+\varepsilon,b-\delta)} , \qquad a+\varepsilon<x<b-\delta .$$

Because ρ is continuous on $[a+\varepsilon, b-\delta]$, it is clear that Q_ε^δ has a smooth extension to a neighborhood of $[a+\varepsilon, b-\delta]$. Apply Lemma 2.2 to see that

$$Q_\varepsilon^\delta \geq V_\varepsilon^\delta, \tag{4.5}$$

where V_ε^δ is the value function for the problem on $[a+\varepsilon, b-\delta]$. If (4.2) holds, then

$$Q(x) = \lim_{\varepsilon \downarrow 0} \left[\lim_{\delta \downarrow 0} Q_\varepsilon^\delta(x)\right] \tag{4.6}$$

for every $x \in (a,b)$. If (4.3) holds, then the equality (4.6) is still valid, because $Q_\varepsilon^\delta \longrightarrow 0$ as $\delta \downarrow 0$ for each ε. Also, if $X \in \Sigma(x)$ then, because X has continuous paths,

$$P[X \text{ reaches } b-\delta \text{ before } a+\varepsilon] \geq P[X \text{ reaches } b \text{ before } a+\varepsilon]$$

and

$$P[X \text{ reaches } b \text{ before } a] = \lim_{\varepsilon \downarrow 0} P[X \text{ reaches } b \text{ before } a+\varepsilon] .$$

Hence $V_\varepsilon^\delta \geq V_\varepsilon^0$ and $V_\varepsilon^0 \longrightarrow V$ as $\varepsilon \downarrow 0$, where V_ε^0 denotes the value function for the goal problem on $[a+\varepsilon, b]$. Using (4.5) and (4.6), the conclusion $Q \geq V$ is obtained. ∎

A comparison of the first paragraph of this section with Lemma 2.3 suggests the inequality $Q \leq V$ should also hold and that a process $X \in \Sigma(x)$ will reach 1 with probability nearly $Q(x)$ if its controls (μ,σ) are selected so that the supremum in (4.1) is nearly achieved. In a general goal problem, however, the control sets $C(x)$ may be too wild to allow good, measurable selectors.

Definition. Let μ and σ be Borel-measurable, real-valued functions which have domain (a,b) and which are bounded on closed subsets of (a,b). If $(\mu(x),\sigma(x)) \in C(x)$ for every $x \in (a,b)$, then (μ,σ) is a Borel C-selector.

Assumption II. For every $\varepsilon \in (0,b-a)$, there is a Borel C-selector $(\mu_\varepsilon,\sigma_\varepsilon)$ such that

(a) $$\frac{\mu_\varepsilon(x)}{\sigma_\varepsilon(x)^2} \geq \rho(x) - \varepsilon \quad \text{for} \quad a+\varepsilon \leq x < b \ ,$$

(b) $$\inf\{\sigma_\varepsilon(x) : a+\varepsilon \leq x < b\} > 0.$$

Lemma 4.2 Under assumption II, $Q \leq V$.

Proof: Let $x \in (a,b)$. If $S(x,b) = \infty$, then $Q \equiv 0$ by definition, and so $Q \leq V$. Next, suppose $S(x,b) < \infty$. For $0<\varepsilon<x-a$, let $(\mu_\varepsilon,\sigma_\varepsilon)$ be as given in assumption II and $X^{(\varepsilon)}$ be a process solving

$$X_0^{(\varepsilon)} = x, \qquad dX_t^{(\varepsilon)} = \mu_\varepsilon(X_t^{(\varepsilon)})dt + \sigma_\varepsilon(X_t^{(\varepsilon)})dW_t$$

and let

$$\xi_\varepsilon(w) = \exp\left[-2\int_x^w \frac{\mu_\varepsilon(y)}{\sigma_\varepsilon^2(y)}dy\right] .$$

It follows from (a) and (b) and the assumption $S(x,b) < \infty$ that for each ε,

$$\int_x^b \frac{1}{\xi_\varepsilon(y)\sigma_\varepsilon^2(y)}\left[\int_y^b \xi_\varepsilon(w)\,dw\right]dy < \infty . \tag{4.7}$$

With the iterated integral in (4.7) playing the role of Ψ, use Lemma 2.3 and the fact that $\mu_\varepsilon/\sigma_\varepsilon^2 \longrightarrow \rho$ as $\varepsilon\downarrow 0$ to see that

$$V(x) \geq P[X^{(\varepsilon)} \text{ reaches b before } a+\varepsilon] \longrightarrow Q(x)$$

as $\varepsilon\downarrow 0$. ■

We now give two examples where assumption II(b) does not hold. In the first, Q is strictly larger than V, contrary to the conclusion of lemma 4.2, while in the second, Q and V are equal.

Example 4.1 Let $a=0$, $b=1$, and for $0<x<1$, let $C(x) = \{(0,1-x)\}$. If $x_0 \in (0,1)$, any process X in $\Sigma(x_0)$ must satisfy

$$dX_t = (1-X_t)dW_t , \qquad X_0 = x_0 . \tag{4.8}$$

Then $\rho(x) = 0$ and $\xi(x) = 1$ for $0<x<1$, and Q satisfies (4.2), with $Q(x) = x$. However, in the language of Karlin and Taylor ([6], Section 15.6), the goal 1 is "unattainable" because

$$\int_{x_0}^{1} \frac{S(y,1)}{\xi(y)\sigma^2(y)}\, dy = \int_{x_0}^{1} \frac{1}{1-y}\, dy = +\infty ,$$

for $0<x_0<1$. Hence $V(x) = 0$ for each $x \in (0,1)$, and the conclusion of Lemma 4.2 fails. Notice that assumption II(b) is not satisfied because $\sigma(x) = 1-x \longrightarrow 0$ as $x \longrightarrow 1$. ∎

Example 4.2 Let $a=0$, $b=1$, and for $0<x<1$, let $C(x) = \{(-1, 2(1-x)^{½})\}$. Now if $0<x_0<1$, any $Y \in \Sigma(x_0)$ satisfies

$$\text{(4.9)} \qquad dY_t = -dt + 2(1-Y_t)^{½} dW_t , \qquad Y_0 = x_0 .$$

Then, using formula (4.2), it can be seen that

$$Q(x) = 1 - (1-x)^{½}, \qquad 0 \le x \le 1 .$$

Also, the goal 1 is "attainable" because

$$\int_{x_0}^{1} \frac{S(y,1)}{\xi(y)\sigma^2(y)}\, dy = \int_{x_0}^{1} \frac{2(1-y)^{½}}{(1-y)^{-½}\, 4(1-y)}\, dy = \int_{x_0}^{1} \frac{1}{2}\, dy < \infty .$$

Therefore $V = Q$, in spite of the fact that $\sigma(x) \longrightarrow 0$ and $\mu(x)/\sigma^2(x) \longrightarrow -\infty$ as $x \longrightarrow 1$. ∎

The assumption made in Lemma 4.1 that ρ is continuous is not necessary and could be replaced by the assumption that ρ is Borel measurable and bounded on closed subsets of $(0,1)$. The proof could be carried out using techniques from [9] and the verification lemma of [4]. However, as will now be shown, the results already proved are sufficiently general to apply to subfair casino problems.

<u>Lemma 4.3</u>. A subfair casino satisfies assumptions I and II with $a=0$ and $b=1$.

<u>Proof</u>: Let $0<x<y<1$. Then $\rho(x) \leq \rho(y)$ because $C(x) \subset C(y)$ by (1.5). It follows that $\rho(x)$ converges to $\rho(1)$ as $x \longrightarrow 1$, where $\rho(1)$ is defined in (3.2). Also $\rho(x) \geq xy^{-1}\rho(y)$ because $xy^{-1}C(y) \subset C(x)$ by (1.5). Hence,

$$0 \leq \rho(y) - \rho(x) \leq \rho(y)(1 - xy^{-1}) \leq \rho(1)(1 - xy^{-1}) .$$

Let $y \downarrow x$ and $x \uparrow y$ to get right and left continuity of ρ, respectively. That ρ is finite follows from the finiteness of $\rho(1)$ and the inequalities $x\rho(1) \leq \rho(x) \leq \rho(1)$.

To construct the C-selector, first use the uniform continuity of ρ on $[\varepsilon,1]$ to choose $\alpha > 0$ such that $|x-y| < \alpha$, $x,y \in [\varepsilon,1]$ implies $|\rho(x)-\rho(y)| < \varepsilon/2$. Consider the set

$$E = \{\varepsilon,\varepsilon+\alpha,\ldots,\varepsilon+n\alpha\} \cup \{1\}$$

where $\varepsilon+n\alpha \leq 1 < \varepsilon+(n+1)\alpha$. For $x \in E$, choose $(\mu(x),\sigma(x)) \in C(x)$ so that $\mu(x)/\sigma^2(x) > \rho(x)-\varepsilon/2$. For y between two points of E, take $(\mu(y),\sigma(y))$ to have the same value as at the left-hand point. Define (μ,σ) to equal any Borel C-selector on $(0,\varepsilon)$, say

$$(\mu(x),\sigma(x)) = x\cdot(\mu(\varepsilon),\sigma(\varepsilon)) .$$

Finally, take $\mu_\varepsilon = \mu$, $\sigma_\varepsilon = \sigma$. ∎

Theorem 4.1 is immediate from the three lemmas.

5. Proportional houses.

For a proportional house as defined in section 1, the function ρ satisfies

$$(5.1) \qquad \rho(x) = \frac{\hat{\rho}}{x}, \qquad 0 < x < 1,$$

where the control sets are given by (1.6) and

$$\hat{\rho} = \sup\left\{ \frac{\mu}{\sigma^2} : (\mu,\sigma) \in \hat{C} \right\}.$$

If $\hat{\rho}$ is finite, then ρ is finite and continuous on $(0,1)$. Also, for $(\mu,\sigma) \in \hat{C}$, if $\mu(x) = x\mu$ and $\sigma(x) = x\sigma$, then $(\mu(x),\sigma(x)) \in C(x)$. Furthermore, if $\mu/\sigma^2 \geq \hat{\rho}-\varepsilon^2$, then $\mu(x)/\sigma^2(x) \geq \rho(x)-\varepsilon$ on $[\varepsilon,1]$ for $0<\varepsilon<1$. Therefore, Lemmas 4.1 and 4.2 apply to show $V = Q$ where Q, as defined by (4.2) with $a=0$ and $b=1$, is easily calculated with the aid of (5.1). The next theorem records the result of that calculation.

Theorem 5.1. A proportional house has

$$V(x) = x^{-2\hat{\rho}+1} \qquad \text{on } (0,1) \quad \text{if } -\infty < \hat{\rho} < \tfrac{1}{2}$$

and

$$V(x) = 1 \qquad \text{on } (0,1) \quad \text{if } \hat{\rho} \geq \tfrac{1}{2} \quad .$$

Notice that if $-\infty < \hat{\rho} < 0$, the rich man's casino (Example

3.1) is a casino which is also a proportional house and has the value function given by Theorem 5.1. However, no casino has the value function above when $0 < \hat{\rho} < \frac{1}{2}$.

For proportional houses for which $\hat{\rho} \geq \frac{1}{2}$ and for superfair casinos, we have now seen that a player can reach 1 with probability one and the question of how to reach 1 in minimal expected time arises. This question is completely resolved for proportional houses in [4], but remains open for general, superfair casinos.

6. Optimal strategies.

The standard approach to the study of optimality in stochastic control, as presented in [2] and [6] for example, uses the Bellman equation, which can be written for a goal problem as

$$(6.1) \qquad \sup D(\mu,\sigma)V(x) = 0$$

where $D(\mu,\sigma)V(x) = \mu V'(x) + \frac{1}{2}\sigma^2 V''(x)$ and the supremum is over all $(\mu,\sigma) \in C(x)$ for $a<x<b$. Now the value function V of a casino does satisfy (6.1) as can be easily checked using the exact formulas for the four types of casino.

One then seeks, for each x, controls $(\mu(x),\sigma(x))$ which achieve the supremum in (6.1), that is, such that

$$(6.2) \qquad D(\mu(x),\sigma(x))V(x) = 0 .$$

Assuming the pair (μ,σ) is a Borel C-selector, one hopes that the process X defined by

$$X_0 = x, \qquad dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

will be optimal at x. It follows from Theorem 6.1 below that X is optimal for a subfair or fair casino if μ and σ are bounded on closed subsets of $(0,1)$ and $\inf \sigma > 0$ on such sets. (Use Lemma 2.3.). However, (6.2) always has the solution $\mu(x) \equiv 0$, $\sigma(x) \equiv 0$, available.

Furthermore, if the casino is superfair or trivial, then V is constant on $(0,1)$, and every pair (μ,σ) satisfies (6.2). Now in the trivial case, every available X reaches 1 with probability zero and is optimal. In the superfair case, X is optimal at x if and only if $P[X \text{ reaches } 1] = 1$, and a sufficient condition for optimality can be found in Lemma 2.3.

The theorem of this section characterizes optimal strategies for fair and subfair casinos, and for proportional houses for which $\hat{\rho} < 1/2$.

Let λ be Lebesgue measure on $[0,\infty)$. It was assumed in section 1 that if X is given by (1.1) then $\mu(t) = \sigma(t) = 0$ when $X_t = a$ or b. Here, we further set $D(0,0)V(a) = 0$ and $D(0,0)V(b) = 0$.

<u>Theorem 6.1</u>. Suppose a goal problem on $[a,b]$ satisfies assumptions I and II of section 4 and has value function V which is strictly less than 1 and not identically 0 on (a,b). Let $a < x < b$ and let $X \in \Sigma(x)$ satisfy (1.1). Then X is optimal at x if and only if

$$\text{(i)} \qquad D(\mu(t),\sigma(t))V(X_t) = 0 \qquad \lambda \times P\text{ - a.e.} \ ,$$

$$\text{(ii)} \qquad P[X \text{ reaches } b \text{ or } \lim_{t\to\infty} X_t = a] = 1 \ .$$

Condition (i) is equivalent to

$$\text{(i')} \quad \left[\frac{\mu(t)}{\sigma^2(t)} = \rho(X_t) \quad \text{or} \quad (\mu(t),\sigma(t)) = (0,0)\right] \quad \lambda \times P\text{ - a.e.}$$

This equivalence is an easy consequence of (4.2).

The techniques used to prove Theorem 6.1 are adapted from Chapter 3 of Dubins and Savage [1]. They are widely used in the discrete-time theory ([10],[11]), but seem not to be well-known in continuous-time.

Lemma 6.1. For each x such that $a<x<b$ and each (μ,σ) in $C(x)$,

$$D(\mu,\sigma)V(x) \leq 0 .$$

Proof. Immediate from (4.1) - (4.4) and Lemmas 4.1 and 4.2. ∎

Fix $x \in (a,b)$ and $X \in \Sigma(x)$. Let u be the indicator function of $\{b\}$ and define

$$\begin{aligned} u(X) &= E[\lim_{t\to\infty} u(X_t)] \\ &= P[X \text{ reaches } b] \quad . \end{aligned}$$

The $\lim_{t\to\infty} u(X_t)$ exists almost surely and equals the indicator of [X reaches b] because X is absorbed at b. Also define

$$V(X) = E[\overline{\lim_{t\to\infty}}\, V(X_t)] \quad .$$

For each positive integer n, define the stopping time τ_n by

$$\tau_n = \inf\left\{t: \quad X_t \geq b-\frac{1}{n} \quad \text{or} \quad X_t \leq a+\frac{1}{n}\right\}$$

Lemma 6.2. $u(X) \leq V(X) \leq V(x)$.

Proof. The first inequality is trivial because $u \leq V$. For the second inequality, with the hypotheses of Theorem 6.1 in force, Lemmas 4.1 and 4.2 imply that for each n, the function V is twice-continuously-differentiable on a neighborhood of the closed interval $[a+(1/n),b-(1/n)]$. Therefore Ito's Lemma and Lemma 6.1 can be applied to (1.1) to get

$$(6.3) \qquad V(X_{t\wedge\tau_n}) = V(x) + \int_0^{t\wedge\tau_n} D(\mu(s),\sigma(s))V(X_s)\,ds + \int_0^{t\wedge\tau_n} V'(X_s)\sigma(s)\,dW_s$$

$$\leq V(x) + \int_0^{t\wedge\tau_n} V'(X_s)\sigma(s)\,dW_s \quad .$$

Let τ be an almost-surely-finite stopping time and apply Lemma 2.1 to the local martingale

$$(6.4) \qquad M_t^{(n)} = \int_0^{t\wedge\tau_n} V'(X_s)\sigma(s)\,dW_s$$

to obtain

(6.5) $$EV(X_{\tau\wedge\tau_n}) \le V(x)$$

for each n. Notice that $X_{\tau\wedge\tau_n} \longrightarrow X_\tau$ almost-surely as $n\longrightarrow\infty$ because X has continuous paths. Hence (6.5), the continuity of V, and the Lebesgue dominated convergence theorem give

(6.6) $$EV(X_\tau) \le V(x) \quad .$$

The second inequality of the lemma now follows from the Fatou equation of [5] and [8]:

(6.7) $$V(X) = \overline{\lim_{\tau\to\infty}}\, EV(X_\tau) \quad . \qquad \blacksquare$$

Lemma 6.3. $V(X) = V(x)$ if and only if condition (i) of Theorem 6.1 holds.

Proof: Assume condition (i) and let τ be an almost-surely-finite stopping time. It follows from (6.3) that

$$V(X_{t\wedge\tau_n}) = V(x) + M_t^{(n)}$$

for all n with probability 1, where $M^{(n)}$ is the local martingale from (6.4). Because $|M_t^{(n)}| \le 2\sup\{|V(y)|: a\le y\le b\}$, Lemma 2.1 now implies that

$$EV(X_{\tau\wedge\tau_n}) = V(x)$$

for each n. Then $EV(X_\tau) = V(x)$ by dominated convergence, and the Fatou equation (6.7) gives the desired equality.

Assume now that $V(X) = V(x)$. Suppose (i) does not hold. By Lemma 6.1, there exist positive numbers ε, δ, and t such that

$$s \le t \quad \text{and} \quad D(\mu(s),\sigma(s))V(X_s) < -\varepsilon$$

on a set of $\lambda x P$-measure at least 2δ. Thus there is a positive integer N such that if $n \ge N$, then

$$\lambda x P\left[\left\{(s,\omega): s \le t \text{ and } D[\mu(s,\omega),\sigma(s,\omega)]V(X_{s\wedge\tau_n}(\omega)) < -\varepsilon\right\}\right] \ge \delta .$$

It follows, using (6.3) and Lemma 2.1, that for each a.s. finite stopping time τ satisfying $\tau \ge t$,

$$EV(X_\tau) = \lim_{n\to\infty} EV(X_{\tau\wedge\tau_n}) \le V(x) - \varepsilon\delta .$$

The Fatou equation then implies that $V(X) \le V(x) - \varepsilon\delta$, a contradiction. ∎

<u>Lemma 6.4</u>. $u(X) = V(X)$ if and only if condition (ii) of Theorem 6.1 holds.

<u>Proof</u>: Recall that $u(X_t)$ converges a.s. and let $B =$ [X reaches b]. Then

$$V(X) - u(X) = E[\overline{\lim}\, V(X_t)] - E[\lim u(X_t)]$$

$$= E[\overline{\lim}\, (V-u)(X_t)]$$

$$= \int_B 0 + \int_{B^c} \overline{\lim}\, V(X_t) \quad .$$

So $V(X) = u(X)$ if and only if $V(X_t) \to 0$ a.s. on B^c. Because of Lemmas 4.1 and 4.2 and the hypothesis that V is strictly less than 1 and not identically 0 on (a,b), V must satisfy formula (4.2). Then V is strictly increasing and continuous on (a,b) and hence $V(X_t) \to 0$ if and only if $X_t \to a$. ■

Putting together Lemmas 6.2, 6.3, and 6.4, the proof of Theorem 6.1 is finished.

By analogy with the terminology of Dubins and Savage [1], we say that $X \in \Sigma(x)$ is <u>thrifty</u> if $V(X) = V(x)$ and <u>equalizing</u> if $V(X) = u(X)$. We have seen that, under the hypotheses of Theorem 6.1, X is optimal if and only if X is thrifty and equalizing. Lemma 6.3 characterizes thrifty processes as being those governed by controls which almost always achieve the supremum in the Bellman equation (6.1). These are the same processes X for which $\{V(X_t), \mathcal{F}_t\}$ is a martingale, or, in other words, the processes X with the property that the player's expected potential winnings never decrease. Lemma 6.4 characterizes equalizing processes as being those for which the player's potential winnings at time t, namely $V(X_t)$, and his actual utility at time t, $u(X_t)$, merge as t approaches infinity.

These ideas are no doubt applicable to quite general problems and not only to the goal problems studied in this paper.

In standard approaches to the study of optimal strategies, some ad hoc condition is often imposed which guarantees that all processes X will be equalizing. For example, we could require that $\inf \sigma$ be positive on (a,b). Then every X would leave (a,b) with probability one and be equalizing.

We conclude with a simple example of a control problem on $[0,1]$ for which every control set contains two elements. There is a thrifty strategy and an equalizing strategy, but no process X is both thrifty and equalizing.

<u>Example 6.1</u> For $0<x<1$, let

$$C(x) = \{(0,1-x),(-1,2(1-x)^{\frac{1}{2}}\} .$$

Here $C(x)$ is the union of the control sets in Examples 4.1 and 4.2, and $\rho(x) = 0$ for all $x \in (0,1)$. Thus the control pair $(\mu(x),\sigma(x)) = (0,1-x)$ satisfies (6.2) and a process X satisfying (4.8) is thrifty by Lemma 6.3. However, X is not equalizing and, in fact, $u(X) = 0$ by the discussion of Example 4.1. On the other hand, a process Y, which uses the control pair $(-1,2(1-x)^{\frac{1}{2}})$ and is defined by (4.9), is equalizing but not thrifty. To get a nearly optimal process at $x_0 \in (0,1)$, let $0<\varepsilon<1$ and take $X^{(\varepsilon)}$ to be a solution to

$$
\begin{aligned}
dX_t^{(\varepsilon)} &= \mu_\varepsilon(X_t^{(\varepsilon)})dt + \sigma_\varepsilon(X_t^{(\varepsilon)})dW_t , \\
X_0^{(\varepsilon)} &= x_0
\end{aligned}
$$

where

$$
(\mu_\varepsilon(x),\sigma_\varepsilon(x)) = \begin{cases} (0,1-x) & \text{if } 0<x<1-\varepsilon \\ (-1,2(1-x)^{1/2}) & \text{if } 1-\varepsilon<x<1 \end{cases} .
$$

Then $u(X^{(\varepsilon)}) \uparrow V(x_0)$ as $\varepsilon \downarrow 0$. Loosely speaking, a good strategy uses $\mu/\sigma^2 = 0$ until the goal 1 is almost reached and then switches to μ/σ^2 which is large negatively. ■

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