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## AN ABSTRACT LINEAR DUALITY MODEL

by

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## ABSTRACT

We investigate an abstract linear duality model which has as special instances several duality systems of interest in combinatorial optimization: subspace orthogonality, cone polarity, lattice duality, blocking polyhedra and antiblocking polyhedra. The descriptive duality present in the model is that of specifying a set in terms of linear constraints or viewing a set as being generated by certain types of linear combinations. We define properties of Weyl, Farkas and Minkowski for the general model by analogy with classical results in cone polarity and we investigate relationships among these properties and further properties of Lehman and Fulkerson, defined by analogy with results on dual pairs of polyhedra of the blocking or antiblocking type. In particular, we show that, for any given duality system, the Weyl property is equivalent to the combined properties of Farkas and Minkowski (or Lehman), that the Farkas property implies the Fulkerson property and that the Minkowski property is equivalent to the combined properties of Lehman and Fulkerson. We also adapt results of Dixon [5] in order to obtain general sufficient conditions for validity of the Weyl property.

In the second part of the paper, specific instances of the model are examined in more detail. The instances studied here are "integral duality" and "nonnegative integral duality", which are related to the problem of finding integral solutions and nonnegative integral solutions, respectively, for linear systems. For each instance, the validity of the Weyl, Minkowski, Farkas, Lehman and Fulkerson properties is examined. We also characterize the constrained sets under each duality.

## Introduction

There is an apparent formal similarity in the definition of the orthogonal complement of a vector subspace of  $\mathbb{R}^n$  and the cone polar to a given convex cone in  $\mathbb{R}^n$ : both prescribe that the inner product between the elements of the initial object and elements of its corresponding "dual partner" belong to a particular subset of the reals. This subset is  $\{0\}$  for the subspace case and is the nonnegative reals  $\mathbb{R}_+$  in the cone case. In [17] a general model for linear duality was proposed which encompasses these similarities. In this paper we develop this model further and we study in detail specific instances of the model for integral and nonnegative integral duality. The development here closely follows that in [2] and [17].

### 1. Duality Model

An abstraction of the linear algebraic features of orthogonality and polarity requires a framework in which "inner product" makes sense. We thus assume an underlying commutative ring  $R$ , which for all applications in this paper will be the real numbers with the usual addition and multiplication. All sets involved are assumed to be subsets of  $R^n$ , the free  $R$ -module of rank  $n$ , for some  $n$ , and the inner product between elements of  $R^n$  is the ordinary "dot product"; i.e., where  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ , we define  $xy = x_1y_1 + \dots + x_ny_n$ .

**1.1 Definition:** Let  $R$  be a commutative ring with  $D \subseteq X \subseteq R$  and  $\{0,1\} \subseteq X$ . For any subset  $S \subseteq X^n$ , the  $(X,D)$ -dual of  $S$  is

$$S^* = \{x \in X^n: Sx \in D\},$$

where  $Sx \in D$  denotes  $sx \in D, \forall s \in S$ .  $\square$

Thus for  $R = \mathbb{R}$ , when  $X = \mathbb{R}$  and  $D = \{0\}$ , the  $(X,D)$ -dual of subset  $S \subseteq \mathbb{R}^n$  is the subspace orthogonal to  $S$ , and when  $X = \mathbb{R}$  and  $D = \mathbb{R}_+$ , the  $(X,D)$ -dual of  $S$  is the cone

polar to  $S$ . In the following we will assume that  $R$ ,  $X$  and  $D$  have been fixed and we will refer to  $S^*$  simply as the "dual of  $S$ ".

Definition 1.1 stipulates that members of  $S^*$  satisfy constraints defined by the elements of  $S$ . We thus use the following terminology.

**1.2 Definition:** A subset  $S \subseteq X^n$  is *constrained* provided  $S = T^*$  for some  $T \subseteq X^n$ . Moreover, when  $S = T^*$  and  $|T|$  is finite, then  $S$  is *finitely constrained*.  $\square$

Thus finitely constrained sets are of the form  $\{x \in X^n: Ax \in D^m\}$  for some  $m \times n$  matrix  $A \in X^{m \times n}$ .

**1.3 Definition:** A subset  $S \subseteq X^n$  is *closed* (under  $(S,D)$ -duality) if  $S = S^{**}$ , where  $S^{**}$  denotes  $(S^*)^*$ .  $\square$

The following properties are easily derived from Definition 1.3 (see [2] or [17] for the proofs).

**1.4 Proposition:** Suppose  $S$  and  $T$  are subsets of  $X^n$ . Then:

- (a)  $S \subseteq T \Rightarrow S^* \supseteq T^*$ .
- (b)  $(S \cup T)^* = S^* \cap T^*$ .
- (c)  $S \subseteq S^{**}$ .
- (d)  $S^* = S^{***}$ .
- (e)  $S$  is closed  $\Leftrightarrow S$  is constrained.
- (f)  $S^{**}$  is the smallest closed set containing  $S$ .  $\square$

Emphasis thus far has been placed on defining certain subsets of  $X^n$ , the constrained or closed subsets, from the "outside", i.e., in terms of constraints. We observe also that any subspace or convex cone can be equally well described from the "inside", i.e., in terms of linear combinations of a set of generators in the subspace case or of nonnegative combinations in the cone case. This descriptive equivalence between

external and internal representations of the same set is the "duality" which serves as the focus for the remainder of our development.

We thus require an appropriate definition of "generated sets" in terms of certain types of linear combinations. Note that in the subspace and cone cases, the types of linear combinations considered are those under which the constrained sets are closed ( $S \subseteq X^n$  is *closed* with respect to  $T$ -linear combinations, where  $T \subseteq X^m$ , provided  $(y_1 a_1 + \dots + y_m a_m) \in S$  whenever  $a_1, \dots, a_m \in S$  and  $y \in T$ ). That is, sets of the form  $\{x \in \mathbb{R}^n: Sx = 0\}$  and  $\{x \in \mathbb{R}^n: Sx \geq 0\}$ , where  $S \subseteq \mathbb{R}^n$ , are closed under arbitrary and nonnegative linear combinations, respectively. We now show that with  $(D^m)^*$ -linear combinations, where  $(D^m)^*$  is the  $(X,D)$ -dual of  $D^m$ , we achieve closure for constrained sets in the general model.

**1.5 Theorem:** For each  $m \geq 1$ ,  $(D^m)^*$  is the set of those  $m$ -vectors providing coefficients for linear combinations under which all constrained sets are closed; i.e., for each  $m \geq 1$ ,

$$(D^m)^* = \{y \in X^m: \forall n \geq 1, S \subseteq X^n \text{ and } a_1, \dots, a_m \in S^* \Rightarrow (y_1 a_1 + \dots + y_m a_m) \in S^*\}.$$

**Proof:** Let  $y \in (D^m)^*$ ,  $S \subseteq X^n$  and  $a_1, \dots, a_m \in S^*$ , where  $m \geq 1$ . Then for any  $s \in S$  we have

$$(y_1 a_1 + \dots + y_m a_m)s = y_1(a_1 s) + \dots + y_m(a_m s) = yz,$$

where we denote  $z_i = a_i s$ , for  $1 \leq i \leq m$ . Since  $a_i \in S^*$  and  $s \in S$ , we have  $z_i \in D$ ,  $1 \leq i \leq m$ . Hence  $z \in D^m$  and since  $y \in (D^m)^*$ , we obtain  $yz \in D$ . That is,  $(y_1 a_1 + \dots + y_m a_m)s \in D, \forall s \in S$ . Thus, as required,  $(y_1 a_1 + \dots + y_m a_m) \in S^*$ .

On the other hand, since  $0, 1 \in X$  by Definition 1.1, we have that  $e_i \in X^m$ ,  $1 \leq i \leq m$ , where  $e_i = (0, \dots, 0, 1, 0, \dots, 0)$  denotes the  $i^{\text{th}}$  unit vector. Clearly  $e_i \in (D^m)^*$ .

Thus for  $y \in X^m$ , applying the stipulation in the statement of the theorem with  $m = n$ ,  $S = D^m$  and  $e_i = a_i$ ,  $1 \leq i \leq m$ , we obtain that  $y \in (D^m)^*$ , which completes the proof.  $\square$

Note the explicit use of the fact that  $0, 1 \in X$ , as stipulated in Definition 1.1, in the proof of Theorem 1.5. As discussed above, this theorem motivates the following definition.

**1.6 Definition:** For  $S \subseteq X^n$ , the set *generated* by  $S$ , denoted  $\sigma(S)$ , is the set

$$\sigma(S) = \{x \in X^n: x = y_1 a_1 + \dots + y_m a_m, \text{ where } m \geq 1, a_1, \dots, a_m \in S \text{ and } y \in (D^m)^*\}.$$

$S$  is itself a *generated* set provided  $S = \sigma(T)$  for some  $T \subseteq X^n$ ; when  $S = \sigma(T)$  and  $|T|$  is finite, then  $S$  is *finitely generated*.  $\square$

Since  $(D^m)^*$  contains the unit vectors we have that  $S \subseteq \sigma(S)$ . It is also obvious that  $S \subseteq T$  implies  $\sigma(S) \subseteq \sigma(T)$ . We also remark that finitely generated sets are of the form  $S = \{yA: y \in (D^m)^*\}$ , where  $A \in X^{m \times n}$ .

For the particular cases discussed above we have that for  $(\mathbb{R}, \{0\})$ -duality (subspace orthogonality),  $(D^m)^* = \mathbb{R}^m$ , and for  $(\mathbb{R}, \mathbb{R}_+)$ -duality (cone polarity),  $(D^m)^* = \mathbb{R}_+^m$ . In Sections 4, 5 and 6 the properties of  $(\mathbb{Q}, \mathbb{Z})$ -,  $(\mathbb{Z}, \mathbb{Z}_+)$ - and  $(\mathbb{Q}, \mathbb{Z}_+)$ -duality are investigated, with particular attention given to characterizing the constrained (closed) sets for these cases; here  $(D^m)^*$  is given by  $\mathbb{Z}^m$ ,  $\mathbb{Z}_+^m$  and  $\mathbb{Z}_+^m$ , respectively, with  $\mathbb{Q}$ ,  $\mathbb{Z}$  and  $\mathbb{Z}_+$  denoting the rationals, the integers and the nonnegative integers, respectively. Additional instances of interest in combinatorial optimization are obtained by taking  $X = \mathbb{R}_+$  with  $D = [1, \infty)$ , "blocking" duality, or  $D = [0, 1]$ , "antiblocking" duality (see [7]). For the latter two instances one verifies that  $(D^m)^* = \{y \in \mathbb{R}_+^m: y_1 + \dots + y_m \geq 1\}$  and  $(D^m)^* = \{y \in \mathbb{R}_+^m: y_1 + \dots + y_m \leq 1\}$ , respectively.

The next proposition relates generated sets and closure under  $(D^m)^*$ -linear combinations.

**1.7 Proposition:** Let  $S \subseteq X^n$ . The following are equivalent:

- (a)  $\sigma(S) \subseteq S$ .
- (b)  $\sigma(S) = S$ .
- (c)  $S$  is a generated set.

**Proof:** Since it is generally true that  $\sigma(S) \supseteq S$ , (a) and (b) are clearly equivalent. It is also obvious that (b)  $\Rightarrow$  (c).

To show that (c)  $\Rightarrow$  (a), suppose  $S$  is generated by  $T$  and let  $m \geq 1$ ,  $s_1, \dots, s_m \in S$  and  $y \in (D^m)^*$ . Since  $S$  is generated by  $T$ , there exist elements  $a_1, \dots, a_p \in T$  taken as the rows of matrix  $A$ , for which each  $s_i = z_i A$  with  $z_i \in (D^p)^*$ . Thus, taking the  $z_i$  as the rows of matrix  $Z$ , we have

$$y_1 s_1 + \dots + y_m s_m = y(ZA) = (yZ)A.$$

Note that  $yZ = y_1 z_1 + \dots + y_m z_m$  is a  $(D^m)^*$ -linear combination of elements of  $(D^p)^*$ . Thus, since  $(D^p)^*$  is constrained, Theorem 1.5 implies that  $(yZ) \in (D^p)^*$ . Hence  $(yZ)A \in S$ , which proves that  $S$  is closed under  $(D^m)^*$ -linear combinations.  $\square$

**1.8 Corollary:** Let  $S \subseteq X^n$ . Then  $\sigma(S)$  is the smallest generated set containing  $S$ .

**Proof:** Let  $T$  be a generated set such that  $S \subseteq T$ . Then, since  $\sigma(T) = T$ , we have that  $\sigma(S) \subseteq \sigma(T) = T$ .  $\square$

The following corollary relates the notions of generated and constrained sets via Theorem 1.5 and Proposition 1.7.

**1.9 Corollary:** Every constrained set is generated.  $\square$

Note the similarity which this corollary bears to the well-known theorem of Minkowski for "finite" cones, i.e., that any polyhedral (finitely constrained) cone is finitely generated. However, Corollary 1.9 requires no finiteness and, in general, nothing special can be said about the generation properties of a finitely constrained set. In the following section we discuss the relation between constrained and generated sets when finiteness is stipulated. The following example shows that the converse of Corollary 1.9 is generally false.

**1.10 Example:** Let  $R = X = \mathbb{R}$  and  $D = \mathbb{R}_+$  (cone polarity) and define  $S = \{(0,0)\} \cup \{(x_1, x_2) \in \mathbb{R}^2: x_1 > 0, x_2 > 0\}$ . Then  $S$  is a generated set, since it is closed under linear combinations with nonnegative coefficients. However,  $S$  is not constrained, as

$$S^{**} = (S^*)^* = (\{x \in \mathbb{R}^2: x \geq 0\})^* = \{x \in \mathbb{R}^2: x \geq 0\} \neq S. \quad \square$$

We conclude this section by showing that the function  $\sigma(\cdot)$  has no effect on the duality operation.

**1.11 Proposition:** Let  $S \subseteq X^n$ . Then  $(\sigma(S))^* = S^* = \sigma(S^*)$ .

**Proof:** Since  $S^{**}$  is a generated set (by 1.9) containing  $S$  (by 1.4c), we obtain from Corollary 1.8 that  $S \subseteq \sigma(S) \subseteq S^{**}$ . Hence 1.4a implies that  $S^* \supseteq (\sigma(S))^* \supseteq S^{***}$  and we may apply 1.4d to obtain  $S^* = (\sigma(S))^*$ . Furthermore, Corollary 1.9 and Proposition 1.7 immediately imply that  $S^* = \sigma(S^*)$ .  $\square$

**1.12 Corollary:** Let  $S, T \subseteq X^n$  with  $\sigma(S) = T$ . Then  $S^* = T^*$ .

**Proof:** By 1.11,  $(\sigma(S))^* = S^*$  and by assumption,  $\sigma(S) = T$ .  $\square$

**1.13 Corollary:** Let  $A \in X^{m \times n}$  and define  $T = \{yA: y \in (D^m)^*\}$ . Then  $T^* = \{x \in X^n: Ax \in D^m\}$ . Thus any finitely generated set has a finitely constrained dual.



**Proof:** Apply 1.12, taking  $S$  as the rows of matrix  $A$ .  $\square$

## 2. Finiteness Properties

The development of the previous section is in terms of properties which hold for the general  $(X,D)$ -duality model. The settings of subspace orthogonality and cone polarity which motivated the general model, however, suggest that other important properties are valid for certain  $(X,D)$  pairs. These properties deal with finitely generated and finitely constrained subsets, i.e., sets of the form  $\{yA: y \in (D^m)^*\}$  and  $\{x \in X^n: Ax \in D^m\}$ , respectively, where  $A \in X^{m \times n}$ . In order to facilitate the statement of the properties which we now study, we assume throughout this section that all generated sets under consideration are nonempty.

The following three definitions are motivated by classical results in cone polarity (see, e.g., [14,16]).

**2.1 Farkas property:** For any  $A \in X^{m \times n}$  and any  $c \in X^n$ , exactly one of the following holds:

- (a)  $\exists y \in (D^m)^*$  such that  $yA = c$ .
- (b)  $\exists x \in X^n$  such that  $Ax \in D^m$  and  $cx \notin D$ .  $\square$

**2.2 Minkowski property:** Every finitely constrained set is finitely generated.  $\square$

**2.3 Weyl property:** Every finitely generated set is finitely constrained.  $\square$

The following additional properties are motivated by results established for the respective dualities associated with blocking polyhedra and antiblocking polyhedra (see [7]).

**2.4 Lehman property:** Every finitely constrained set has a finitely constrained dual.  $\square$

**2.5 Fulkerson property:** Every finitely constrained set has a finitely generated dual.  $\square$

Note the similarity between the Lehman and Fulkerson properties and Corollary 1.13, which asserts that finitely generated sets have finitely constrained duals. The latter property holds for *any*  $(X,D)$ -duality, however, in contrast to 2.4 and 2.5, which hold only in certain instances. For example, with  $X = \mathbb{Q}$  and  $D = \mathbb{Z}$  we have that since  $\mathbb{Q}^n = \{x \in \mathbb{Q}^n: 0x_1 + \dots + 0x_n \in \mathbb{Z}\}$ , the set  $S = \mathbb{Q}^n$  is finitely constrained. But  $S^* = \{x \in \mathbb{Q}^n: Sx \in \mathbb{Z}\} = \{0\}$  and  $S^*$  cannot be represented by finitely many constraints of the form  $sx \in \mathbb{Z}$ , where  $s \in S$ . Thus  $S^*$  is not finitely constrained and so the Lehman property fails for  $(\mathbb{Q}, \mathbb{Z})$ -duality. Similarly, for  $X = \mathbb{R}_+$  and  $D = [1, \infty)$ , the set  $S = \{x \in \mathbb{R}_+^2: x_1 \geq 1\}$  is finitely constrained, while the dual  $S^* = S$  is not finitely generated. Thus the Fulkerson property fails for  $(\mathbb{R}_+, [1, \infty))$ -duality.

We now show that the combination of the properties of Lehman and Fulkerson is equivalent to the Minkowski property.

**2.6 Proposition:** For a specific  $(X,D)$ -duality, the Minkowski property holds if and only if the Lehman and Fulkerson properties both hold.

**Proof:** Suppose the properties of Lehman and Fulkerson are valid. Then  $S$  finitely constrained implies (Lehman) that  $S^*$  is finitely constrained, which in turn implies (Fulkerson) that  $S^{**}$  is finitely generated. But now 1.4(e) implies  $S = S^{**}$ , since  $S$  is constrained. Thus the Minkowski property holds. Conversely, if the Minkowski property holds and  $S$  is finitely constrained, then  $S$  is also finitely generated and 1.13 implies that  $S^*$  is finitely constrained, so that the Lehman property holds. Applying the Minkowski property to  $S^*$  shows that  $S^*$  is finitely generated, hence the Fulkerson property also is valid.  $\square$

We note that the property, "Every finitely generated set has a finitely generated dual.", is equivalent to the Minkowski property; this fact is an easy consequence of Corollary 1.13.

The Farkas property has well-known algorithmic implications for decision problems of the following form.

**2.7 Problem:** Given  $A \in X^{m \times n}$  and  $c \in X^n$ , is there a solution  $y \in (D^m)^*$  for the linear system  $yA = c$ ?

When one can demonstrate in polynomial-time that a given vector  $y$  satisfies alternative 2.1a, Problem 2.7 is in the class NP of decision problems. When the Farkas property holds and one can validate in polynomial-time that a given vector  $x$  satisfies alternative 2.1b, Problem 2.7 is in the class co-NP of decision problems. Thus validity of the Farkas property suggests that Problem 2.7 is in the problem class  $NP \cap co-NP$ , which in turn suggests the likelihood that Problem 2.7 is in the class P of polynomial-time solvable decision problems. For a more detailed discussion we refer the reader to [8].

In order to relate the Farkas property to the others, we first establish equivalent statements for it.

**2.8 Theorem:** For a specific  $(X,D)$ -duality, the following are equivalent:

- (a) The Farkas property holds.
- (b) Every finitely generated set is constrained.
- (c) If  $S = \{x \in X^n: Ax \in D^m\}$ , where  $A \in X^{m \times n}$ , then  $S^* = \{yA: y \in (D^m)^*\}$ .

**Proof:** Since (c) stipulates that any finitely generated set is in fact the dual of a finitely constrained set, clearly (c)  $\Rightarrow$  (b). Also, if a finitely generated set  $T = \{yA: y \in (D^m)^*\}$  is constrained, then by Proposition 1.4e and Corollary 1.13 we have  $T = (T^*)^* = (\{x \in X^n: Ax \in D^m\})^*$ . Thus (b)  $\Rightarrow$  (c).

We may restate alternatives 2.1a and 2.1b as (a')  $c \in \{yA: y \in (D^m)^*\}$  and (b')  $c \notin (\{x \in X^n: Ax \in D^m\})^*$ . Since the Farkas property asserts that 2.1a holds if and only if 2.1b fails, we conclude that the Farkas property is equivalent to

$$c \in \{yA: y \in (D^m)^*\} \Leftrightarrow c \in (\{x \in X^n: Ax \in D^m\})^*.$$

That is, the Farkas property holds if and only if

$$\{yA: y \in (D^m)^*\} = (\{x \in X^n: Ax \in D^m\})^*.$$

Hence (a) and (c) are equivalent.  $\square$

Note that 2.8c is actually the "dual" result to that expressed in Corollary 1.13. Furthermore, the form of 2.8c shows immediately that the Farkas property implies the Fulkerson property.

**2.9 Corollary:** For a specific  $(X,D)$ -duality, if the Farkas property holds then the Fulkerson property also holds.  $\square$

Note also that 2.8b serves to highlight the distinction between the Farkas and Weyl properties, indicating that the Weyl property is stronger (see Theorem 2.10 below), since it concludes that finitely generated sets are *finitely* constrained. This distinction is evident if one considers the  $(\mathbb{R}, \mathbb{R}_+)$ -duality of convex cones in geometric terms. Here both the Farkas and Weyl properties are valid. The Farkas property asserts that either a vector  $c$  is in the cone  $K$  generated by the rows of matrix  $A$  (i.e.,  $\exists y \geq 0$  s.t.  $yA = c$ ) or it is not, in which case some hyperplane defined by  $x$  separates  $c$  from  $K$  (i.e.,  $\exists x$  s.t.  $Ax \geq 0, cx < 0$ ). On the other hand, the Weyl property makes the stronger assertion that, since  $K$  is (finitely) generated by the rows of  $A$ , it can also be described by finitely many constraints (i.e.,  $K = \{x: Bx \geq 0\}$  for some matrix  $B$ ); thus when  $c \notin K$ , one of the *finitely many* rows of  $B$  may be used to separate  $c$  from  $K$ .

**2.10 Theorem:** For a specific  $(X,D)$ -duality, the Weyl property holds if and only if the Farkas and Minkowski properties both hold.

**Proof:** Assume the Weyl property holds. To see that the Minkowski property is also valid, suppose  $S = \{x \in X^n: Ax \in D^m\}$  is a finitely constrained set. By 2.8c,  $S^*$  is finitely generated, and so the Weyl property implies that  $S^*$  is finitely constrained. Applying 2.8c to  $S^*$  now shows that  $S^{**}$  is finitely generated. Since  $S^{**} = S$ , the Minkowski property follows.

For the reverse implication, assume that the Farkas and Minkowski properties are valid and let  $S = \{yA: y \in (D^m)^*\}$  be a finitely generated set. Suppose  $T = \{x \in X^n: Ax \in D^m\}$ . The Minkowski property now implies that for some matrix  $B \in X^{p \times n}$ ,  $T = \{yB: y \in (D^p)^*\}$ . By applying 2.8c and then 1.13 we obtain that

$$S = T^* = (\{yB: y \in (D^p)^*\})^* = \{x \in X^n: Bx \in D^p\}.$$

Thus the Weyl property is valid.  $\square$

We note that Theorem 2.10 provides abstract justification for the observation of Stoer and Witzgall [16, p. 57] concerning the classical results of cone polarity: "The theorems of Minkowski and Farkas may be derived from the theorem of Weyl. However, the theorems of Farkas and Minkowski must be combined in order to yield the theorem of Weyl." Proposition 2.6 shows that the Lehman property is implied by the Minkowski property. This relation is actually strict, since in blocking duality  $(X = \mathbb{R}_+, D = [1, \infty))$ , for example, the Lehman property holds but the Minkowski property fails. Thus it is of interest that the Minkowski property can be replaced by the Lehman property in Theorem 2.10, thereby providing a sharper characterization.

**2.11 Corollary:** For a specific  $(X,D)$ -duality, the Weyl property holds if and only if the Farkas and Lehman properties both hold.

**Proof:** The "only if" assertion is immediate from 2.6 and 2.10. To see the "if" part, note that when  $S$  is finitely generated, 2.8b implies that  $S$  is constrained, hence  $S = S^{**}$ . Also, 1.13 implies that  $S^*$  is finitely constrained, hence the Lehman property implies that  $S^{**} (=S)$  is finitely constrained.  $\square$

The relationships just derived actually describe all logical dependence among properties 2.1-2.5. Indeed, one can verify that, with respect to properties 2.1-2.5, an  $(X,D)$ -duality may:

- i) satisfy only the Fulkerson property (e.g.,  $X = \mathbb{R}_+$ ,  $D = [0,1]$ , "anti-blocking duality");
- ii) satisfy only the Lehman property (e.g.,  $X = \mathbb{R}_+$ ,  $D = [1,\infty)$ , "blocking duality" or  $X = \mathbb{Q}$ ,  $D = \mathbb{Z}_+$ , see Section 6);
- iii) satisfy only the Fulkerson, Lehman and Minkowski properties (e.g.,  $X = \mathbb{Z}$ ,  $D = \mathbb{Z}_+$ , see Section 5);
- iv) satisfy only the Farkas and Fulkerson properties (e.g.,  $X = \mathbb{Q}$ ,  $D = \mathbb{Z}$ , see Section 4);
- v) satisfy all five properties (e.g.,  $X = \mathbb{R}$ ,  $D = \{0\}$  (or  $D = \mathbb{R}_+$ )).

As seen above, the Weyl property (2.3) is the strongest of those stated here. In the following section we investigate sufficient conditions for the validity of the Weyl property. We conclude this section with a brief summary of the behavior of subspace and cone dualities with respect to the properties investigated above.

For subspace duality we have  $X = \mathbb{R}$ ,  $D = \{0\}$  and  $(D^m)^* = \mathbb{R}^m$ . Thus the generated subsets of  $\mathbb{R}^n$  are exactly the vector subspaces in  $\mathbb{R}^n$ . Here all generated sets are, of course, finitely generated. The dual of any set  $S$  is the orthogonal complement of the subspace generated by  $S$ . Since any nonempty subspace satisfies  $(S^*)^* = S$ , it follows that any nonempty generated set is finitely constrained by a basis of its orthogonal complement.

Hence the notions of generated set, finitely generated set, constrained set and finitely constrained set are all equivalent in this case. Thus the five properties 2.1-5 are valid for subspace duality. This discussion remains valid if one considers subspace duality in the field of rationals (i.e.,  $(\mathbb{Q},\{0\})$ -duality).

For cone duality, with  $X = \mathbb{R}$ ,  $D = \mathbb{R}_+$  and  $(D^m)^* = \mathbb{R}_+^m$ , the generated sets are the subsets of  $\mathbb{R}^n$  which are closed under nonnegative linear combinations, that is, the convex cones. In contrast to the subspace case, not all generated sets are finitely generated; "circular" cones, for instance, are not finitely generated. Also, not all convex cones are constrained, as is demonstrated in Example 1.10. It can be shown (see [14, p.121] for a proof) that a cone is constrained if and only if it is topologically closed. Furthermore, not all constrained cones are finitely constrained. However, properties 2.1-5 are all valid for cones and in the following section we indicate how to prove the validity of the Weyl property (and hence of all the others) in this setting. This proof remains valid if one considers cone duality in the rationals  $((\mathbb{Q},\mathbb{Q}_+)$ -duality). However, closure under this duality is not equivalent to topological closure. As shown by Hartmann [11], a set  $S$  is closed under  $(\mathbb{Q},\mathbb{Q}_+)$ -duality if and only if  $S$  is topologically closed *and* its lineality space has a basis of rational vectors.

### 3. Validation of Duality Properties

We now consider the question of determining whether the properties 2.1-5 are valid for a specific  $(X,D)$ -duality. Since the Weyl property (2.3) has been shown to imply the others, we focus attention on means for validating it. One approach is based on using an "elimination procedure" under which finitely generated sets have their representation altered until the validity of the Weyl property becomes obvious. We discuss this now for cone duality, using Fourier-Motzkin elimination.

Suppose that  $K \subseteq \mathbb{R}^n$  is a nonempty, finitely generated cone; i.e., where the rows of  $A \in \mathbb{R}^{m \times n}$  constitute a set of generators for  $K$ ,  $K = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \text{ such that}$

$yA = x, y \geq 0$  is consistent}. By applying Fourier-Motzkin elimination (see, e.g., [16]), we eliminate  $y$  to obtain an equivalent (in terms of consistency) homogeneous linear inequality system in  $x$  only, say  $\{Bx \geq 0\}$  for some  $B \in \mathbb{R}^{p \times n}$ . Thus we have  $K = \{x \in \mathbb{R}^n: Bx \geq 0\}$ ; hence  $K$  is finitely constrained, establishing Weyl's result.

Gauss-Jordan elimination can be similarly used to establish the Weyl property for subspace duality (see [17]) and in Section 4 below we make similar use of "unimodular" elimination in studying  $(\mathbb{Q}, \mathbb{Z})$ -duality.

In [5] Dixon presents an algebraic framework for linear duality similar to that studied here and obtains general conditions under which the Farkas property (2.1) holds. We now discuss his result and show that it may be altered slightly to obtain a corresponding theorem for the Weyl property. The development here closely follows that of [5].

Throughout the remainder we assume that  $X = \mathbb{R}$ . Following [5], we call any surjective  $X$ -linear function  $f: X^n \rightarrow X^m$  (where  $n \geq m$ ) a *retraction*; i.e., a retraction  $f$  is of the form  $f(x) = Ax$  for some matrix  $A \in X^{m \times n}$  with rank  $m$ . Let  $\mathcal{S}$  be a collection of subsets of  $(X^1 \cup X^2 \cup \dots)$  and denote  $\mathcal{S}_n = \{S \in \mathcal{S} : S \subseteq X^n\}$  for  $n = 1, 2, \dots$ . We say that  $\mathcal{S}$  is *closed under retractions* provided  $f(S) \in \mathcal{S}$  whenever  $n \geq 1$ ,  $S \in \mathcal{S}_n$  and  $f$  is a retraction defined on  $X^n$ . The family of all finitely generated sets provides an important example; this family is closed under retractions because  $f(\sigma(S)) = \sigma(f(S))$  for any  $S \subseteq X^n$  and any retraction  $f$  defined on  $X^n$ . The following theorem provides conditions under which all sets in a family  $\mathcal{S}$  closed under retractions are constrained. In particular, by taking  $\mathcal{S}$  as the family of all finitely generated sets, it yields conditions under which the Farkas property holds.

**3.1 Theorem** (Dixon [5]): Let  $\mathcal{S}$  be closed under retractions. Then every  $S \in \mathcal{S}$  is constrained provided:

- (a)  $S^{**} = S$  for all  $S \in \mathcal{S}_1$  (the one-dimensional case).



- (b) For each  $n \geq 2$ ,  $S \in \mathcal{S}_n$  and  $z \in X^n \setminus S$ , there exists an  $X$ -basis of  $X^n$ , say  $u_1, \dots, u_n$ , for which  $(z + Xu_1) \cap S = \emptyset$  (there is a line through  $z$  which does not intersect  $S$ ).

Conversely, if every  $S \in \mathcal{S}$  is constrained and  $R (=X)$  is a principal ideal domain, then (a) and (b) must hold.  $\square$

In [5] Dixon uses Theorem 3.1 to obtain validations of the Farkas property for several  $(X, D)$  pairs, including subspace ( $X = \mathbb{R}$ ,  $D = \{0\}$ ), cone ( $X = \mathbb{R}$ ,  $D = \mathbb{R}_+$ ) and integral ( $X = \mathbb{Q}$ ,  $D = \mathbb{Z}$ ) duality. He also shows that the Farkas property holds for the case in which  $D$  is a principal ideal domain and  $X$  is its field of quotients (see, e.g., [13]); this case is treated in more detail in [2].

We now modify Theorem 3.1 to insure that all sets in a family  $\mathcal{S}$  closed under retractions are *finitely* constrained, which, when  $\mathcal{S}$  is taken as the family of all finitely generated sets, amounts to establishing conditions under which the Weyl property (2.3) holds.

**3.2 Theorem:** Let  $\mathcal{S}$  be closed under retractions. Then every  $S \in \mathcal{S}$  is finitely constrained provided:

- (a) Every  $S \in \mathcal{S}_1$  is finitely constrained (the one-dimensional case).
- (b) For each  $n \geq 2$  and  $S \in \mathcal{S}_n$ , there exists a finite collection of  $X$ -bases of  $X^n$ , say  $\{u_1^1, \dots, u_n^1\}, \dots, \{u_1^k, \dots, u_n^k\}$ , such that to each  $z \in X^n \setminus S$  there corresponds an index  $j$ ,  $1 \leq j \leq k$ , for which  $(z + Xu_1^j) \cap S = \emptyset$  (there is a finite set of directions  $\{u_1^1, \dots, u_1^k\}$  such that the line through  $z$  in a least one of these directions does not intersect  $S$ ).

Conversely, if every  $S \in \mathcal{S}$  is finitely constrained and  $R (=X)$  is a principal ideal domain, then (a) and (b) must hold.

**Proof:** Our proof follows that given in [5] for Theorem 3.1. First, assume that (a) and (b) hold. We will use induction on  $n$  to prove that every  $S \in \mathcal{S}$  is finitely constrained. By (a), this is true for  $n = 1$ . Assume that the property holds for  $1, \dots, n-1$ , where  $n \geq 2$ , and let  $S \in \mathcal{S}_n$ . Condition (b) now provides a (finite) collection of  $X$ -bases,  $\{u_1^1, \dots, u_n^1\}, \dots, \{u_1^k, \dots, u_n^k\}$  such that for every  $z \in X^n \setminus S$  we have  $(z + Xu_1^j) \cap S = \emptyset$  for some  $j \in \{1, 2, \dots, k\}$ .

For  $j = 1, 2, \dots, k$ , consider the retractions  $f_j: X^n \rightarrow X^{n-1}$  defined by  $f_j(\alpha_1 u_1^j + \dots + \alpha_n u_n^j) = (\alpha_2, \dots, \alpha_n)$ . Then  $f_j(S) \in \mathcal{S}_{n-1}$ , and for any  $x \in X^n$  we have:

$$f_j(x) \in f_j(S) \Leftrightarrow (x - s) \in Xu_1^j \text{ for some } s \in S \Leftrightarrow (x + Xu_1^j) \cap S \neq \emptyset.$$

Thus  $z \notin S$  implies that for some  $j$ ,  $f_j(z) \notin f_j(S)$ . By the induction hypothesis, there is a finite set  $\{a_1^j, \dots, a_{p(j)}^j\} \subseteq X^{n-1}$  such that

$$f_j(S) = \{x \in X^{n-1} : a_i^j x \in D, 1 \leq i \leq p(j)\}.$$

Let  $A_j \in X^{(n-1) \times n}$  be matrices for which  $f_j(x) = A_j x$ ,  $1 \leq j \leq k$ . We claim then that

$$S = \{x \in X^n : (a_i^j A_j)x \in D, 1 \leq j \leq k, 1 \leq i \leq p(j)\}.$$

Clearly,  $z \in S$  implies  $f_j(z) = A_j z \in f_j(S)$  for any  $j$  and hence  $(a_i^j A_j)z = a_i^j(A_j z) \in D$ , for  $1 \leq j \leq k$  and  $1 \leq i \leq p(j)$ . On the other hand, when  $z \notin S$ ,  $A_j z = f_j(z) \notin f_j(S)$  for some index  $j \in \{1, 2, \dots, k\}$ . Thus the claim is valid and hence every  $S \in \mathcal{S}_n$  is finitely constrained, completing the induction.

For the converse assertion, assume now that  $X$  is a principal ideal domain and that every  $S \in \mathcal{S}$  is finitely constrained. Then condition (a) holds trivially and to establish (b), let  $S \in \mathcal{S}_n$ , where  $n \geq 2$ . Since  $S$  is finitely constrained, there are elements  $v_1, \dots, v_k \in X^n$  such that  $S = \{x \in X^n : v_j x \in D, 1 \leq j \leq k\}$ . Corresponding to each  $j$ ,  $1 \leq j \leq k$ , we now construct an  $X$ -basis for  $X^n$ ,  $\{u_1^j, \dots, u_n^j\}$  for which  $v_j u_1^j = 0$ . Note that this will complete

the proof, since then for  $z \in X^n \setminus S$ , we have  $v_j z \notin D$  for some index  $j$ ,  $1 \leq j \leq k$ ; hence for any vector of the form  $z + \delta u_1^j$  with  $\delta \in X$ , we have  $v_j(z + \delta u_1^j) = v_j z \notin D$  so that

$(z + Xu_1^j) \cap S = \emptyset$ , establishing (b). To construct the desired bases, fix  $j$ ,  $1 \leq j \leq k$ , and write  $v_j = \alpha_1 e_1 + \dots + \alpha_n e_n$ , where  $e_i$ ,  $1 \leq i \leq n$ , are the unit vectors of  $X^n$  and each  $\alpha_i \in X$ . Let  $d$  be the greatest common divisor of  $\alpha_1$  and  $\alpha_2$  and choose  $\beta_1, \beta_2$  so that  $\alpha_1 = d\beta_2$ ,  $\alpha_2 = d(-\beta_1)$ . Clearly  $\alpha_1\beta_1 + \alpha_2\beta_2 = 0$ . Moreover, the greatest common divisor of  $\beta_1$  and  $\beta_2$  is 1 and hence there exist  $\gamma_1, \gamma_2 \in X$  such that  $\beta_1\gamma_1 - \beta_2\gamma_2 = 1$ . Now define  $u_1^j, \dots, u_n^j$  by

$$u_1^j = \beta_1 e_1 + \beta_2 e_2, u_2^j = \gamma_2 e_1 + \gamma_1 e_2 \text{ and } u_i^j = e_i \text{ for } i > 2.$$

Since

$$\det \begin{bmatrix} \beta_1 & \beta_2 & 0 \\ \gamma_2 & \gamma_1 & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix} = \beta_1\gamma_1 - \beta_2\gamma_2 = 1,$$

(here  $I_{n-2}$  is the  $(n-2) \times (n-2)$  identity matrix) it follows that the matrix which transforms  $(e_1, \dots, e_n)$  into  $(u_1^j, \dots, u_n^j)$  is invertible over  $X$ . Hence,  $\{u_1^j, \dots, u_n^j\}$  is an  $X$ -basis of  $X^n$ .

Note also that the relation  $\alpha_1\beta_1 + \alpha_2\beta_2 = 0$  implies that  $v_j u_1^j = 0$ , as required.  $\square$

To illustrate the manner in which Theorem 3.2 may be used to establish the Weyl property, we consider again subspace and cone duality. For subspace duality, let  $\mathcal{S}$  denote the family of all (nonempty) vector subspaces of  $\mathbb{R}^n$ , for  $n = 1, 2, \dots$ . The only nonempty subspaces of  $\mathbb{R}$  are  $\{0\} = \{x \in \mathbb{R}: 1x = 0\}$  and  $\mathbb{R} = \{x \in \mathbb{R}: 0x = 0\}$ ; hence 3.2a holds. Now let  $S$  be a subspace of  $\mathbb{R}^n$  for some  $n \geq 2$ . If  $S \neq \{0\}$ , then let  $u_1 \in S \setminus \{0\}$  and extend  $u_1$  to a basis  $\{u_1, \dots, u_n\}$  for  $\mathbb{R}^n$ . For any  $z \in \mathbb{R}^n \setminus S$  we have

$$(z + \mathbb{R}u_1) \cap S \neq \emptyset \Leftrightarrow s - z \in \mathbb{R}u_1 \text{ for some } s \in S \Leftrightarrow z \in S.$$

Therefore  $z \notin S$  implies  $(z + \mathbb{R}u_1) \cap S = \emptyset$ , and hence 3.2b holds in this case (a single basis suffices in this case). If  $S = \{0\}$ , let  $u_1$  and  $u_2$  be any two linearly independent vectors of  $\mathbb{R}^n$  (recall  $n \geq 2$ ). For each  $z \neq 0$ , at least one of the lines  $z + \mathbb{R}u_1$ ,  $z + \mathbb{R}u_2$  does

not contain the origin, so that 3.2b also holds in this case. The validity of the Weyl property for  $(\mathbb{R}, \{0\})$ -duality thus follows.

For the cone case, let  $\mathcal{S}$  be the family of all (nonempty) finitely generated cones in  $\mathbb{R}^n$ , for  $n = 1, 2, \dots$ . Again, 3.2a is trivially true, as the only finitely generated cones in  $\mathbb{R}$  are  $\{0\} = \{x \in \mathbb{R}: 1x \geq 0, (-1)x \geq 0\}$ ,  $\mathbb{R}_+ = \{x \in \mathbb{R}: 1x \geq 0\}$ ,  $\mathbb{R}_- = \{x \in \mathbb{R}: (-1)x \geq 0\}$  and  $\mathbb{R} = \{x \in \mathbb{R}: 0x \geq 0\}$ . Now suppose  $K \in \mathcal{S}_n$ , where  $n \geq 2$ . If  $K$  is simply a ray (half-line), say  $K = \{yu_1: y \geq 0\}$ , then let  $u_2$  be any nonzero vector of  $\mathbb{R}^n$  not parallel to  $u_1$ . Then it is easy to check that for each  $z \notin K$ , at least one of the lines  $z + \mathbb{R}u_1$ ,  $z + \mathbb{R}u_2$  fails to intersect  $K$ . If  $K$  is not a ray, suppose  $K$  has generators  $G = \{u_1, \dots, u_k\}$ , where  $k \geq 2$ . Here,  $(z + \mathbb{R}u_i) \cap K \neq \emptyset$  if and only if  $z \in \mathcal{K}(G \cup \{-u_i\})$ , the cone generated by  $G \cup \{-u_i\}$ . Stoer and Witzgall [16, p. 58] show that  $\bigcap_{i=1}^k \mathcal{K}(G \cup \{-u_i\}) = K$ , provided  $K$  is not a ray. Hence for every  $z \notin K$ , at least one of the lines  $z + \mathbb{R}u_i$ ,  $1 \leq i \leq k$ , does not intersect  $K$ . Thus 3.2b holds and the Weyl property is valid for cone duality ( $X = \mathbb{R}$ ,  $D = \mathbb{R}_+$ ). It is of interest to observe that the proof obtained here is essentially the same as that given in [16].

#### 4. Integral duality

In this and in the following sections we examine specific instances of  $(X, D)$ -duality which are related to integer programming. As is customary in this subject we will restrict attention to the case of rational data. In particular, when we refer to subspace and cone duality in the remaining sections we mean  $(\mathbb{Q}, \{0\})$ - and  $(\mathbb{Q}, \mathbb{Q}_+)$ -duality, respectively. In order to distinguish among the duals of a set  $S$  under each duality we will use the symbols  $S^\perp$ ,  $S^+$  and  $S^\#$  to denote the dual of a set  $S \subseteq \mathbb{Q}^n$  under  $(\mathbb{Q}, \{0\})$ -,  $(\mathbb{Q}, \mathbb{Q}_+)$ - and  $(\mathbb{Q}, \mathbb{Z})$ -duality.

We term *integral duality* (or simply  *$\mathbb{Z}$ -duality*) the  $(X, D)$ -duality obtained by taking  $R = X = \mathbb{Q}$  and  $D = \mathbb{Z}$ . In this case

$$(D^m)^* = \{x \in Q^m: xy \in Z, \forall y \in Z^m\} = Z^m.$$

Hence, the *generated* sets here are exactly those subsets of  $Q^n$  which are closed under *integral linear combinations*, that is, *modules* over the ring of integers  $Z$  (or  $Z$ -modules for short). Given  $S \subseteq Q^n$ , the set  $\mathcal{M}(S)$  of all finite integral linear combinations of elements of  $S$  is called the  $Z$ -module *generated* by  $S$ . If  $|S|$  is finite,  $\mathcal{M}(S)$  is *finitely generated*. Hence, finitely generated  $Z$ -modules are sets of the form

$$M = \{yA: y \in Z^m\}, \text{ where } A \in Q^{m \times n}.$$

Not all  $Z$ -modules are finitely generated. The set of the dyadic rationals  $D = \{m/2^n: m, n \in Z\}$  is an example of a  $Z$ -module which is not finitely generated. Vector subspaces of  $Q^n$  (except for  $\{0\}$ ) constitute another important example of such  $Z$ -modules.

We will denote the  $(Q, Z)$ -dual of a set  $S \subseteq Q$  by  $S^\#$ . Hence  $S^\#$  is defined by  $S^\# = \{x \in Q^n: Sx \in Z\}$ . Recall that by Corollary 1.9  $S^\#$  is a generated set, i.e., a  $Z$ -module, for every  $S \subseteq Q^n$ .  $Z$ -modules of the form  $M = S^\#$  for some  $S \subseteq Q^n$  are called *constrained*  $Z$ -modules. If, in addition,  $|S|$  is finite,  $M$  is said to be *finitely constrained*. Hence, finitely constrained  $Z$ -modules are sets of the form  $\{x \in Q^n: Ax \in Z^m\}$ , where  $A \in Q^{m \times n}$ .

The general duality properties derived in Section 1 immediately imply the following properties for integral duality.

**4.1 Proposition:** Let  $S, T$  denote subsets of  $Q^n$ .

- (a)  $S \subseteq T \Rightarrow S^\# \supseteq T^\#$ .
- (b)  $(S \cup T)^\# = S^\# \cap T^\#$ .
- (c)  $S \subseteq S^{\#\#}$ .
- (d)  $S^\# = S^{\#\#\#}$ .
- (e)  $S = S^{\#\#} \Leftrightarrow S$  is a constrained  $Z$ -module.
- (f)  $S^{\#\#}$  is the smallest constrained  $Z$ -module containing  $S$ .
- (g) If  $S$  generates the  $Z$ -module  $M$ , then  $S^\# = M^\#$ .

- (h) If  $S = \{yA: y \in \mathbb{Z}^m\}$ , where  $A \in \mathbb{Q}^{m \times n}$ , then  
 $S^\# = \{x \in \mathbb{Q}^n: Ax \in \mathbb{Z}^m\}$ .
- (i) If  $S$  and  $T$  are  $\mathbb{Z}$ -modules then  $(S+T)^\# = S^\# \cap T^\#$ .  $\square$

Proposition 4.1(h) provides an expression for the  $(\mathbb{Q}, \mathbb{Z})$ -dual of a finitely generated  $\mathbb{Z}$ -module. It is also of interest to obtain an expression for the  $(\mathbb{Q}, \mathbb{Z})$ -dual of  $\mathbb{Z}$ -modules which are the sum of a finitely generated  $\mathbb{Z}$ -module and a subspace (as we will see, in Theorem 4.9 below, these are exactly the constrained sets for  $\mathbb{Z}$ -duality).

**4.2 Proposition:** Let  $S = \{yA + zB: y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\}$ , where  $A \in \mathbb{Q}^{m \times n}$  and  $B \in \mathbb{Q}^{p \times n}$ . Then

$$S^\# = \{x \in \mathbb{Q}^n: Ax = 0, Bx \in \mathbb{Z}^p\}.$$

**Proof:**  $S = T + U$ , where  $T = \{yA: y \in \mathbb{Q}^m\}$  and  $U = \{zB: z \in \mathbb{Z}^p\}$ . By Proposition 4.1(h),  $U^\# = \{x \in \mathbb{Q}^n: Bx \in \mathbb{Z}^p\}$ . Now

$$\begin{aligned} T^\# &= \{x \in \mathbb{Q}^n: (yA)x \in \mathbb{Z}, \forall y \in \mathbb{Q}^m\} \\ &= \{x \in \mathbb{Q}^n: y(Ax) \in \mathbb{Z}, \forall y \in \mathbb{Q}^m\} \\ &= \{x \in \mathbb{Q}^n: Ax = 0\}. \end{aligned}$$

Hence, by Proposition 4.1(i),

$$S^\# = T^\# \cap U^\# = \{x \in \mathbb{Q}^n: Ax = 0, Bx \in \mathbb{Z}^p\}. \quad \square$$

Next we shall examine the properties of finitely generated and finitely constrained  $\mathbb{Z}$ -modules. As seen in Section 2, the study of these properties for subspaces was greatly simplified by the fact that the properties of being *generated*, *finitely generated*, *constrained* and *finitely constrained* are all equivalent. For  $\mathbb{Z}$ -modules, as in the cone case, this is not true. We have already seen that the set  $\mathbb{D}$  of dyadic rationals provides an example of a  $\mathbb{Z}$ -module which is not finitely generated.  $\mathbb{D}$  also gives an example of a  $\mathbb{Z}$ -module which is

*not constrained* (just observe that  $\mathbf{D}^\# = \{x \in \mathbb{Q} : mx/2^n \in \mathbb{Z}, \forall m, n \in \mathbb{Z}\} = \{0\}$ , which implies  $\mathbf{D}^{\#\#} = \mathbb{Q} \neq \mathbf{D}$ ). It is also easy to see that not all *constrained*  $\mathbb{Z}$ -modules are *finitely constrained*. Let  $S = \{x \in \mathbb{Q}^n : Bx = 0\}$ , where  $B \in \mathbb{Q}^{m \times n}$ , be a subspace of  $\mathbb{Q}^n$  of dimension less than  $n$ . Then  $S$  is constrained (since  $S = \{yB : y \in \mathbb{Q}^m\}^\#$ ) but not finitely constrained, since the restrictions  $Bx = 0$  cannot be replaced by a finite number of restrictions of the type  $ax \in \mathbb{Z}$  (see [2] for more details).

Actually, this example shows that the Weyl and Minkowski properties, as described in Section 2, *cannot hold for  $\mathbb{Z}$ -duality*. Indeed  $\{0\}$  is an example of a finitely generated  $\mathbb{Z}$ -module which is *not* finitely constrained. Also,  $\mathbb{Q}^n$  is a finitely constrained  $\mathbb{Z}$ -module (note that  $\mathbb{Q}^n = \{x \in \mathbb{Q}^n : 0x \in \mathbb{Z}\}$ ) which is *not* finitely generated. The Lehman property also fails:  $\mathbb{Q}^n$  is finitely constrained but its dual  $\{0\}$  is not.

The difference between the duality properties of  $\mathbb{Z}$ -modules and cones results from the fact that subspaces behave differently with respect to these two dualities. Subspaces can be viewed both as finitely generated and finitely constrained cones. However, subspaces are neither finitely generated nor finitely constrained  $\mathbb{Z}$ -modules (unless their dimension is 0 or  $n$ , respectively). In order to obtain an equivalence property between finitely generated and finitely constrained  $\mathbb{Z}$ -modules, we have to treat their *span* and *lineality* explicitly (the *lineality* of a  $\mathbb{Z}$ -module  $M \subseteq \mathbb{Q}^n$  is the largest subspace contained in  $M$ ; i.e., it is the set of all vectors  $x \in \mathbb{Q}^n$  such that  $\lambda x \in M, \forall \lambda \in \mathbb{Q}$ ). That is, we must deal with sets of the form  $\{yA + zB : y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\}$  on the one hand and  $\{x \in \mathbb{Q}^n : Cx = 0, Dx \in \mathbb{Z}^m\}$  on the other. We shall show that in fact these two forms are equivalent, which means that Weyl-, Farkas-, Minkowski-, Lehman- and Fulkerson-type properties are valid for these "extended" concepts of finitely generated and finitely constrained sets.

To prove the equivalence between the extended notions of finitely generated and finitely constrained  $\mathbb{Z}$ -modules, we use the concept of *unimodular elimination*, a refinement of Gaussian elimination which preserves integrality.

A matrix  $P \in \mathbb{Z}^{m \times n}$  is said to be *unimodular* if  $\det P = \pm 1$ . Observe that if  $P$  is unimodular, then  $P^{-1}$  exists and has integral entries. Hence, for every  $y \in \mathbb{Q}^m$  we have  $y \in \mathbb{Z}^m \Leftrightarrow yP \in \mathbb{Z}^m$ .

Now, let  $A \in \mathbb{Q}^{m \times n}$ , let  $P \in \mathbb{Z}^{m \times m}$  be unimodular and define  $B = PA$ . Then, using the fact that  $y \in \mathbb{Z}^m \Leftrightarrow yP \in \mathbb{Z}^m$ , we have

$$\{yA: y \in \mathbb{Z}^m\} = \{(yP)A: y \in \mathbb{Z}^m\} = \{yB: y \in \mathbb{Z}^m\}.$$

Thus, the module generated by the rows of a matrix  $A$  *does not change* when  $A$  is premultiplied by a unimodular matrix.

This fact can be used to row-reduce a rational matrix to a triangular form without altering the  $\mathbb{Z}$ -module which it generates. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

be a rational matrix. Let  $L$  be an integer such that  $La_{11}$  and  $La_{21}$  are integers. Let  $g = \gcd(La_{11}, La_{21})$ . It follows from the Euclidean algorithm (see [13]) that there are integers  $p$  and  $q$  such that  $pLa_{11} + qLa_{21} = g$ . Consider the matrix

$$P = \begin{bmatrix} p & q \\ -La_{21}/g & La_{11}/g \end{bmatrix}.$$

$P$  has integral entries, its determinant is 1 (i.e.,  $P$  is unimodular) and  $PA$  is of the form

$$\begin{bmatrix} a_{11}' & a_{12}' \\ 0 & a_{22}' \end{bmatrix}.$$

By repeatedly applying this process, an arbitrary rational matrix can be row-reduced to an upper triangular matrix which generates the same  $\mathbb{Z}$ -module. We call this process *unimodular elimination*.

By combining Gaussian and unimodular elimination we obtain the following:



**4.3 Theorem:** Let  $M = \{yA + zB : y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\}$  where  $A \in \mathbb{Q}^{m \times n}$  and  $B \in \mathbb{Q}^{p \times n}$ . Let  $r = \text{rank}(A)$  and  $s = \text{rank}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) - \text{rank}(A)$ . Assume that the first  $r+s$  columns of

$\begin{bmatrix} A \\ B \end{bmatrix}$  are linearly independent. Then there are matrices  $A' \in \mathbb{Q}^{r \times n}$  and  $B' \in \mathbb{Q}^{s \times n}$  such that

$M = \{yA' + zB' : y \in \mathbb{Q}^r, z \in \mathbb{Z}^s\}$ , with

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} A_1 & & A_2 \\ 0 & B_1 & B_2 \end{bmatrix},$$

where  $A_1 \in \mathbb{Q}^{r \times r}$  and  $B_1 \in \mathbb{Q}^{s \times s}$  are upper triangular matrices with nonzero diagonal entries.

**Proof:** By using Gaussian elimination, we can row reduce  $\begin{bmatrix} A \\ B \end{bmatrix}$  to a matrix of the form

$$\begin{matrix} r \\ m-r \\ p \end{matrix} \left\{ \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \\ 0 & \tilde{B} \end{bmatrix} \right\},$$

where  $A_1 \in \mathbb{Q}^{r \times r}$  is upper triangular.

This is equivalent to premultiplying  $\begin{bmatrix} A \\ B \end{bmatrix}$  by a matrix of the form

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & I_p \end{bmatrix},$$

where  $P_1 \in \mathbb{Q}^{m \times m}$  is a nonsingular matrix and  $I_p$  denotes the  $p \times p$  identity matrix.

Now, by using unimodular elimination, we can find a unimodular matrix  $Q$  such that  $Q\tilde{B}$  is of the form

$$\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix},$$

where  $B_1 \in \mathbb{Q}^{s \times s}$  is upper triangular.

Hence, premultiplying  $\begin{bmatrix} A \\ B \end{bmatrix}$  by the product

$$\begin{bmatrix} I_m & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ P_2 & I_p \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ QP_2 & Q \end{bmatrix}$$

yields a matrix of the form:

$$\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \\ 0 & B_1 & B_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A' \\ 0 \\ B' \\ 0 \end{bmatrix}.$$

The correspondence

$$[y, z] \Leftrightarrow [y', z'] \begin{bmatrix} P_1 & 0 \\ QP_2 & Q \end{bmatrix}$$

defines a bijection on  $Q^m \times Z^p$ , since:

$$z' \in Z^p \Leftrightarrow z'Q \in Z^p \Leftrightarrow z \in Z^p.$$

Thus:

$$x \in M \Leftrightarrow \exists y \in Q^m, \exists z \in Z^p \text{ such that } [y, z] \begin{bmatrix} A \\ B \end{bmatrix} = x$$

$$\Leftrightarrow \exists y' \in Q^m, \exists z' \in Z^p \text{ such that } [y', z'] \begin{bmatrix} P_1 & 0 \\ QP_2 & Q \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = x$$

$$\Leftrightarrow \exists y' \in Q^m, \exists z' \in Z^p \text{ such that } [y', z'] \begin{bmatrix} A' \\ 0 \\ B' \\ 0 \end{bmatrix} = x$$

$$\Leftrightarrow \exists y'' \in Q^r, \exists z'' \in Z^s \text{ such that } [y'', z''] \begin{bmatrix} A' \\ B' \end{bmatrix} = x. \quad \square$$

We can now state and prove the following:

**4.4 Theorem:** Let  $S = \{yA + zB: y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\}$ , where  $A \in \mathbb{Q}^{m \times n}$  and  $B \in \mathbb{Q}^{p \times n}$ .

Then there are integers  $r, s$  and matrices  $C \in \mathbb{Q}^{r \times n}$  and  $D \in \mathbb{Q}^{s \times n}$  such that

$$S = \{x \in \mathbb{Q}^n: Cx = 0, Dx \in \mathbb{Z}^s\}.$$

(Note that by Proposition 4.2 this implies that  $S$  is constrained.)

**Proof:** By Theorem 4.3 we can assume, without loss of generality, that

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A_1 & \vdots & A_2 & \vdots \\ 0 & B_1 & \vdots & B_2 \end{bmatrix},$$

where  $A_1 \in \mathbb{Q}^{m \times m}$  and  $B_1 \in \mathbb{Q}^{p \times p}$  are upper triangular matrices having nonzero diagonal entries.

Hence, using Gaussian elimination, we can find an invertible matrix  $Q$  such that

$$\begin{bmatrix} A \\ B \end{bmatrix} Q = \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_p & 0 \end{bmatrix}.$$

Therefore

$$\begin{aligned} S &= \{x: x = yA + zB, y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\} \\ &= \{x: xQ = (yA + zB)Q, y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\} \\ &= \{x: xQ = (y, z, 0), y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\} \\ &= \{x: xQ^j \in \mathbb{Z}, \text{ for } m+1 \leq j \leq m+p \text{ and} \\ &\quad xQ^j = 0, \text{ for } j > m+p\} \\ &\quad (\text{where } Q^j \text{ is the } j^{\text{th}} \text{ column of } Q) \\ &= \{x: Dx \in \mathbb{Z}^p, Cx = 0\}, \text{ where} \\ D &= [Q^{m+1}, \dots, Q^{m+p}]^t \text{ and } C = [Q^{m+p+1}, \dots, Q^n]^t. \quad \square \end{aligned}$$

Theorem 4.4 is a Weyl-type property. By an analysis similar to that used to establish the results in Section 2, corresponding Farkas-, Minkowski-, Lehman- and Fulkerson-type properties are obtained. Hence, we have

#### 4.5 Corollary:

- (a) For any  $A \in \mathbb{Q}^{m \times n}$ ,  $B \in \mathbb{Q}^{p \times n}$  and  $c \in \mathbb{Q}^n$  exactly one of the following is true:
- (i)  $\exists y \in \mathbb{Q}^m, z \in \mathbb{Z}^p$  such that  $yA + zB = c$ ;
  - (ii)  $\exists x \in \mathbb{Q}^n$  such that  $Ax = 0$ ,  $Bx \in \mathbb{Z}^p$  and  $cx \notin \mathbb{Z}$ .
- (b) If  $S = \{x \in \mathbb{Q}^n: Ax = 0, Bx \in \mathbb{Z}^p\}$ , where  $A \in \mathbb{Q}^{m \times n}$  and  $B \in \mathbb{Q}^{p \times n}$ , then  $S^\# = \{yA + zB: y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\}$ .
- (c) If  $S = \{x \in \mathbb{Q}^n: Ax = 0, Bx \in \mathbb{Z}^p\}$ , where  $A \in \mathbb{Q}^{m \times n}$  and  $B \in \mathbb{Q}^{p \times n}$ , then there are integers  $r$  and  $s$  and matrices  $C \in \mathbb{Q}^{r \times n}$  and  $D \in \mathbb{Q}^{s \times n}$  such that  $S = \{yC + zD: y \in \mathbb{Q}^r, z \in \mathbb{Z}^s\}$ .  $\square$

Now, let us again consider the Weyl, Minkowski, Farkas, Lehman and Fulkerson properties for  $\mathbb{Z}$ -duality, in the strict sense defined in Section 2. Recall that the Weyl, Minkowski and Lehman properties fail. However, the validity of the extended versions of these properties (derived in Theorem 4.4 and Corollary 4.5) is sufficient to show that the Farkas (hence also the Fulkerson) property still holds in the original sense. Indeed, taking  $A$  vacuous (i.e.,  $m = 0$ ) in 4.5(a) yields:

**4.6 Corollary:** The Farkas property holds for  $\mathbb{Z}$ -duality. That is, for any  $B \in \mathbb{Q}^{p \times n}$  and  $c \in \mathbb{Q}^n$ , exactly one of the following is true:

- (i)  $\exists z \in \mathbb{Z}^p$  such that  $zB = c$ ;
- (ii)  $\exists x \in \mathbb{Q}^n$  such that  $Bx \in \mathbb{Z}^p$  and  $cx \notin \mathbb{Z}$ .  $\square$

We now focus on the problem of characterizing constrained (or closed)  $\mathbb{Z}$ -modules, i.e.,  $\mathbb{Z}$ -modules with the property that  $S^{\#\#} = S$ .

We will show that the  $\mathbb{Z}$ -modules which are constrained are precisely the ones which are the sum of a subspace and a finitely generated  $\mathbb{Z}$ -module, i.e., modules of the form  $\{yA + zB: y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\}$  or, equivalently, of the form  $\{x \in \mathbb{Q}^n: Cx = 0, Dx \in \mathbb{Z}^p\}$ . Observe that, in this aspect,  $\mathbb{Z}$ -duality differs from cone duality, since not all

constrained cones can be expressed as a sum of a subspace and a finitely generated cone (recall that circular cones are examples of cones which are constrained but not finitely generated).

We will use the fact that a *submodule of a finitely generated  $\mathbb{Z}$ -module is again finitely generated*. This is a well-known theorem in algebra. However, we present here an elementary proof of this result which uses standard techniques in integer programming (see [1, p. 496] for another simple proof).

First we need the following:

**4.7 Lemma:** Let  $M \subseteq \mathbb{Q}^n$  be a  $\mathbb{Z}$ -module. Then  $M$  is *finitely generated* if and only if there is a nonzero  $k \in \mathbb{Z}$  such that  $kM \subseteq \mathbb{Z}^n$ .

**Proof:**

(a) ( $\Rightarrow$ ) Let  $M = \{yA: y \in \mathbb{Z}^m\}$ , where  $A \in \mathbb{Q}^{m \times n}$ . Since all entries in  $A$  are rational numbers, we can choose a nonzero integer  $k$  such that  $kA \in \mathbb{Z}^{m \times n}$ . Hence  $kM = \{y(kA): y \in \mathbb{Z}^m\} \subseteq \mathbb{Z}^n$ .

(b) ( $\Leftarrow$ ) Suppose that  $N = kM \subseteq \mathbb{Z}^n$ . Let  $S$  be the subspace generated by  $N$ . Let  $a_1, \dots, a_r$  denote elements of  $N$  such that  $\{a_1, \dots, a_r\}$  constitute a basis of  $S$ . For any  $x \in N$ , there exist rationals  $\alpha_1, \dots, \alpha_r$  (uniquely determined) such that  $x = \sum_{i=1}^r \alpha_i a_i$ . Define  $T = \{x \in N: 0 \leq \alpha_i \leq 1, \forall i\}$ . (Observe that  $T \supseteq \{a_1, \dots, a_r\}$ ). Since  $N \subseteq \mathbb{Z}^n$ , it is clear that  $T$  is finite. Let  $T = \{t_1, \dots, t_s\}$ . Clearly,  $\{\sum_{i=1}^s y_i t_i: y_i \in \mathbb{Z}\} \subseteq N$ , since  $N$  is closed under integer linear combinations.

Now, let  $x \in N$ . We can express  $x$  as

$$x = \sum_{i=1}^n \alpha_i a_i = \sum_{i=1}^n \lfloor \alpha_i \rfloor a_i + \sum_{i=1}^n (\alpha_i - \lfloor \alpha_i \rfloor) a_i ,$$

which shows that every  $x \in N$  is an integer combination of elements of  $T$ . Hence,  $N \subseteq \{\sum_{i=1}^s y_i t_i : y_i \in \mathbb{Z}\}$ .

Therefore, we proved that  $N = \{\sum_{i=1}^s y_i t_i : y_i \in \mathbb{Z}\}$ , which shows that  $N$  (and hence  $M$ ) is finitely generated.  $\square$

**4.8 Proposition:** Let  $M \subseteq \mathbb{Q}^n$  be a finitely generated  $\mathbb{Z}$ -module. Then any submodule of  $M$  is also finitely generated.

**Proof:** Let  $N$  be a submodule of  $M$ . By Lemma 4.7, there exists a nonzero  $k \in \mathbb{Z}$  such that  $kN \subseteq kM \subseteq \mathbb{Z}^n$ . But, again by Lemma 4.7, this implies that  $N$  is finitely generated.  $\square$

Now we are ready for the main result of this section.

**4.9 Theorem:** Let  $M \subseteq \mathbb{Q}^n$ . Then  $M$  is a constrained  $\mathbb{Z}$ -module if and only if there exist matrices  $A \in \mathbb{Q}^{m \times n}$  and  $B \in \mathbb{Q}^{p \times n}$  such that  $M = \{yA + zB : y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\}$ .

**Proof:** The "if" part was established in Theorem 4.4. To show the converse, suppose that  $M$  is a constrained  $\mathbb{Z}$ -module. That is,  $\exists S \subseteq \mathbb{Q}^n$  such that  $M = S^\# = \{x \in \mathbb{Q}^n : Sx \in \mathbb{Z}\}$ .

Let  $m$  be the dimension of the subspace spanned by  $S$ . We consider two cases:

(a)  $m = n$

Since  $S$  is full dimensional, we can choose  $n$  linearly independent elements  $s_1, \dots, s_n \in S$ . Let  $S_0$  be the matrix which has rows  $s_1, \dots, s_n$  and let  $M_0 = \{s_1, \dots, s_n\}^\# = \{x \in \mathbb{Q}^n : S_0 x \in \mathbb{Z}^n\}$ .

Then

$$x \in M_0 \Leftrightarrow$$

$$S_0 x \in \mathbb{Z}^n \Leftrightarrow$$

$$\exists y \in \mathbb{Z}^n \text{ s.t. } y = S_0 x \Leftrightarrow$$

$$\exists y \in \mathbb{Z}^n \text{ s.t. } x = S_0^{-1}y.$$

Thus,  $M_0 = \{(S_0)^{-1}y: y \in \mathbb{Z}^n\}$ , which shows that  $M_0$  is finitely generated.

But since  $\{s_1, \dots, s_n\} \subseteq S$  we have  $M \subseteq M_0$ . Hence Proposition 4.8 implies that  $M$  is finitely generated.

(b)  $m < n$

Let  $T = [t_1 t_2 \dots t_m t_{m+1} \dots t_n] \in \mathbb{Q}^{n \times n}$  be a matrix with columns  $t_1, \dots, t_n$ , where  $\{t_{m+1}, \dots, t_n\}$  is a basis of the subspace orthogonal to  $S$  and  $\{t_1, \dots, t_m\}$  is a basis of  $\mathbb{Q}^n$ .

Let  $ST$  denote the set of all vectors of the form  $sT$ , where  $s \in S$ . Since the last  $n-m$  columns of  $T$  are orthogonal to  $S$ , the last  $n-m$  components of all elements of  $ST$  are zero.

We will represent this by writing

$$ST = \left[ \underbrace{S'}_m \quad \underbrace{0}_{n-m} \right], \text{ where } S' \subseteq \mathbb{Q}^m.$$

(Actually, "post-multiplying"  $S$  by  $T$  is equivalent to doing "column operations" in  $S$  to reduce the last  $n-m$  "columns" to 0.)

Define  $N = T^{-1}M = \{T^{-1}m: m \in M\}$ . Then

$$\begin{aligned} N &= \{y \in \mathbb{Q}^n: Ty \in M\} \\ &= \{y \in \mathbb{Q}^n: STy \in \mathbb{Z}\} \\ &= \{y \in \mathbb{Q}^n: [S' \ 0]y \in \mathbb{Z}\} \\ &= \{y = (y_1, y_2): S'y_1 \in \mathbb{Z} \text{ and } y_2 \in \mathbb{Q}^{n-m}\} \\ &= \{(y_1, 0): S'y_1 \in \mathbb{Z}\} + \{(0, y_2): y_2 \in \mathbb{Q}^{n-m}\}. \end{aligned}$$

Since  $\text{rank } S' = m$ , by part (a) we conclude that  $\{(y_1, 0): S'y_1 \in \mathbb{Z}\}$  is finitely generated (say by the columns of  $\begin{bmatrix} A \\ 0 \end{bmatrix}$ , where  $A \in \mathbb{Q}^{m \times p}$ ).

Thus

$$N = \left\{ \begin{bmatrix} A \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ I \end{bmatrix} y_2 : w \in \mathbb{Z}^p, y_2 \in \mathbb{Q}^{n-m} \right\},$$

and hence

$$M = TN = \left\{ T \begin{bmatrix} A \\ 0 \end{bmatrix} w + T \begin{bmatrix} 0 \\ I \end{bmatrix} y_2 : w \in \mathbb{Z}^p, y_2 \in \mathbb{Q}^{n-m} \right\},$$

which is the desired result.  $\square$

In [2] we also give an alternative proof of Theorem 4.9, based on geometrical properties of  $\mathbb{Z}^n$ . As a by-product of that proof one can easily derive an additional characterization for the  $\mathbb{Z}$ -modules which are constrained under  $\mathbb{Z}$ -duality: these are precisely the  $\mathbb{Z}$ -modules which are *topologically closed* (this result was also pointed out to us by a referee, who suggested a shorter, more algebraically oriented proof). The approach used here, which is fundamentally based on Proposition 4.8, has, however, the advantage of being applicable to more general dualities, namely dualities in which  $X = \mathbb{Q}$  and  $D$  is any subring of  $\mathbb{Q}$ . (Observe that  $D$  contains 0 and 1, as required in Section 1.)

Note, first, that unimodular elimination is still appropriate for  $D$ -modules. That is, given a unimodular matrix  $P \in \mathbb{Z}^{m \times m}$ , we have  $\{yA : y \in D^m\} = \{y(PA) : y \in D^m\}$ , for any  $A \in \mathbb{Q}^{m \times n}$  (this follows from the fact that, for  $P$  unimodular,  $y \in D^m \Leftrightarrow yP \in D^m$ ). Therefore, an analogue of Theorem 4.4 remains true for this case; i.e., sets of the form  $S = \{yA + zB : y \in \mathbb{Q}^m, z \in D^p\}$  are constrained under  $(\mathbb{Q}, D)$ -duality whenever  $D$  is a subring of  $\mathbb{Q}$ .

The basic result used to prove the converse was Proposition 4.8. A similar result holds for modules over any subring  $D$  of the rationals. That is, submodules of finitely generated  $D$ -modules are again finitely generated (see [2] for an elementary proof). This implies that the proof of Theorem 4.9 remains valid with  $\mathbb{Z}$  replaced by any subring  $D$  of  $\mathbb{Q}$ . Hence, we have the following:

**4.10 Theorem:** Let  $D$  be a subring of  $\mathbb{Q}$ . Then  $S \subseteq \mathbb{Q}^n$  is constrained under  $(\mathbb{Q}, D)$ -duality if and only if there exist matrices  $A \in \mathbb{Q}^{m \times n}$  and  $B \in \mathbb{Q}^{p \times n}$  such that  $S = \{yA + zB : y \in \mathbb{Q}^m, z \in D^p\}$ .  $\square$



We end this section by remarking that Theorem 4.9 can be stated in an even more general context. Actually, the results concerning unimodular elimination remain true whenever  $D$  is a principal ideal domain and  $X$  is its field of quotients (see [18], for example). On the other hand, principal ideal domains are special cases of Noetherian rings, which have the property that submodules of finitely generated modules are again finitely generated (see [10], Section 18). Hence the proofs of both implications of Theorem 4.9 proceed as before, allowing us to state the following:

**4.11 Theorem:** Let  $D$  be a principal ideal domain and  $X$  its field of quotients. Then  $S \subseteq X^n$  is constrained under  $(X,D)$ -duality if and only if there exist matrices  $A \in X^{m \times n}$  and  $B \in X^{p \times n}$  such that  $S = \{yA + zB : y \in X^m, z \in D^p\}$ .  $\square$

## 5. Nonnegative integral duality in $\mathbb{Z}$

As seen in Section 2, the  $(X,D)$ -duality model is relevant to the study of the problem of deciding whether there is a vector  $y \in (D^m)^*$  such that  $yA = c$ . If  $X$  and  $D$  are chosen in such a way that  $(D^m)^* = \mathbb{Z}_+^m$ , the duality thus obtained is useful to the study of the problem of existence of *nonnegative* integral solutions to a linear system. In this section we investigate one such model: the case  $X = \mathbb{Z}$ ,  $D = \mathbb{Z}_+$  ( $(\mathbb{Z}, \mathbb{Z}_+)$ -duality or simply  $\mathbb{Z}_+$ -duality). In the next section we examine the case  $X = \mathbb{Q}$ ,  $D = \mathbb{Z}_+$ .

The problem of deciding whether a linear system has a nonnegative integral solution (i.e., the *feasibility* problem for *integer programming*) is "hard", in the sense that it is NP-complete and, hence, no polynomial algorithm is known to solve it. Our duality models will reflect this fact through the failure of the Farkas property.

First, we note that if  $X = \mathbb{Z}$  and  $D = \mathbb{Z}_+$  we have:

$$(D^m)^* = \{x \in \mathbb{Z}^m : xy \in \mathbb{Z}_+, \forall y \in \mathbb{Z}_+^m\} = \mathbb{Z}_+^m.$$

Hence, the generated sets here are the subsets of  $\mathbb{Z}^n$  which are closed under *nonnegative integral* linear combinations, which we will call  $\mathbb{Z}_+$ -modules. Given  $S \subseteq \mathbb{Z}^n$ , the  $\mathbb{Z}_+$ -

module *generated by*  $S$  is the set  $\sigma(S)$  of all nonnegative integral combinations of elements of  $S$ . *Finitely generated*  $\mathbb{Z}_+$ -modules are those of the form  $\sigma(S)$  for some finite  $S$ , i.e., sets of the form  $\{yA: y \in \mathbb{Z}_+^m\}$  where  $A \in \mathbb{Z}^{m \times n}$ .

The  $\mathbb{Z}_+$ -dual of a set  $S \subseteq \mathbb{Z}^n$  is the set  $S^*$  defined by  $S^* = \{x \in \mathbb{Z}^n: Sx \in \mathbb{Z}_+^m\}$ ; i.e.,  $S^*$  is the set of all vectors in  $\mathbb{Z}^n$  which make a *nonnegative* inner product with each element of  $S$  (observe that the integrality of the inner product is guaranteed by the fact that all sets involved are contained in  $\mathbb{Z}^n$ ). In other words,  $S^*$  is the set of all integer-valued points in the cone  $S^+ \equiv \{x \in \mathbb{Q}^n: Sx \geq 0\}$  *polar* to  $S$ . A set  $S \subseteq \mathbb{Z}^n$  is said to be (finitely) *constrained* if  $S = T^*$  for some (finite)  $T \subseteq \mathbb{Z}^n$ . Note that finitely  $\mathbb{Z}_+$ -constrained sets are of the form  $\{x \in \mathbb{Z}^n: Ax \geq 0\}$  for some  $A \in \mathbb{Z}^{m \times n}$ . Also recall that Corollary 1.9 implies that every  $\mathbb{Z}_+$ -constrained set is a  $\mathbb{Z}_+$ -module.

From the results in Section 1 it immediately follows that:

**5.1 Proposition:** Let  $S, T$  denote subsets of  $\mathbb{Z}^n$ . Then:

- (a)  $S \subseteq T \Rightarrow S^* \supseteq T^*$ .
- (b)  $(S \cup T)^* = S^* \cap T^*$ .
- (c)  $S \subseteq S^{**}$ .
- (d)  $S^* = S^{***}$ .
- (e)  $S = S^{**} \Leftrightarrow S$  is a constrained  $\mathbb{Z}_+$ -module.
- (f)  $S^{**}$  is the smallest constrained  $\mathbb{Z}_+$ -module containing  $S$ .
- (g) If  $T$  generates the  $\mathbb{Z}_+$ -module  $S$ , then  $T^* = S^*$ .
- (h) If  $S = \{yA: y \in \mathbb{Z}_+^m\}$ , for some  $A \in \mathbb{Z}^{m \times n}$ , then

$$S^* = \{x \in \mathbb{Z}^n: Ax \in \mathbb{Z}_+^m\}.$$

In the definition of the  $\mathbb{Z}_+$ -dual of a set  $S \subseteq \mathbb{Z}^n$  we observed that  $S^*$  is the set of all integer-valued points in the cone  $S^+$  polar to  $S$ . The next proposition emphasizes the relationship between constrained sets under  $\mathbb{Z}_+$ -duality and constrained cones.

**5.2 Proposition:** Let  $S \subseteq \mathbb{Z}^n$ .  $S$  is (finitely) constrained under  $\mathbb{Z}_+$ -duality if and only if  $S = \mathbb{Z}^n \cap K$  for some (finitely) constrained cone  $K \subseteq \mathbb{Q}^n$ .

**Proof:** First, suppose that  $S = T^*$  is a  $\mathbb{Z}_+$ -constrained set. Then  $S = \{x \in \mathbb{Z}^n: Tx \geq 0\} = T^+ \cap \mathbb{Z}^n$ , where  $T^+ = \{x \in \mathbb{Q}^n: Tx \geq 0\}$  is a constrained cone. Also, it is clear that if  $S$  is finitely constrained (i.e.,  $|T|$  is finite), then  $T^+$  is also finitely constrained.

Now, assume that  $S = \mathbb{Z}^n \cap K$ , where  $K = \{x \in \mathbb{Q}^n: Tx \geq 0\}$  for some  $T \subseteq \mathbb{Q}^n$ . By properly scaling each element of  $T$  we can assume, without loss of generality, that  $T \subseteq \mathbb{Z}^n$ . Thus  $S = \{x \in \mathbb{Z}^n: Tx \geq 0\}$ , which shows that  $S$  is  $\mathbb{Z}_+$ -constrained. If, in particular,  $K$  is finitely constrained, it is clear that  $S$  is also finitely constrained.  $\square$

The previous proposition related constrained  $\mathbb{Z}_+$ -modules to constrained cones. Now, we establish the relationship between  $\mathbb{Z}_+$ -modules and cones from the point of view of generation.

**5.3 Proposition:** Let  $K \subseteq \mathbb{Q}^n$  be a cone, let  $S = K \cap \mathbb{Z}^n$  and let  $T \subseteq \mathbb{Z}^n$ . If the  $\mathbb{Z}_+$ -module generated by  $T$  is  $S$ , then the cone generated by  $T$  is  $K$ .

**Proof:** Since  $T \subseteq K$  it suffices to show that every  $x \in K$  can be expressed as a nonnegative combination of elements of  $T$ . So, let  $x \in K$ . Since  $K \subseteq \mathbb{Q}^n$ , there is some  $\lambda > 0$  such that  $\lambda x \in \mathbb{Z}^n$ , which implies  $\lambda x \in S$ . But  $T$  generates  $S$  as a  $\mathbb{Z}_+$ -module. Thus,  $\lambda x$  can be expressed as a nonnegative integral combination of elements of  $T$ . This implies that  $x$  is a nonnegative combination of elements of  $T$ , as desired.  $\square$

We can now improve the characterization of the  $\mathbb{Z}_+$ -constrained sets given in Proposition 5.2.

**5.4 Proposition:** Let  $S \subseteq \mathbb{Z}^n$  and let  $K \subseteq \mathbb{Q}^n$  be the cone generated by  $S$ . Then  $S$  is a (finitely) constrained  $\mathbb{Z}_+$ -module if and only if (i)  $K$  is a (finitely) constrained cone and (ii)  $S = K \cap \mathbb{Z}^n$ .

**Proof:** By Proposition 5.2, (i) and (ii) immediately imply that  $S$  is  $\mathbb{Z}_+$ -constrained (finitely constrained if  $K$  is).

Now, suppose that  $S$  is (finitely)  $\mathbb{Z}_+$ -constrained. By Proposition 5.2,  $S = K' \cap \mathbb{Z}^n$  for some (finitely) constrained cone  $K'$ . But by Proposition 5.3,  $K'$  is the cone generated by  $S$ , which shows that  $K = K'$  and hence that (i) and (ii) hold.  $\square$

Propositions 5.3 and 5.4 together with the theorems of Weyl and Minkowski for cones, imply an immediate relationship between finitely generated and finitely constrained  $\mathbb{Z}_+$ -modules. If a  $\mathbb{Z}_+$ -module  $S$  is constrained but *not* finitely constrained, then it *cannot* be finitely generated (since this would imply that the cone generated by  $S$  is finitely generated and hence finitely constrained, by Weyl's theorem). Another way of stating the same property is that if a finitely generated  $\mathbb{Z}_+$ -module *is* constrained, then it must be *finitely* constrained.

We are now ready to discuss the validity of the Farkas, Weyl, Minkowski, Lehman and Fulkerson properties for  $\mathbb{Z}_+$ -duality. Recall that, by Theorem 2.8, the Farkas property holds for  $\mathbb{Z}_+$ -duality if and only if every finitely generated  $\mathbb{Z}_+$ -module is constrained. Let  $S = \{yA: y \in \mathbb{Z}_+^m\}$ , where  $A \in \mathbb{Z}^{m \times n}$ , be a finitely generated  $\mathbb{Z}_+$ -module. It is easily shown that the cone generated by  $S$  is  $K = \{yA: y \in \mathbb{Q}_+^m\}$ , which is constrained by the Farkas property for cones. Hence, Proposition 5.4 implies that  $S$  is a constrained  $\mathbb{Z}_+$ -module if and only if  $S$  contains all integral points in  $K$ . This is not always the case. For example, let  $S = \{2y_1 + 3y_2: y_1, y_2 \in \mathbb{Z}_+\} \subseteq \mathbb{Z}^1$ . The cone generated by  $S$  is  $K = \mathbb{Q}_+$ . It is easy to see that  $1 \notin S$ ; thus,  $S$  does not contain all integral points in  $K$ .

Therefore, the Farkas property (and hence the Weyl property) *fails* for  $\mathbb{Z}_+$ -duality. That is, given  $A \in \mathbb{Z}^{m \times n}$  and  $c \in \mathbb{Z}^n$ , it is *not necessarily true* that exactly one of the following alternatives holds:

- (i)  $\exists y \in \mathbb{Z}_+^m$  such that  $yA = c$ ;
- (ii)  $\exists x \in \mathbb{Z}^n$  such that  $Ax \geq 0$  and  $cx < 0$ .

The Minkowski property, on the other hand, does hold for  $\mathbb{Z}_+$ -duality (see Trotter [17]). It is an immediate consequence of the following result usually attributed to Hilbert [12] (see, e.g., Giles and Pulleyblank [9]). See Schrijver [15, p. 232] for a brief discussion on the origin of this result.

**5.5 Theorem** (Hilbert's finite basis theorem): Let  $K = \{x \in \mathbb{Q}^n: Ax \geq 0\}$ , where  $A \in \mathbb{Q}^{m \times n}$ , be a finitely constrained cone. Let  $S = K \cap \mathbb{Z}^n$ . Then there is a finite set  $H \subseteq S$  such that  $H$  generates  $S$  as a  $\mathbb{Z}_+$ -module.  $\square$

**5.6 Corollary:** The Minkowski property (and hence the Lehman and Fulkerson properties) holds for  $\mathbb{Z}_+$ -duality.  $\square$

Motivated by Theorem 5.5, we say that a set  $H = \{h_1, \dots, h_k\} \subseteq \mathbb{Z}^n$  is a *Hilbert basis* if every integral vector in the cone generated by  $H$  can be expressed as a nonnegative integral combination of  $h_1, \dots, h_k$ . Observe that Proposition 5.4 immediately implies the following result.

**5.7 Proposition:** A finitely generated  $\mathbb{Z}_+$ -module  $S = \{yA: y \in \mathbb{Z}_+^m\}$  is constrained if and only if the set of row-vectors of  $A$  constitutes a Hilbert basis (in this case, we say that  $A$  is a *Hilbert matrix*).  $\square$

A rational linear system  $\{Ax \leq b\}$  is *totally dual integral* (see Edmonds and Giles [6]) provided the linear programming problem  $\{\min yb: yA = c, y \geq 0\}$  has an integral optimal solution for each  $c$  for which an optimal solution exists. Thus the Hilbert basis

requirement of Proposition 5.7 is equivalent to the requirement that the system  $\{Ax \leq 0\}$  be totally dual integral (see also Cook [4]). That is, homogeneous totally dual integral systems with integral coefficients correspond precisely to Hilbert bases which, in turn, correspond (by 5.7) precisely to those finitely generated  $\mathbb{Z}_+$ -modules for which the Weyl property holds.

We remark further that the failure of the Farkas property for  $\mathbb{Z}_+$ -duality can be seen as another indication that integer programming is "hard". By the same token, we may also expect that finitely generated  $\mathbb{Z}_+$ -modules which are also constrained (and in this case they must be finitely constrained, as pointed out earlier) should be associated with instances of integer programming problems which are "easy" to solve. The following theorem of Chandrasekaran [3] shows that this is indeed the case.

#### 5.8 Theorem (Chandrasekaran [3]):

There is a polynomial time algorithm that, given a linear system  $yA = c$ , where  $A \in \mathbb{Z}^{m \times n}$  and  $c$  is an integral vector in the cone generated by the rows of  $A$ , either finds a nonnegative integral solution  $y$  or shows that  $A$  is not a Hilbert matrix (i.e., that the  $\mathbb{Z}_+$ -module generated by the rows of  $A$  is not constrained).  $\square$

It follows that Theorem 5.8 can be used along with any polynomial-time algorithm for linear programming (see [15]) in order to validate (in polynomial-time) which alternative of the Farkas property holds, given a Hilbert matrix  $A$  and an integral vector  $c$ .

### 6. Nonnegative integral duality in $\mathbb{Q}$

We now investigate another  $\mathbb{Z}_+$ -duality model, which differs from  $(\mathbb{Z}, \mathbb{Z}_+)$ -duality only in that  $X$  is taken as  $\mathbb{Q}$  instead of  $\mathbb{Z}$ . This means that we are allowing rational data, instead of all-integral data. In this section we compare the properties of these two dualities.

We start by observing that for  $X = \mathbb{Q}$  and  $D = \mathbb{Z}_+$  we still have

$$(D^m)^* = \{x \in \mathbb{Q}^n: xy \in \mathbb{Z}_+, \forall y \in \mathbb{Z}_+^m\} = \mathbb{Z}_+^m.$$

Hence, as in  $(\mathbb{Z}, \mathbb{Z}_+)$ -duality, the generated sets for  $(\mathbb{Q}, \mathbb{Z}_+)$ -duality are those sets (now contained in  $\mathbb{Q}^n$ ) which are closed under nonnegative integral combinations (we will continue to call such sets  $\mathbb{Z}_+$ -modules).

Recall that, under  $(\mathbb{Z}, \mathbb{Z}_+)$ -duality, the dual  $S^*$  of a set  $S \subseteq \mathbb{Z}^n$  is given by  $S^* = S^+ \cap \mathbb{Z}^n$ , where  $S^+$  is the cone polar to  $S$ . Under  $(\mathbb{Q}, \mathbb{Z}_+)$ -duality, the dual of a set  $S \subseteq \mathbb{Q}^n$  is given by

$$\begin{aligned} S^* &= \{x \in \mathbb{Q}^n: Sx \in \mathbb{Z}_+\} \\ &= \{x \in \mathbb{Q}^n: Sx \in \mathbb{Z}\} \cap \{x \in \mathbb{Q}^n: Sx \geq 0\} \\ &= S^\# \cap S^+, \end{aligned}$$

where  $S^\#$  and  $S^+$  denote the duals of  $S$  under  $(\mathbb{Q}, \mathbb{Z})$ - and  $(\mathbb{Q}, \mathbb{Q}_+)$ -duality, respectively.

Therefore, every constrained set under  $(\mathbb{Q}, \mathbb{Z}_+)$ -duality is the intersection of a constrained  $\mathbb{Z}$ -module and a constrained cone. A natural question to ask at this point is whether the converse is true; i.e., is the intersection of a constrained  $\mathbb{Z}$ -module  $M^\#$  and a constrained cone  $K^+$  always constrained under  $(\mathbb{Q}, \mathbb{Z}_+)$ -duality? The answer is *no*. For example, let  $M^\# = \mathbb{Z}$  and  $K^+ = \mathbb{Q}$ . Then  $M^\# \cap K^+ = \mathbb{Z}$  is not  $(\mathbb{Q}, \mathbb{Z}_+)$ -constrained, since  $\mathbb{Z}^* = \{x \in \mathbb{Q}: xy \in \mathbb{Z}_+, \forall y \in \mathbb{Z}\} = \{0\}$  and hence  $\mathbb{Z}^{**} = \{0\}^* = \mathbb{Q} \neq \mathbb{Z}$ .

To obtain conditions under which  $M^\# \cap K^+$  is  $(\mathbb{Q}, \mathbb{Z}_+)$ -constrained we first establish some notation. For any  $T \subseteq \mathbb{Q}^n$  we denote by  $\mathcal{S}(T)$ ,  $\mathcal{K}(T)$  and  $\mathcal{M}(T)$ , respectively, the subspace, the cone and the  $\mathbb{Z}$ -module generated by  $T$ . If  $T$  is closed under addition we denote by  $s(T)$  the *lineality space* of  $T$ , i.e., the set of all  $x \in \mathbb{Q}^n$  such that  $\lambda x \in T$  for every  $\lambda \in \mathbb{Q}$ . It is easy to see that for every  $T \subseteq \mathbb{Q}^n$  we have  $s(T^+) = s(T^\#) = s(T^*) = T^\perp$ , where  $T^\perp$  is the subspace orthogonal to  $T$ .

**6.1 Lemma:** Let  $M \subseteq \mathbb{Q}^n$  be a  $\mathbb{Z}$ -module and let  $K \subseteq \mathbb{Q}^n$  be a cone. Assume that  $M$  and  $K$  generate the same subspace  $S$ . Then

a)  $\mathcal{K}(M \cap K) = K;$

b)  $\mathcal{M}(M \cap K) = M.$

**Proof:** (a) Clearly,  $\mathcal{K}(M \cap K) \subseteq K$ . To prove the reverse inclusion, let  $u \in K$ . Since  $\mathcal{S}(K) = \mathcal{S}(M)$ , we have  $u \in \mathcal{S}(M)$ . Hence, there are elements  $m_1, \dots, m_k \in M$  and rationals  $\lambda_1, \dots, \lambda_k$  such that  $u = \sum_{i=1}^k \lambda_i m_i$ . Let  $\lambda$  be a positive integer such that  $\lambda \lambda_i \in \mathbb{Z}$ ,  $i = 1, \dots, k$ . Then  $\lambda u \in M \cap K$ , which implies that  $u \in \mathcal{K}(M \cap K)$ . Thus,  $K \subseteq \mathcal{K}(M \cap K)$ .

(b) Clearly,  $\mathcal{M}(M \cap K) \subseteq M$ . Now, select elements  $m_1, \dots, m_k \in M$  such that they form a basis for the subspace  $S$  generated by  $M$ . Let  $T = \{x \in M: x = \sum_{i=1}^k \alpha_i m_i, \text{ with } 0 \leq \alpha_i \leq 1\}$ . Each element  $m \in M$  can be expressed as

$$m = \sum_{i=1}^k \alpha_i m_i = \sum_{i=1}^k \lfloor \alpha_i \rfloor m_i + \left[ \sum_{i=1}^k (\alpha_i - \lfloor \alpha_i \rfloor) m_i \right],$$

which is an integral combination of the elements  $m_1, \dots, m_k, (\sum_{i=1}^k (\alpha_i - \lfloor \alpha_i \rfloor) m_i)$ , all of which are in  $T$ . Thus,  $T$  generates  $M$ . Also,  $T$  is *bounded*, since for any  $x \in T$  we have

$$\|x\| = \left\| \sum_{i=1}^k \alpha_i m_i \right\| \leq \sum_{i=1}^k \|m_i\| = \delta.$$

Choose any relatively interior element  $u$  of  $K$ . As in part (a), there is a positive integer  $\lambda_1$  such that  $\lambda_1 u \in M \cap K$ . Since  $\lambda_1 u$  is again in the relative interior of  $K$ , we can select a positive integer  $\lambda_2$  such that the intersection of the sphere of center  $m_0 = \lambda_2 \lambda_1 u$  and radius  $\delta$  with  $S$  is contained in  $K$ . Since  $T \subseteq S$  and  $\|t\| \leq \delta$  for all  $t \in T$ , this implies that  $m_0 + T = \{m_0 + t: t \in T\} \subseteq K$ . In fact,  $m_0 + T \subseteq M \cap K$ , since  $m_0 \in M$  and  $T \subseteq M$ . Finally notice that each  $t \in T$  is the difference  $(m_0 + t) - m_0$  of two elements of  $m_0 + T$ . Hence  $\mathcal{M}(m_0 + T) \supseteq T$  and we have



$$\mathfrak{M}(M \cap K) \supseteq \mathfrak{M}(m_0 + T) \supseteq \mathfrak{M}(T) = M. \quad \square$$

Using Lemma 6.1 we can now establish necessary and sufficient conditions under which the  $\mathbb{Z}$ -module  $M$  and the cone  $K$  satisfy  $(M \cap K)^* = M^\# \cap K^+$ .

**6.2 Theorem:** Let  $M \subseteq \mathbb{Q}^n$  and  $K \subseteq \mathbb{Q}^n$  denote a  $\mathbb{Z}$ -module and a cone, respectively. Then  $(M \cap K)^* = M^\# \cap K^+$  if and only if  $\mathfrak{S}(M) = \mathfrak{S}(K)$ .

**Proof:** First, assume that  $M$  and  $K$  satisfy  $\mathfrak{S}(M) = \mathfrak{S}(K)$ . By Lemma 6.1 we have  $\mathfrak{K}(M \cap K) = K$  and  $\mathfrak{M}(M \cap K) = M$ . Therefore,

$$\begin{aligned} (M \cap K)^* &= (M \cap K)^\# \cap (M \cap K)^+ = (\mathfrak{M}(M \cap K))^\# \cap (\mathfrak{K}(M \cap K))^+ \\ &= M^\# \cap K^+. \end{aligned}$$

Now, suppose that  $(M \cap K)^* = M^\# \cap K^+$ . By considering the lineality spaces of the sets on each side of this equality we have:

$$\begin{aligned} s(M^\#) \cap s(K^+) &= s((M \cap K)^*), \text{ which is equivalent to} \\ M^\perp \cap K^\perp &= (M \cap K)^\perp. \end{aligned}$$

Using the fact that for any sets  $T_1, T_2 \subseteq \mathbb{Q}^n$  we have  $T_1^\perp \cap T_2^\perp = (\mathfrak{S}(T_1) + \mathfrak{S}(T_2))^\perp$ , this last equality implies that  $\mathfrak{S}(M) + \mathfrak{S}(K) = \mathfrak{S}(M \cap K)$ , which shows that  $\mathfrak{S}(M) = \mathfrak{S}(K) = \mathfrak{S}(M \cap K)$ .  $\square$

We can now obtain characterizations for constrained sets under  $(\mathbb{Q}, \mathbb{Z}_+)$ -duality.

**6.3 Proposition:** Let  $T \subseteq \mathbb{Q}^n$ . Then  $T$  is  $(\mathbb{Q}, \mathbb{Z}_+)$ -constrained if and only if there exist a constrained  $\mathbb{Z}$ -module  $M$  and a constrained cone  $K$  such that (i)  $s(M) = s(K)$  and (ii)  $T = M \cap K$ .

**Proof:** ( $\Rightarrow$ ) Assume  $T = U^*$  for some  $U \subseteq \mathbb{Q}^n$ . Let  $M = U^\#$  and  $K = U^+$ . Clearly,  $T = U^\# \cap U^+ = M \cap K$ . So, (ii) holds. Also, we have  $s(M) = s(U^\#) = U^\perp$  and  $s(K) = s(U^+) = U^\perp$ . Thus, (i) also holds.

( $\Leftarrow$ ) Assume  $T = M \cap K$ , where  $M \subseteq \mathbb{Q}^n$  is a  $\mathbb{Z}$ -module and  $K \subseteq \mathbb{Q}^n$  is a cone. Suppose that  $s(M) = s(K)$ . Since  $M = M^{\#\#}$  and  $K = K^{++}$ , this last condition implies that  $(M^\#)^\perp = (K^+)^\perp$ , which in turn implies that  $\mathcal{S}(M^\#) = \mathcal{S}(K^+)$ . Now, by Proposition 6.2, we have

$$(M^\# \cap K^+)^* = (M^\#)^\# \cap (K^+)^+ = M \cap K = T.$$

Hence  $T$  is  $(\mathbb{Q}, \mathbb{Z}_+)$ -constrained.  $\square$

The next proposition improves the characterization given in Proposition 6.3 by showing that  $M$  and  $K$  can be chosen as the  $\mathbb{Z}$ -module and the cone generated by  $T$ , respectively.

**6.4 Proposition:** Let  $T \subseteq \mathbb{Q}^n$ . Then  $T$  is  $(\mathbb{Q}, \mathbb{Z}_+)$ -constrained if and only if (i)  $\mathcal{M}(T)$  is a constrained  $\mathbb{Z}$ -module, (ii)  $\mathcal{K}(T)$  is a constrained cone, (iii)  $s(\mathcal{M}(T)) = s(\mathcal{K}(T))$  and (iv)  $T = \mathcal{M}(T) \cap \mathcal{K}(T)$ .

**Proof:** The "if" part is immediately implied by Proposition 6.3.

Now, suppose that  $T$  is  $(\mathbb{Q}, \mathbb{Z}_+)$ -constrained. By Proposition 6.3, there exist a constrained  $\mathbb{Z}$ -module  $M$  and a constrained cone  $K$  satisfying  $T = M \cap K$  and  $s(M) = s(K)$ . Define  $M' = M \cap \mathcal{S}(T)$  and  $K' = K \cap \mathcal{S}(T)$ . We have

$$M' \cap K' = M \cap \mathcal{S}(T) \cap K \cap \mathcal{S}(T) = T \cap \mathcal{S}(T) = T.$$

Also, since  $M$  and  $K$  have the same lineality space, we have  $s(M') = s(K')$ . Moreover,  $M'$  and  $K'$  are a constrained  $\mathbb{Z}$ -module and a constrained cone, respectively, since  $M'$  is the intersection of two constrained  $\mathbb{Z}$ -modules and  $K'$  is the intersection of two

constrained cones (recall that subspaces are special cases of constrained cones and  $\mathbb{Z}$ -modules).

Hence, if we show that  $M' = \mathfrak{M}(T)$  and  $K' = \mathfrak{K}(T)$  the proof will be complete. It is clear that  $\mathfrak{S}(M') \subseteq \mathfrak{S}(T)$  and  $\mathfrak{S}(K') \subseteq \mathfrak{S}(T)$ . Since  $T = M' \cap K'$  we also have  $\mathfrak{S}(T) \subseteq \mathfrak{S}(M')$  and  $\mathfrak{S}(T) \subseteq \mathfrak{S}(K')$ . Therefore  $\mathfrak{S}(M') = \mathfrak{S}(K') = \mathfrak{S}(T)$  and Lemma 6.1 implies

$$\mathfrak{M}(T) = \mathfrak{M}(M' \cap K') = M'$$

and

$$\mathfrak{K}(T) = \mathfrak{K}(M' \cap K') = K'. \quad \square$$

Applying Proposition 6.4 to the special case where  $S$  is a finitely generated  $\mathbb{Z}_+$ -module yields the following (a *pointed* cone has lineality  $\{0\}$ ):

**6.5 Proposition:** Let  $S = \{yA: y \in \mathbb{Z}_+^m\}$ , where  $A \in \mathbb{Q}^{m \times n}$ , be a finitely generated  $\mathbb{Z}_+$ -module. Then  $S$  is finitely constrained if and only if:

- (i) the cone  $\{yA: y \in \mathbb{Q}_+^m\}$  is pointed;
- (ii)  $S = \{yA: y \in \mathbb{Q}_+^m\} \cap \{yA: y \in \mathbb{Z}^m\}$  (i.e.,  $S$  contains all points in the

intersection of the  $\mathbb{Z}$ -module and the cone generated by the rows of  $A$ ).

**Proof:** This follows from Proposition 6.4 by observing that  $\mathfrak{M}(S) = \{yA: y \in \mathbb{Z}^m\}$  is a finitely generated  $\mathbb{Z}$ -module and hence has lineality space  $\{0\}$ . Thus, the condition  $s(\mathfrak{M}(S)) = s(\mathfrak{K}(S))$  holds if and only if  $\mathfrak{K}(S)$  is pointed.  $\square$

Proposition 6.5 establishes the conditions under which a finitely generated  $\mathbb{Z}_+$ -module  $S = \{yA: y \in \mathbb{Z}_+^m\}$ , where  $A \in \mathbb{Q}^{m \times n}$ , is constrained. First of all, the cone  $\mathfrak{K}(S) = \{yA: y \geq 0\}$  must be pointed. Also  $S$  must contain all points in  $\{yA: y \geq 0\} \cap \{yA: y \in \mathbb{Z}^m\}$ . Neither of these properties holds in general, as the following examples show:

- a) Let  $S = \mathbf{Z} = \{1y_1 + (-1)y_2: y_1, y_2 \in \mathbf{Z}_+\}$ . Then  $S$  is a finitely generated  $\mathbf{Z}_+$ -module. Since  $\mathcal{K}(S) = \mathbf{Q}$  is not pointed,  $S$  cannot be constrained.
- b) Let  $S = \{2y_1 + 3y_2: y_1, y_2 \in \mathbf{Z}_+\} = \{0, 2, 3, 4, \dots\}$ . In this case  $\mathcal{K}(S) = \mathbf{Q}_+$  is a pointed cone. But  $\mathcal{K}(S) \cap \mathcal{M}(S) = \mathbf{Q}_+ \cap \mathbf{Z} = \mathbf{Z}_+ \neq S$ . Thus,  $S$  is not constrained.

Therefore, the Farkas and Weyl properties fail again for  $(\mathbf{Q}, \mathbf{Z}_+)$ -duality, as they did for  $(\mathbf{Z}, \mathbf{Z}_+)$ -duality. In contrast to  $(\mathbf{Z}, \mathbf{Z}_+)$ -duality, however,  $(\mathbf{Q}, \mathbf{Z}_+)$ -duality *does not have* the Minkowski and Fulkerson properties. To see this, note that for  $S = \{0\}$  we have  $S^* = \mathbf{Q}$ , and here  $S$  is finitely constrained since  $S = \{x: x \in \mathbf{Z}_+, -x \in \mathbf{Z}_+\}$ , while  $S^*$  is *not* finitely generated (over  $\mathbf{Z}_+$ ). Thus the Fulkerson property, and hence also the Minkowski property, fails for  $(\mathbf{Q}, \mathbf{Z}_+)$ -duality. The Lehman property is valid for  $(\mathbf{Q}, \mathbf{Z}_+)$ -duality, as we discuss following Theorem 6.6 below. As done for  $\mathbf{Z}$ -duality, we now describe any finitely constrained  $\mathbf{Z}_+$ -module as the sum of a subspace and a finitely generated  $\mathbf{Z}_+$ -module.

**6.6 Theorem:** Let  $A \in \mathbf{Q}^{m \times n}$  and let  $T = \{x \in \mathbf{Q}^n: Ax \in \mathbf{Z}_+^m\}$  be a finitely constrained  $\mathbf{Z}_+$ -module. Then there exist integers  $r$  and  $h$  and matrices  $C \in \mathbf{Q}^{r \times n}$  and  $H \in \mathbf{Q}^{h \times n}$  such that  $T = \{yC + wH: y \in \mathbf{Q}^r, w \in \mathbf{Z}_+^h\}$  (i.e.,  $T$  is the sum of a subspace and a finitely generated  $\mathbf{Z}_+$ -module).

**Proof:** By Corollary 4.5(c) there exist matrices  $C \in \mathbf{Q}^{r \times n}$  and  $D \in \mathbf{Q}^{s \times n}$  such that  $\{x: Ax \in \mathbf{Z}^m\} = \{yC + zD: y \in \mathbf{Q}^r, z \in \mathbf{Z}^s\}$ . Thus, we have

$$T = \{yC + zD: y \in \mathbf{Q}^r, z \in \mathbf{Z}^s, (yC + zD)A^t \in \mathbf{Z}_+^m\}.$$

Since  $CA^t = 0$  (because the subspace  $\{yC: y \in \mathbf{Q}^r\}$  must be orthogonal to each row of  $A$ ), we have

$$T = \{yC + zD: y \in \mathbf{Q}^r, z \in \mathbf{Z}^s, z(DA^t) \in \mathbf{Z}_+^m\}$$

$$\begin{aligned}
&= \{yC: y \in \mathbb{Q}^r\} + \{zD: z \in \mathbb{Z}^s, z(DA^t) \in \mathbb{Z}_+^m\} \\
&= \{yC: y \in \mathbb{Q}^r\} + \{zD: z \in \mathbb{Z}^s, z(DA^t) \geq 0\},
\end{aligned}$$

this last step being justified by the fact that  $(DA^t)$  must be an integral matrix, since the rows of  $D$  are in the  $(\mathbb{Q}, \mathbb{Z})$ -dual of the  $\mathbb{Z}$ -module generated by the rows of  $A$ .

By observing that  $\{z \in \mathbb{Z}^s: z(DA^t) \geq 0\}$  is the set of integral points in a cone, we conclude, by Theorem 5.5, that there exists a matrix  $H' \in \mathbb{Z}^{h \times s}$  such that  $\{z \in \mathbb{Z}^s: z(DA^t) \geq 0\} = \{wH': w \in \mathbb{Z}_+^h\}$ . Hence, defining  $H = H'D$ , we obtain

$$\begin{aligned}
T &= \{yC: y \in \mathbb{Q}^r\} + \{w(H'D): w \in \mathbb{Z}_+^h\} \\
&= \{yC: y \in \mathbb{Q}^r\} + \{wH: w \in \mathbb{Z}_+^h\},
\end{aligned}$$

which is the desired result.  $\square$

Suppose  $T = \{x \in \mathbb{Q}^n: Ax \in \mathbb{Z}_+^m\}$ , where  $A \in \mathbb{Q}^{m \times n}$ . Then by Theorem 6.6 we have that  $T = \{yC + wH: y \in \mathbb{Q}^r, w \in \mathbb{Z}_+^h\}$  for matrices  $C \in \mathbb{Q}^{r \times n}$ ,  $H \in \mathbb{Q}^{h \times n}$ . As in

Proposition 4.2 (the corresponding result holds in the present setting), we then have that  $T^* = \{x \in \mathbb{Q}^n: Cx = 0, Hx \in \mathbb{Z}_+^h\}$ . Hence  $T^* = \{x \in \mathbb{Q}^n: Cx \in \mathbb{Z}_+^r, -Cx \in \mathbb{Z}_+^r, Hx \in \mathbb{Z}_+^h\}$ , and so  $T^*$  is finitely constrained. Thus the Lehman property holds for  $(\mathbb{Q}, \mathbb{Z}_+)$ -

duality. More generally, if  $T = \{x \in \mathbb{Q}^n: Bx = 0, Ax \in \mathbb{Z}_+^m\}$  with  $B \in \mathbb{Q}^{p \times n}$ , then we have  $T = \{x \in \mathbb{Q}^n: Bx \in \mathbb{Z}_+^p, -Bx \in \mathbb{Z}_+^p, Ax \in \mathbb{Z}_+^m\}$ , so we may still express  $T = \{yC + wH: y \in \mathbb{Q}^r, w \in \mathbb{Z}_+^h\}$  and  $T^* = \{x \in \mathbb{Q}^n: Cx = 0, Hx \in \mathbb{Z}_+^h\}$ . Applying Theorem 6.6

now to  $T^*$  shows that  $T^*$  itself may be represented as the sum of a subspace and a finitely generated  $\mathbb{Z}_+$ -module. Hence explicit consideration of orthogonality restrictions and lineality spaces leads to valid analogues of the Minkowski and Fulkerson properties, even though in the strict sense these properties fail for  $(\mathbb{Q}, \mathbb{Z}_+)$ -duality.

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