Controlling a Process to a Goal In Finite Time

by

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Abstract

A player starts at x in (0,1) and seeks to reach 1 by time t_0. The process $\{X(t),\ 0 \le t \le t_0\}$ of the player's positions is a diffusion process (or an Ito process) whose infinitesimal parameters μ,σ are chosen by the player at each instant of time from a set depending on the current position. The probability of reaching 1 by time t_0 is maximized if the player can and does choose the parameters so that σ and μ/σ^2 are maximized, at least when these maxima are sufficiently regular. This result implies that bold play is optimal for subfair, continuous-time red-and-black and roulette when there is a limit on playing time.

Key words: Stochastic control, gambling, red-and-black

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1. Introduction

The problem considered here is that of controlling a stochastic process $X = \{X(t), t \geq 0\}$ with state space an interval $[a,b]$ $(-\infty \leq a < b < \infty)$ so as to maximize $P[X(t_0) = b]$ where t_0 is fixed. Assume X satisfies a stochastic differential equation

$$(1.1) \qquad X(0) = x, \quad dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

where $\{W(t)\}$ is a standard Brownian motion process on (Ω, F, P) adapted to a filtration $\{\underline{F}_t\}$ and where each $\underline{F}_t$ is independent of $\{W(t+s) - W(t), s \geq 0\}$ and contains all P-null sets. The control processes $\mu(t) = \mu(t,\omega)$ and $\sigma(t) = \sigma(t,\omega)$ are assumed to be real-valued, progressively measurable and to satisfy

$$(1.2) \qquad \begin{aligned} \int_0^t |\mu(s)|ds &< \infty \qquad \text{a.s.} \\ \int_0^t \sigma^2(s)\, ds &< \infty \qquad \text{a.s.} \end{aligned}$$

for every $t > 0$. Associated to every $y \in (a,b)$ is a <u>control</u> <u>set</u> $A(y)$ which is a non-empty subset of $R \times R^+$. The player is required to choose the value of (μ,σ) from $A(y)$ when the current position is y. More precisely, it is assumed that

$$(\mu(t),\sigma(t)) \in A(X(t))$$

whenever $a < X(t) < b$. It is also assumed that $\mu(t) = \sigma(t) = 0$ when $X(t)=a$ or $X(t)=b$ so that X is absorbed at a and b.

Let $\Sigma(x)$ be the collection of all such processes $X = \{X(t)\}$ starting at $x \in (a,b)$ and assume $\Sigma(x)$ is non-empty for every x. Call such a control problem a <u>goal problem with time limit</u> t_0 and define its value function to be

$$U_{t_0}(x) = \sup_{X\in\Sigma(x)} P[X(t_0) = b].$$

Goal problems without a time limit were studied by Pestien and Sudderth [6,7], who showed, under mild regularity conditions, that in order to maximize $P[X(t) = b \text{ for some } t \geq 0]$, the player should choose (μ,σ) so that μ/σ^2 attains the supremum

$$\rho(x) = \sup \{\mu/\sigma^2 : (\mu,\sigma) \in A(x)\}, \quad a < x < b. \tag{1.3}$$

(Here, 0/0 is taken to be $-\infty$). Such a choice of (μ,σ) will not in general be optimal for a goal problem with a time limit. It is easy to give examples for which the optimal controls depend on time as well as position. However, there is an interesting condition which results in optimal controls which are stationary in time. To state the condition, define for $a < x < b$,

$$\sigma_0(x) = \sup \{\sigma : (\mu,\sigma) \in A(x)\}. \tag{1.4}$$

<u>Condition C.</u> The function ρ of (1.3) can be written in the form

$$\rho(x) = \mu_0(x)/\sigma_0^2(x) \tag{1.5}$$

where ρ, μ_0 and σ_0 are bounded continous functions on (a,b) and σ_0 is everywhere positive.

Let $Y = \{Y(t),\ t \geq 0\}$ be a diffusion process which starts at $x \in (a,b)$, is absorbed at the endpoints a and b and satisfies

$$(1.6) \qquad dY(t) = \mu_0(Y(t))dt + \sigma_0(Y(t))dW(t)$$

prior to absorption. Define

$$(1.7) \qquad Q_t(x) = P[Y(t) = b].$$

Here is our main result.

<u>Theorem</u> 1.1 If condition C holds, then $Q_t(x) \geq U_t(x)$ for all $t \geq 0$.

The proof will be given in sections 2 and 3. Section 2 presents a verification lemma which is then applied in section 3 to prove the theorem.

Here is a simple corollary which identifies Y as the optimal process.

<u>Corollary</u>. If condition C holds and if $(\mu_0(y), \sigma_0(y)) \in A(y)$ for $a < y < b$, then $Y \in \Sigma(x)$ and $Q_t(x) = U_t(x)$ for all $t \geq 0$.

The rest of this section is devoted to some applications of the theorem to gambling theory. In section 4 a final application will be made to prove two comparison theorems related to one of Hajek [3].

Our first application is to a continuous time version of roulette which includes continuous-time red-and-black and which was introduced in [6]. In roulette a gambler chooses at each stage an amount s to stake and a particular bet. For a given stake, all bets have the same expected return but they may have different variances. Here is a continuous-time formulation. Let $a = 0$ and $b = 1$. Suppose λ is a fixed real number and that s_1 and σ_1 are bounded, continuous functions from (0,1) to the positive real numbers. For $0 < x < 1$, define the control set A(x) by

$$A(x) = \{(s\lambda,\ s\sigma):\ 0 \le s \le s_1(x),\ 0 \le \sigma \le \sigma_1(x)\}. \tag{1.8}$$

If the problem is <u>subfair</u> in the sense that $\lambda \le 0$, then, by (1.3)

$$\rho(x) = (s_1(x)\lambda)/(s_1(x)\sigma_1(x))^2 = \lambda/(s_1(x)\sigma_1^2(x)).$$

Furthermore, condition C holds with $\mu_0(x) = \lambda s_1(x)$, $\sigma_0(x) = s_1(x)\sigma_1(x)$. So Theorem 1.1 applies to show that <u>bold play</u> is optimal even when there is a limit on playing time; the gambler should select the maximum stake $s_1(x)$ and the maximum variance $\sigma_1(x)$ whenever $X(t) = x \in (0,1)$. The fact that bold play is optimal for subfair continuous-time roulette without a time limit is an immediate corollary. (However, the corresponding result in [6] is proved without the assumption made here that s_1 and σ_1 are continuous.) The theorem does not apply if the problem is <u>superfair</u> in the sense that $\lambda > 0$ and we do not know the optimal strategy in this case.

Red-and-black corresponds to the special case where the gambler is allowed to choose a stake s but the variance for a given stake and position is fixed. Thus

$$A(x) = \{(s\lambda,\ s\sigma_1(x)) : 0 \le s \le s_1(x)\} \tag{1.9}$$

where s_1 and σ_1 satisfy the same assumptions as before. If $\lambda \le 0$, it follows from the theorem or from the result for roulette that the gambler should play boldly and make the maximum stake $s_1(x)$ whenever $X(t) = x$. It was shown by A. Dvoretzky that bold play is optimal for discrete-time red-and-black with a time limit (cf. Dubins and Savage [1, section 5.5]). Perhaps the same is true for discrete-time roulette.

2. A verification lemma. The lemma which we will use to prove the theorem of section 1 is a refinement of a result of Orey, Pestien, and Sudderth ([5], Proposition 1.1). The formulation of the lemma uses the framework of continuous -time gambling theory which will be explained briefly. For a more general and more detailed exposition consult references [4], [5], and [6].

Let F be a Borel subset of the Euclidean space R^d and assume that the interior F^0 of F is not empty. For each $x \in F$, let $C(x)$ be a non-empty collection of pairs (a,b) where $a \in R^d$ and b is a $d \times m$ matrix of real numbers. Assume $C(x) = \{(0,0)\}$ for $x \in F - F^0$.

Consider next an Ito process $X = \{X(t),\ t \ge 0\}$ in F defined by a stochastic differential equation

$$X(0) = x,\ dX(t) = \alpha(t)dt + \beta(t)dW(t) \tag{2.1}$$

where $W = \{W(t),\ t \ge 0\}$ is a standard m-dimensional Brownian motion adopted to increasing, right-continous σ-fields $\{\underline{F}_t\}$, and each $\underline{F}_t$ is independent of $\{W_{t+s}$ -

W_t, $s \geq 0\}$. The function $\alpha = \alpha(t,\omega)$ is to be R^d -valued, progressively measurable, adapted to $\{\underline{F}_t\}$, and such that

$$(2.2) \qquad \int_0^t |\alpha(s)| d\, s < \infty \quad \text{a.s.} \quad \text{for all } t.$$

The function $\beta = \beta(t,\omega)$ has as values real d×m matrices, is progressively measurable, adapted to $\{\underline{F}_t\}$ and satisfies

$$(2.3) \qquad \int_0^t |\beta(s)|^2 ds < \infty \quad \text{a.s.} \quad \text{for all } t.$$

Assume X is absorbed at the time T of its first exit from F^0.

For each $x \in F$, the collection $\Sigma(x)$ of processes X available at x consists of those X satisfying (2.1), (2.2) and (2.3) together with

$$(\alpha(t,\omega),\ \beta(t,\omega)) \in C(X(t,\omega)),$$
$$(\alpha(t,\omega),\ \beta(t,\omega)) = (0,0) \text{ for } t \geq T(\omega).$$

Let u, the <u>utility function</u>, be a Borel function from F to the real line. The utility of a process $X \in \Sigma(x)$ is defined to be

$$u(X) = E\,[\limsup_{t \to \infty} u(X_t)].$$

The expected value occuring on the right is assumed to be well-defined for all available X.

The triple (F, Σ, u) is a <u>continuous-time gambling problem</u> in the sense of Pestien and Sudderth [6]. Its value function V is defined by

$$V(x) = \sup \{u(X): X \in \Sigma(x)\}.$$

Let $Q: R^d \to R$ and define, for a a $d\times 1$ vector and b a $d\times m$ matrix,

$$D(a,b)Q(y) = Q_x(y)a + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} Q_{x_i x_j}(y)(bb')_{ij}$$

where

$$Q_x = \left(\frac{\partial Q}{\partial x_1}, \ldots, \frac{\partial Q}{\partial x_d}\right), \quad Q_{x_i x_j} = \frac{\partial^2 Q}{\partial x_i \partial x_j} \quad .$$

Here is our verification lemma, which extends Proposition 1.1 of [5].

<u>Lemma</u>. Let $Q: F \to R$. Assume Q is continuous on F and has continuous second order derivatives on F^0. Assume also that for $x \in F^0$ and $X \in \Sigma(x)$,

(i) $E[\limsup_{t \to \infty} Q(X(t))] \geq E[\limsup_{t \to \infty} u(X(t))]$

(ii) $P[D(\alpha(t),\beta(t))Q(X(t)) \leq 0$ for all $t < T] = 1$, where α and β are related to X as in (2.1),

(iii) there is an integrable random variable Y such that for all $t \geq 0$, $Q(X(t)) \geq Y$.

Then $Q \geq V$.

Proof: Let $x \in F^0$ and $X \in \Sigma(x)$. For each positive ϵ which is smaller than the distance from x to the complement of F, let

$$F_\epsilon = \{y \in F : |y-z| \geq \epsilon \text{ for all } z \in F^c\}$$

and

$$T_\epsilon = \inf \{t \geq 0 : X(t) \notin F_\epsilon\}.$$

Let τ be an almost surely finite $\{\underline{F}_t\}$ stopping time. Now Q is smooth on the open set F^0 which contains F_ϵ and the proof of Proposition 1.1 of [5] shows

$$EQ(X_{\tau \wedge T_\epsilon}) \leq Q(x).$$

Let ϵ approach 0 and use Fatou's inequality and the continuity of Q on F to see that

$$\begin{aligned} EQ(X(\tau)) &= EQ(X(\tau \wedge T)) \\ &= E(\lim_{\epsilon \to 0} Q(X(\tau \wedge T_\epsilon))) \\ &\leq \liminf_{\epsilon \to 0} EQ(X(\tau \wedge T_\epsilon)) \\ &\leq Q(x). \end{aligned}$$

By condition (i) and by Lemma 1 of [6], it follows that $Q \geq V$. □

3. <u>Proof of Theorem 1.1.</u> The goal problem described in section 1 will now be reformulated as a continuous-time gambling problem in R^2. The first coordinate, x_1, of the state vector x will correspond to the player's position in [a,b]. The second coordinate, x_2, will represent time. It is convenient to allow negative as well as positive times up to the time limit t_0. So we define

$$F = \{x \in R^2 \colon a \le x_1 \le b,\ x_2 \le t_0\}. \tag{3.1}$$

For $x \in F^0 = (a,b) \times (-\infty,\ t_0)$, the set of controls is

$$C(x) = \{((\begin{smallmatrix}\mu\\1\end{smallmatrix}),\ (\begin{smallmatrix}\sigma\\0\end{smallmatrix}))\colon (\mu,\sigma) \in A(x_1)\} \tag{3.2}$$

where $A(x_1)$ is a non-empty subset of $R \times R^+$ as in section 1. According to our conventions every available process $X = \{X(t)\} = \{X_1(t),\ X_2(t))\} \in \Sigma(x)$ satisfies a stochastic differential equation

$$\begin{aligned} dX_1(t) &= \mu(t)dt + \sigma(t)dW(t) \\ dX_2(t) &= dt \\ X(0) &= x \end{aligned} \tag{3.3}$$

prior to its time T of absorption at the boundary of F. The time T is given by

$$T = \min\{\inf \{t\colon X_1(t) \in \{a,b\}\},\ t_0 - x_2\}.$$

The control functions μ and σ satisfy (1.2) and are equal to zero for $t \ge T$.

The appropriate utility function is

$$\begin{aligned} u(x) &= 1 \text{ if } x_1 = b \text{ and } x_2 \le t_0 \\ &= 0 \text{ otherwise.} \end{aligned} \tag{3.4}$$

Thus, for $X \in \Sigma(x)$,

$$
\begin{aligned}
u(X) &= P[X_1(T) = b] \\
&= P[X_1(t) = b \text{ for some } t \le t_0 - x_2] \\
&= P[X_1(t_0 - x_2) = b].
\end{aligned}
$$

Assume condition C of section 1 and, for $x \in F^0$, let $Y = (Y_1, Y_2)$ satisfy

$$
\begin{aligned}
dY_1(t) &= \mu_0(Y_1(t))dt + \sigma_0(Y_1(t))dW(t) \\
dY_2(t) &= dt \\
Y(0) &= x
\end{aligned}
$$

prior to absorption at the boundary of F.

Set

$$
(3.5) \qquad
\begin{aligned}
Q(x) &= u(Y) \text{ for } x \in F^0 \\
&= u(x) \text{ for } x \in F - F^0
\end{aligned}
$$

To prove Theorem 1, it suffices to show

$$
(3.6) \qquad Q \ge V,
$$

where V is the value function for the continous-time gambling problem (F, Σ, u). We would like to apply the verification lemma of section 2. However, there are two technical difficulties. First, Q may not be sufficiently smooth on F^0.

Second Q is not continuous on F. (In fact, Q is discontinuous at the point (b, t_0) since $Q(b,x_2) = 1$ for $x_2 \le t_0$, and $Q(x_1, t_0) = 0$ for $x_1 < b$.)

To overcome the first difficulty, approximate μ_0 and σ_0 by smooth functions (say C^∞) μ_n and σ_n on (a,b) such that for n = 1, 2, ...,

$$\sigma_n \ge \sigma_0,\ \rho_n = \mu_n/\sigma_n^2 \ge \mu_0/\sigma_0^2 = \rho,$$

and

$$\sup_{a<x_1<b} \{|\mu_n(x_1)-\mu_0(x_1)| + \sigma_n(x_1)-\sigma(x_1)\} \le 1/n \ . \tag{3.7}$$

For $x \in F^0$, let $X^{(n)} = (X_1^{(n)}, X_2^{(n)})$ be a process satisfying

$$\begin{aligned} dX_1^{(n)}(t) &= \mu_n(X_1^{(n)}(t))dt + \sigma_n(X_1^{(n)}(t))dW(t) \\ dX_2^{(n)}(t) &= dt \\ X^{(n)}(0) &= x \end{aligned}$$

prior to absorption at the boundary of F. It follows from (3.7) that $X^{(n)}$ converges weakly to Y as $n \to \infty$. (See, for example, Stroock and Varadhan [8, Theorem 11.1.4 p. 264].

Hence,

$$u(Y) = P[Y_1(t_0 - x_2) = b]$$

$$\geq \limsup_{n\to\infty} P[X_1^{(n)}(t_0 - x_2) = b]$$

$$= \limsup_{n\to\infty} u\,(X^{(n)}).$$

So, to prove (3.6), it suffices to show

$$Q_n \geq V$$

where Q_n is defined as in (3.5) with Y replaced there by $X^{(n)}$. Thus, without loss of generality, we now assume μ_0 and ρ_0 to be C^∞ functions.

Standard arguments will now show that Q satisfies the following partial differential equation on F^0

$$\frac{1}{2}\,\sigma_0^{\,2}(x_1)Q_{x_1x_1} + \mu_0(x_1)Q_{x_1} + Q_{x_2} = 0 \tag{3.8}$$

with boundary conditions

$$Q(b,x_2) = 1,\ Q(a,x_2) = Q(x_1,\ t_0) = 0 \tag{3.9}$$

for $x_1 < b_1$. It follows that Q is C^2 on F^0. (Apply Theorem 6.2.4 of Friedman [2] to a nice neighborhood of any $x \in F^0$.)

However, Q remains discontinuous at the point $x = (b,\ t_0)$. To sidestep this difficulty, define, for $\epsilon > 0$, the function

$$Q^\epsilon(x_1,x_2) = Q(x_1,\ x_2 - \epsilon)$$

for $x = (x_1,x_2) \in F$. Then Q^ϵ is C^2 on F^0 and continuous on F. Also Q^ϵ converges to Q as ϵ decreases to zero. So, to prove (3.6), it suffices to show

$$(3.10) \qquad Q^\epsilon \geq V.$$

This will be established once we check the conditions of the lemma in section 2. Conditions (i) and (iii) are obvious because

$$Q^\epsilon \geq u \geq 0.$$

To check (ii), let $x \in F^0$ and

$$(\alpha,\beta) = (\binom{\mu}{1}, \binom{\sigma}{0}) \in C(x).$$

Then

$$(3.11) \qquad D(\alpha,\beta)Q^\epsilon = \frac{1}{2}\sigma^2 Q^\epsilon_{x_1x_1} + \mu Q^\epsilon_{x_1} + Q^\epsilon_{x_2} .$$

(We omit the argument x here and below.) Now Q and, hence, Q^ϵ is increasing in x_1 and decreasing in x_2; so

$$(3.12) \qquad Q^\epsilon_{x_1} \geq 0 \ , \ Q^\epsilon_{x_2} \leq 0.$$

Consider the two cases: $\sigma = 0$ and $\sigma > 0$. If $\sigma = 0$, then $\mu \leq 0$ because $\mu/\sigma^2 \leq \rho(x_1) < \infty$.

Hence,

$$D(\alpha,\beta)Q^{\epsilon} = \mu Q^{\epsilon}_{x_1} + Q^{\epsilon}_{x_2} \le 0.$$

by (3.12). Now suppose $\sigma > 0$. Then

$$\begin{aligned} \frac{1}{\sigma^2} D(\alpha,\beta)Q^{\epsilon} &= \frac{1}{2} Q^{\epsilon}_{x_1x_1} + \frac{\mu}{\sigma^2} Q^{\epsilon}_{x_1} + \frac{1}{\sigma^2} Q^{\epsilon}_{x_2} \\ &\le \frac{1}{2} Q^{\epsilon}_{x_1x_1} + \frac{\mu_0(x_1)}{\sigma_0(x_1)^2} Q^{\epsilon}_{x_1} + \frac{1}{\sigma_0(x_1)^2} Q^{\epsilon}_{x_2} \\ &= 0. \end{aligned}$$

The inequality is by (1.3), (1.4), (1.5) and (3.12). The final equality is by (3.8) which still holds when Q is replaced by Q^{ϵ}.

Inequality (3.10) now follows from the lemma.

4. <u>Two comparison theorems</u>.

Consider processes X and Y on the real line satisfying (1.1) and (1.6), respectively. Hajek [3, Theorem 1.3] showed that if μ_0 and σ_0 are constants and if

$$X(0) \le Y(0),\ \mu(t) \le \mu_0,\ |\sigma(t)| \le \sigma_0$$

then

$$P[X(t) \ge b] \le 2P[Y(t) \ge b]$$

for every $t \geq 0$ and every b.

We will formulate two related results in terms of a goal problem as in section 1 taking the left hand endpoint a to be $-\infty$. Assume μ_0 and σ_0 are bounded, continuous functions on $(-\infty, b)$ and , for $-\infty < x < b$, define the control sets

$$(4.1) \qquad A(x) = \{(\mu,\sigma) : \mu \leq \mu_0(x),\ 0 \leq \sigma \leq \sigma_0(x)\}.$$

<u>Theorem 4.1</u>. If $\mu_0(x) \leq 0$ and $\sigma_0(x) > 0$ for $-\infty < x < b$, then

$$P[\sup_{0\leq s\leq t} X(s) \geq b] \leq P[\sup_{0\leq s\leq t} Y(s) \geq b]$$

for every $X \in \Sigma(x)$, $x < b$, $t \geq 0$.

Proof: Immediate from Theorem 1.1 because condition C holds. □

Theorem 4.1 depends crucially on the assumption that the processes are subfair ($\mu_0 \leq 0$). We do not know the optimal process in the general superfair case, but our final result does give the solution when μ_0 and σ_0 are both positive constants. To state the result, consider a fixed time limit t_0 and define for $x < b$, $t < t_0$,

$$(4.2) \qquad \begin{aligned} \sigma_1(x,t) &= 0 \text{ if } x + \mu_0(t_0 - t) \geq b, \\ &= \sigma_0 \text{ if not.} \end{aligned}$$

Let Z be a process satisfying

$$(4.3) \qquad Z(0) = x_1, \; dZ(t) = \mu_0 dt + \sigma_1(Z(t),t)dW(t).$$

prior to absorption at b.

<u>Theorem</u> 4.2 If μ_0 and σ_0 are positive constants, $-\infty < x_1 < b$, and $t_0 > 0$, then

$$P[\sup_{0\le t\le t_0} X(t) \ge b] \le P[\sup_{0\le t\le t_0} Z(t) \ge b]$$

for every $X \in \Sigma(x_1)$.

Proof: The proof is somewhat similar to that of Theorem 1.1 given in the previous section. First we reformulate the problem as a gambling problem in R^2 with

$$F = \{x \in R^2: x_1 \le b, \; x_2 \le t_0\}.$$

The control sets C(x) are defined as in (3.2) with $A(x_1)$ given by (4.1). The utility function is defined by (3.4).

Set

$$Q(x_1,x_2) = P[Z(t_0-x_2) = b].$$

We need to show

$$Q \geq V. \tag{4.4}$$

First we will calculate Q explicitly. Let L be the linear function

$$x_1 = L(x_2) = b - \mu_0(t_0 - x_2). \tag{4.5}$$

Clearly,

$$Q(x_1, x_2) = 1 \quad \text{if } x_1 \geq L(x_2) \tag{4.6}$$

for, in this case, by (4.2) and (4.3), Z moves deterministically to the goal at rate μ_0.

On the other hand, if $x_1 < L(x_2)$, then

$$Z(t) = x_1 + \mu_0 t + \sigma_0 W(t)$$

until the first $t \leq t_0 - x_2$ (if any) such that

$$Z(t) \geq L(x_2 + t)$$

or, equivalently,

$$W(t) \geq \frac{L(x_2) - x_1}{\sigma_0}.$$

Thus

$$(4.7) \qquad Q(x_1,x_2) = P[\sup_{0\le t\le t_0-x_2} Z(t) \ge b]$$

$$= P[\sup_{0\le t\le t_0-x_2} W(t) \ge \frac{L(x_2)-x_1}{\sigma_0}]$$

$$= 2P[W(t_0-x_2) \ge \frac{L(x_2)-x_1}{\sigma_0}]$$

$$= 2\Phi\left(\frac{x_1-L(x_2)}{\sigma_0\sqrt{t_0-x_2}}\right) \text{ if } x_1 < L(x_2)$$

where Φ is the standard normal distribution function.

We would now like to use the verification lemma to establish (4.4). As in the proof of Theorem 1.1, there are technical difficulties. First Q is not smooth along the line L of (4.5). Now to the right of L, Q=1, by (4.6) and, hence, V=1. Thus (4.4) is satisfied. This suggests we redefine the state space to be

$$F^1 = \{(x_1,x_2) : x_2 \le t_0, \; x_1 \le L(x_2)\},$$

and the utility function to be

$$\begin{aligned} u^1(x_1, x_2) &= 1 \text{ if } x_1 = L(x_2), \\ &= 0 \text{ if not.} \end{aligned}$$

The control sets remain the same on the interior of F^1 and, clearly, $V^1 = V$ on

F^1. The advantage of the new problem is that Q is obviously C^∞ on the interior of F^1.

Unfortunately Q is not continuous at the point (b, t_0). To handle this difficulty, define for $\epsilon > 0$

$$Q^\epsilon(x_1, x_2) = Q(x_1 - \mu_0\epsilon, x_2 - \epsilon).$$

Then $Q^\epsilon \to Q$ as $\epsilon \to 0$. So it suffices to show

$$Q^\epsilon \geq V.$$

The application of the lemma is now relatively straightforward.

Conditions (i) and (iii) are immediate because

$$Q^\epsilon \geq u^1 \geq 0.$$

To verify (ii), let $x = (x_1, x_2)$ be in the interior of F^1. So $x_2 < t_0$ and $x_1 < L(x_2)$. Let $(\alpha,\beta) = (\binom{\mu}{1}, \binom{\sigma}{0}) \in C(x)$
so that $\mu \leq \mu_0$, $\sigma \leq \sigma_0$. Use (4.7) to calculate

$$D(\alpha,\beta)Q^\epsilon = \frac{1}{2}\sigma^2 Q^\epsilon_{x_1x_1} + \mu Q^\epsilon_{x_1} + Q^\epsilon_{x_2}, \tag{4.8}$$

Set

$$\lambda = \Phi'\left(\frac{x_1 - L(x_2)}{\sigma_0\sqrt{t_0 - x_2 + \epsilon}}\right) / \sigma_0(t_0 - x_2 + \epsilon)^{3/2}.$$

After some calculation, the right-hand-side of (4.8) becomes

$$\lambda[-\frac{\sigma^2}{\sigma_0^2}(x_1-L(x_2)) + 2\mu(t_0-x_2+\epsilon)-2\mu_0(t_0-x_2+\epsilon) + x_1-L(x_2)]$$

$$\leq \lambda[(1-\sigma^2/\sigma_0^2)(x_1-L(x_2))]$$

$$\leq 0.$$

The first inequality holds because $\mu \leq \mu_0$; the second because $\sigma \leq \sigma_0$ and $x_1 < L(x_2)$. The proof is now complete. □

The reader may have noticed that the optimal process Z of Theorem 4.2 is "bang-bang" in the sense that σ is always chosen to be 0 or σ_0. The fact that the optimal control is "bang-bang" was suggested to us by Steven Orey and can be proved for more general superfair problems in which μ_0 and σ_0 are not assumed to be constant.

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