PARAMETER ESTIMATION: THE PROPER WAY TO USE BAYESIAN POSTERIOR PROCESSES WITH BROWNIAN NOISE

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ABSTRACT. This paper studies a problem of Bayesian parameter estimation for a sequence of scaled counting processes whose weak limit is a Brownian motion with an unknown drift. The main result of the paper is that the limit of the posterior distribution processes is, in general, not equal to the posterior distribution process of the mentioned Brownian motion with the unknown drift. Instead, it is equal to the posterior distribution process associated with a Brownian motion with the same unknown drift and a different standard deviation coefficient. The difference between the two standard deviation coefficients can be arbitrarily large. The characterization of the limit of the posterior distribution processes is then applied to a family of stopping time problems. We show that the proper way to find asymptotically optimal solutions to stopping time problems w.r.t. the scaled counting processes is by looking at the limit of the posterior distribution processes rather than by the naive approach of looking at the limit of the scaled counting processes themselves. The difference between the performances can be arbitrarily large.

1. INTRODUCTION

Brownian¹ motion is a fundamental process in modeling various stochastic phenomena. It has practical applications in various fields, such as mathematical finance, physics, queueing networks, and signal processing. Brownian motion is the continuous-time analogue of random walks and it can be obtained as the weak limit of discrete processes.

In this paper we study the relation between a Brownian motion with an unknown drift and a sequence of scaled counting processes in continuous time, which we term as 'discrete processes'. We assume that there exists a random variable θ with a known prior distribution, and a sequence of discrete processes $\{(\tilde{L}^n_{\theta}(t))\}_{n\in\mathbb{N}}$ that converges in distribution to $\tilde{L}(t) = \tilde{L}_{\theta}(t) := \theta t + \sigma W(t)$, where (W(t)) is a standard Brownian motion independent of the drift θ . The decision maker (DM) does not observe the random variable θ , but rather observes continuously $\tilde{L}^n := \tilde{L}^n_{\theta}$. Therefore, for sufficiently large $n \in \mathbb{N}$, the observed process is approximately distributed as a Brownian motion with an unknown drift. For every n, define $\tilde{\pi}^n$ (resp. $\tilde{\pi}$) to be the (Bayesian) posterior distribution process of θ given the observations from \tilde{L}^n (resp. \tilde{L}).

In many optimal control/stopping time problems such as the Bayesian sequential testing problem in its different versions and the Bayesian Brownian bandit problem (see the literature review below) it is possible to formulate both the problem and the solution

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¹This paper is an extended version of the paper with the same title that appears on *Mathematics of Operations Research*. The only difference is that in this version we allow the case that the system was activated before time t = 0.

by using the posterior distribution process. Because in these models the posterior distribution process is of interest, the naive approach of using results taken from optimal stopping problems w.r.t. the posterior distribution process $\tilde{\pi}$, such as the structure of the optimal strategy, and implementing them in optimal stopping problems concerning the process \tilde{L}^n (for sufficiently large n) is not relevant; the right approach should be to find the limit of the posterior distribution processes $\tilde{\pi}^n$ instead of the posterior distribution process of the limit process $\tilde{L} = \lim_{n \to \infty} \tilde{L}^n$. To illustrate this point, in Remark 3.2 below we show that $\tilde{L}(t)$, the value of the process $(\tilde{L}(s))_{0 \le s \le t}$ at time t, is a sufficient statistic for the posterior distribution process $\tilde{\pi}$ at time t. That is, $\tilde{\pi}$ is independent of past observations from \tilde{L} , given the present value of \tilde{L} . However, it appears that, usually, $\tilde{\pi}^n$ depends not only on the present value of \tilde{L}^n , but also on past observations from \tilde{L}^n . Therefore, it uses 'more information' than $\tilde{\pi}$ does and it is 'more accurate'. We show below that this is indeed the case.

1.1. Main Results. The paper's main results are: (1) characterizing the limit of the posterior processes, $\lim_{n\to\infty} \tilde{\pi}^n$, and (2) using this characterization in order to find asymptotically optimal solutions for Bayesian stopping time problems. It might happen that $\lim_{n\to\infty} \tilde{\pi}^n$ is trivial. This case arises, e.g., when the value of θ is detected in an infinitesimal time interval or when the limit is a constant. Under mild assumptions, we find an explicit expression for the limit of the posterior distribution processes, $\lim_{n\to\infty} \tilde{\pi}^n$, and show that in general $\lim_{n\to\infty} \tilde{\pi}^n \neq \tilde{\pi}$. Although, the limit $\lim_{n\to\infty} \tilde{\pi}^n$ has a different distribution than the posterior distribution process $\tilde{\pi}$, we prove that this limit can be expressed as the posterior distribution process of a different Brownian motion with an unknown drift that is given by

(1.1)
$$\hat{M}(t) = \hat{M}_{\theta}(t) := \theta t + \sigma' W'(t), \ t \in [0, \infty),$$

where (W'(t)) is a Brownian motion independent of θ and $0 < \sigma' \leq \sigma$. The quantity σ' depends on the structure of the processes $\{\tilde{L}^n\}_{n\in\mathbb{N}}$. Since $\sigma' \leq \sigma$, the paths of the process $(\hat{M}(t))$ will be more concentrated around the path of the linear drift (θt) than the paths of the process $(\tilde{L}(t))$. In other words, $(\hat{M}(t))$ is less noisy than $(\tilde{L}(t))$. Therefore, it is easier to estimate the parameter θ given $(\hat{M}(t))$ than given $(\tilde{L}(t))$; that is, $\lim_{n\to\infty} \tilde{\pi}^n$ is more informative than $\tilde{\pi}$.

In addition, we identify when the equality $\sigma' = \sigma$ holds. We show that it happens if and only if the processes $\{(\tilde{L}_l^n(t))\}_{l\in S, n\in\mathbb{N}}$ satisfy a memorylessness property and no information, regarding the posterior distribution processes, is lost by looking at the present values of the $\{(\tilde{L}_l^n(t))\}_{l\in S, n\in\mathbb{N}}$ rather than at their past and present values (e.g., Poisson processes with unknown rates that depend on θ and n). This is the same property that holds in the Brownian motion with an unknown drift model. We also show that the difference between the parameters σ' and σ can be arbitrarily large.

Our study thus strengthens the motivation for analyzing the posterior distribution process of a Brownian motion with an unknown drift. Moreover, the fact that the structure of $\lim_{n\to\infty} \tilde{\pi}^n$ is the same as that of $\tilde{\pi}$ is interesting and raises further questions about the structures of posterior processes of more general diffusion processes that involve uncertainty.

We finally show how to find asymptotically optimal solutions for the Bayesian stopping time problems for \tilde{L}^n by using the approximation $\lim_{n\to\infty} \tilde{\pi}^n$ rather than $\tilde{\pi}$. In fact, since the difference between σ' and σ can be arbitrarily large, by using the incorrect approximation $\tilde{\pi}$ in order to calculate the optimal strategy in the *n*-th model, the performance can be arbitrarily bad.

The rest of the paper is organized as follows: The introduction is concluded with a literature review. In Section 2 we introduce some technical preliminaries. In Section 3 we present a model of a Brownian motion with an unknown drift. We give a closed-form formula for the posterior distribution process. In Section 4 we define a sequence of systems (indexed by $n \in \mathbb{N}$) that converges to a Brownian motion with an unknown drift. In Section 5 we present the main results and find the distribution of the limit of the sequence of the posterior distribution processes. In Section 6 we consider a general optimal stopping problem for the *n*-th system and find asymptotically optimal solution by using the presentation we give to the limit of the posterior distribution processes. Summary and directions for future research appear in Section 7. The Appendix contains the proofs of several theorems.

1.2. Literature Review. The model of a DM who observes a Brownian motion with an unknown drift (and known standard deviation) is well explored in the literature and appears in the context of filtering theory, optimal stopping problems, and economics.

A variation of this model was studied in filtering theory by Kalman and Bucy (1961) [17] and Zakai (1969) [27]. These authors analyzed a more general model, where a DM observes a function of a diffusion process with an additional noise, which is formulated as a Brownian motion. They provided equations that the posterior or the unnormalized posterior distribution process satisfies.

Shiryaev (1978) [24] defined a Bayesian sequential testing problem where a DM observes continuously a Brownian motion with an unknown drift and has two hypotheses about the drift together with a prior probability about these hypotheses. In this problem the goal of the DM is to test sequentially the hypotheses with a minimal loss. The choice that the DM should make is to choose a stopping time and at that time to guess which one of the two hypotheses holds. This problem was generalized in several ways. Zhitlukhin and Shiryaev (2011) [28] generalized it to three hypotheses. Gapeev and Peskir (2004) [12] explored the problem with finite horizon. Gapeev and Shiryaev (2011) [14] explored a sequential testing problem where the observed process is a diffusion process satisfying a stochastic differential equation. Buonaguidi and Muliere (2013) [6] studied a sequential testing problem where the observed process is a Lévy process with unknown parameters.

Berry and Friestedt (1985) [3] investigated a Bayesian Brownian bandit problem where a DM operates a two-armed bandit with two available arms; a safe arm that yields a constant payoff, and a risky arm that yields a stochastic payoff, which is a Brownian motion with an unknown drift. There are two hypotheses about the drift together with a prior probability about these hypotheses. The DM has to decide when to switch from the risky arm to the safe arm. Bolton and Harris (1999) [5] investigated a game involving this type of bandit. Cohen and Solan (2013) [8] studied the single DM problem in the case where the observed process is a Lévy process with unknown parameters.

Other statistical Bayesian tests involving hypotheses on a Brownian motion with an unknown drift can be found in the literature. For example, Polson and Roberts (1994)

[22] investigated the likelihood function for a diffusion process with an unknown parameter and provided an example of a Brownian motion with an unknown drift with a normal prior on the drift.

In economic theory, the model of a Brownian motion with two prior hypotheses about the drift was studied, e.g., by Felli and Harris (1996) [10], Bergemann and Valimaki (1997) [2], Bolton and Harris (1999) [5], Keller and Rady (1999) [19], and Moscarini (2005) [20]. In Jovanovic (1979) [16] the prior about the drift is assumed to have the normal distribution. In the listed papers it is assumed that random changes appear after every small time interval and the process of total change can be modeled approximately by a Brownian motion.

Another well-known example of the use of Brownian motion as a continuous-time approximation of a discrete-time processes is in queueuing theory; under heavy traffic, the queue size, which changes by discrete jumps after every random time interval, converges to a reflected Brownian motion with a drift. The uncertainty about the drift can model a situation of a G/G/1 queue in heavy traffic where the rate of service is unknown. Such a case arises, for example, when the number of projects that a server works on and the amount of the effort that it dedicates to each project are unknown. For further examples of queueing models with parameter uncertainty see Whitt (2006) [26] and the references therein.

2. Technical Preliminaries

Let T > 0 and let $\mathcal{D}_T := \mathcal{D}[0, T]$ (resp. $\mathcal{D}_\infty := \mathcal{D}[0, \infty)$) be the space of real-valued RCLL (right-continuous with left limits) functions on [0, T] (resp. $[0, \infty)$).

Fix a Borel set $S \subseteq \mathbb{R}$. Let \mathcal{E}_T (resp. \mathcal{E}_{∞}) be the space of real-valued functions on $S \times [0, T]$ (resp. $S \times [0, \infty)$) that are \mathcal{D}_T (resp. \mathcal{D}_{∞}) with respect to the second variable. The space \mathcal{E}_T is endowed with the metric²

(2.1)
$$e_T(\nu,\kappa) := \sup_{l \in S, t \in [0,T]} |\nu(l,t) - \kappa(l,t)| \wedge 1, \text{ for } \nu, \kappa \in \mathcal{E}_T.$$

By using this metric, we define on the space \mathcal{E}_{∞} the metric

$$e_{\infty}(\nu,\kappa) := \sum_{T=1}^{\infty} e_T(\nu,\kappa) \frac{1}{2^T}, \text{ for } \nu,\kappa \in \mathcal{E}_{\infty}.$$

The metric e_{∞} is a generalization of the standard metric with which one usually defines convergence to a Brownian motion (see Karatzas and Shreve (1991) [18]) for functions of two variables.

Remark 2.1. Let $\{\kappa\} \cup \{\kappa^n\}_{n \in \mathbb{N}} \subset \mathcal{E}_{\infty}$. From the definitions of e_T and e_{∞} it follows that $\{\kappa^n\}_{n \in \mathbb{N}}$ converges to κ if and only if for every $T \in \mathbb{N}$, the restriction of $\{\kappa^n\}_{n \in \mathbb{N}}$ to $S \times [0, T]$ converges to the restriction of κ to $S \times [0, T]$.

Throughout the paper we denote processes with observations in \mathcal{E}_{∞} by bold Greek letters, processes with observations in \mathcal{D}_{∞} by capital Latin letters, and functions from S to \mathbb{R} by small Latin letters.

²All the limiting functions in this paper are in $\mathcal{C}_{\infty} := \mathcal{C}[0,\infty)$ or $\mathcal{C}_T := \mathcal{C}[0,T]$ (the subspaces of continuous functions on $[0,\infty)$ and [0,T], respectively) with respect to their second variable. Therefore, the uniform topology is sufficient for our purpose instead of the often used Skorokhod topology (see Chen and Yao (2001, Ch. 5.1) [7] for further discussion).

2.1. Types of Convergence. Let $\{\boldsymbol{\zeta}\} \cup \{\boldsymbol{\zeta}^n\}_{n \in \mathbb{N}}$ be measurable mappings from a probability space (Ω, \mathcal{F}, P) to $(\mathcal{E}_{\infty}, \mathcal{B}(\mathcal{E}_{\infty}))$. We define two types of convergence $\lim_{n \to \infty} \boldsymbol{\zeta}^n = \boldsymbol{\zeta}$ that are used in this paper.

2.1.1. Uniform Convergence over Compact Sets. We say that $\{\boldsymbol{\zeta}^n\}_{n\in\mathbb{N}}$ converges uniformly over compact sets (u.o.c.) to $\boldsymbol{\zeta}$ if

(2.2)
$$P\left(\lim_{n \to \infty} e_{\infty}(\boldsymbol{\zeta}^{\boldsymbol{n}}, \boldsymbol{\zeta}) = 0\right) = 1.$$

Remark 2.1 implies that Eq. (2.2) is equivalent to the requirement that for every $T \in \mathbb{N}$ one has

(2.3)
$$P\left(\lim_{n\to\infty}e_T(\boldsymbol{\zeta}^n,\boldsymbol{\zeta})=0\right)=1.$$

2.1.2. Convergence in Distribution. We say that $\{\boldsymbol{\zeta}^n\}_{n\in\mathbb{N}}$ converges in distribution to $\boldsymbol{\zeta}$ (and write $\lim_{n\to\infty} \boldsymbol{\zeta}^n \stackrel{\mathrm{d}}{=} \boldsymbol{\zeta}$) if for every bounded and continuous function f (w.r.t. the metric e_{∞}) defined on \mathcal{E}_{∞} one has

$$\lim_{n \to \infty} E[f(\boldsymbol{\zeta}^n)] = E[f(\boldsymbol{\zeta})].$$

As is well known, convergence u.o.c. implies convergence in distribution.

If $X : \Omega \to \mathcal{D}_{\infty}$ or $h : S \to \mathbb{R}$, then one may look at $X(\omega)$ and h as elements in \mathcal{E}_{∞} that are independent of the first and second variables, respectively.

3. AN AUXILIARY MODEL - BROWNIAN MOTION WITH AN UNKNOWN DRIFT

3.1. Formulations and Notations. In this section we study a model of a Brownian motion with an unknown drift. Let θ be a random variable with a countable³ support $S \subset \mathbb{R}$ and a distribution $\pi := {\pi_l}_{l \in S}$. Let (W(t)) be a standard Brownian motion independent of θ . Set $\sigma > 0$ and define

$$X(t) = X_{\theta}(t) := \theta t + \sigma W(t), \ t \in [0, \infty).$$

Suppose that the DM observes the process (X(t)) continuously, but does not observe θ . The drift θ is not known by the DM. For every $l \in S$ define the hypothesis H_l : $\theta = l$. The parameter π_l represents the prior probability that H_l is true. Denote by P_l the probability measure over the space of realized paths under the hypothesis H_l , and by $P := P_{\pi} = \sum_{l \in S} \pi_l P_l$ the probability measure that corresponds to the description above (see Gapeev and Peskir (2004) [12] for a rigorous construction of P).

3.2. The Posterior Distribution Process. At time t = 0, the parameter θ is chosen randomly according to the distribution π . The DM does not observe θ but he knows π and and σ . At each time instant t the DM observes the process (X(t)) and updates his belief about the hypotheses based on this information in a Bayesian fashion. We would like to give a closed-form expression to the *posterior distribution process*⁴

(3.1)
$$\boldsymbol{\pi}(l,t) := P(\theta = l \mid \mathcal{F}_t^X; \pi), \ l \in S, \ t \in [0,\infty),$$

³The results in the section can be extended to a Borel set S with the cardinality of the continuum, see Section 5.4.2.

⁴Note that the process $(\pi(l, t))$ depends on the prior distribution π ; indeed, for every $l \in S$ one has $\pi(l, 0) = \pi_l$. To save cumbersome notation, we omit the dependence on π .

where \mathcal{F}_t^X is the sigma-algebra that is generated by $(X(s))_{0 \le s \le t}$. The value $\pi(l, t)$ is the posterior distribution at time t that H_l is true given the past observations.

Without loss of generality we assume that $0 \in S$, since by taking $m \in S$ one can look at the process

$$X(t) - mt = (\theta - m)t + \sigma W(t), \ t \in [0, \infty).$$

The processes (X(t) - mt) and (X(t)) admit the same filtration; and 0 is in the support of $\theta - m$.

An important auxiliary process is the Girsanov process, also called the Radon–Nikodým density, which is defined by

(3.2)
$$\boldsymbol{\varphi}(l,t) := \frac{d(P_l \mid \mathcal{F}_t^X)}{d(P_0 \mid \mathcal{F}_t^X)}, \ l \in S, \ t \in [0,\infty).$$

The next result connects the process π to the process φ .

Lemma 3.1. For every $l \in S$, and every $t \in [0, \infty)$,

(3.3)
$$\boldsymbol{\pi}(l,t) = \frac{\pi_l \boldsymbol{\varphi}(l,t)}{\sum_{k \in S} \pi_k \boldsymbol{\varphi}(k,t)}$$

For a proof, see Cohen and Solan (2013, Lemma 1) [8]. By Jacod and Shiryaev (1987, Ch. III, Theorems 3.24 and 5.19) [15] the process φ admits the following representation:

(3.4)
$$\varphi(l,t) = \exp\left\{\frac{l}{\sigma^2}X(t) - \frac{1}{2}\left(\frac{l}{\sigma}\right)^2 t\right\}$$
$$= \exp\left\{\frac{l}{\sigma}W(t) - \frac{1}{2}\left(\frac{l}{\sigma}\right)^2 t + \frac{\theta}{\sigma} \cdot \frac{l}{\sigma}t\right\}, \ l \in S, \ t \in [0,\infty).$$

Remark 3.2. Notice that, based on the observed process $(X(s))_{0 \le s \le t}$, the present value at time t, X(t), is a sufficient statistic for θ . That is, for every $l \in S$ and every $t \in [0, \infty)$, the value of the process $(\varphi(l, s))$ at time $t, \varphi(l, t)$, and therefore also $\pi(l, t)$, depends on the process $(X(s))_{0 \le s \le t}$ only through X(t). This means that the Radon–Nikodým density and the posterior distribution process at time t depend on (X(s)) through the present value X(t) and are independent of past values $(X(s))_{0 \le s < t}$.

In order to emphasize the dependence of the processes φ and π on σ , we denote them by φ_{σ} and π_{σ} .

4. Deterministic and Random Parameter Systems

In this section we define a sequence of processes indexed by $n \in \mathbb{N}$ that converges in distribution (w.r.t. n) to a Brownian motion with an unknown drift. For each such process we define a relative posterior distribution process. In Section 4.1 we define a model of a system that consists of arrivals with a known rate. In Section 4.2 we generalize the model to a system that consists of arrivals with an unknown rate. In Section 4.3 we define a sequence of systems with unknown rates. In Section 4.4 we show that under proper assumptions, the scaled number of arrivals to these systems can be approximated by a Brownian motion with an unknown drift. 4.1. Deterministic Parameter System. We define a system that consists of arrivals (each of size 1) that occur according to the random variables $\{v_i\}_{i\geq 1}$. We assume that the system was activated before time t = 0. The parameter t_v is the time passed since the last arrival that occurred before time t = 0. We start the numeration of arrivals from time t = 0. v_1 is interpreted as the time passed from t = 0 until the first arrival; and for every $i \geq 2$, the random variable v_i is interpreted as the interarrival time between the (i - 1)-th and the *i*-th arrivals into the system. We present the interarrival time distribution as $\frac{v}{\mu}$, where v is a nonnegative random variable with expectation 1 and μ is a positive constant.

Formally, a deterministic parameter system

$$\mathcal{S} = (t_v, v, \mu, \{v_i\}_{i \ge 1})$$

is given by

- a nonnegative constant t_v ;
- a nonnegative random variable v;
- a positive constant μ ;
- a sequence of independent random variables $\{v_i\}_{i\geq 1}$. We make the following assumption on $\{v_i\}_{i\geq 1}$.

Assumption 4.1.

For every $t \in [0,\infty)$, one has $P(v_1 + t_v \ge t) = P\left(\frac{1}{\mu}v \ge t \mid \frac{1}{\mu}v \ge t_v\right)$, and for every $i \ge 2, v_i$ is distributed as the random variable $\frac{1}{\mu}v$.

We assume that v has a finite variance and without loss of generality, we assume that it has expectation 1.

Assumption 4.2.

 $\begin{array}{ll} \textit{4.2.1.} \ E[v] = 1. \\ \textit{4.2.2.} \ \sigma_v^2 := \mathrm{Var}[v] < \infty. \end{array}$

The arrival rate is defined by $\frac{1}{E[v_2]}$, which, by the definition of v_i and Assumption 4.2.1, equals μ .

4.1.1. The Counting Processes. Define the process

(4.1)
$$L(t) := \max\left\{m \left| \sum_{i=1}^{m} v_i \le t \right\}, \ t \in [0, \infty)\right.$$

The process (L(t)) counts the number of arrivals during the time interval [0, t], and it is called the *counting process* of the system.

4.2. Random Parameter System. Let θ be a random variable with bounded and countable⁵ support $S \subseteq \mathbb{R}$. For every $l \in S$, let $\pi_l := P(\theta = l)$. Consider a constant t_v and a random variable v that satisfy Assumptions 4.1 and 4.2, respectively. For every $l \in S$, let

$$S_l = (t_v, v, \mu_l, \{v_{i,l}\}_{i \ge 1})$$

be a deterministic parameter system such that the random variables $\{v_{i,l}\}_{i\geq 1}$ are independent of θ . Let $(L_l(t))$ be the corresponding counting process. A random parameter

⁵All the results in the paper can be extended to a bounded set S with cardinality of the continuum, see Subsection 5.4.2.

system is a system where the parameter μ is chosen randomly according to θ . That is, it is a random variable, and its support is the collection of the deterministic parameter systems $\{S_l\}_{l \in S}$. Formally, a random parameter system

$$\mathcal{RS}_{\pi}(\theta) = (t_v, v, \mu_{\theta}, \{v_i\}_{i>1}, \pi)$$

is given by⁶

$$(t_v, v, \mu_{\theta}, \{v_i\}_{i \ge 1}) = \sum_{l \in S} \mathbb{I}_{\{\theta = l\}} (t_v, v, \mu_l, \{v_{i,l}\}_{i \ge 1}),$$

where $\pi := {\{\pi_l\}_{l \in S}}$. The corresponding counting process is

$$L(t) = L_{\theta}(t) = \sum_{l \in S} \mathbb{I}_{\{\theta = l\}} L_l(t), \ t \in [0, \infty).$$

A DM operates a random parameter system. The parameter θ represents the type of the arrival rate and it is unknown to the DM. For every $l \in S$, the parameter π_l represents the probability that the arrival rate's type is $\theta = l$.

For every $l \in S$, define the hypothesis H_l : $\theta = l$. Denote by P_l the probability measure over the space of realized paths under the hypothesis H_l , and by $P := P_{\pi} = \sum_{l \in S} \pi_l P_l$ the probability measure that corresponds to the description above.

4.2.1. The Posterior Distribution Processes. At time t = 0, the DM observes the initial state (t_v, π) without observing θ , and thereafter he observes the counting process (L(t)) continuously. At each time instant t, the DM can update his belief on θ in a Bayesian fashion. Formally, the posterior distribution process is

(4.2)

$$\boldsymbol{\pi}(l,t) := P(\theta = l \mid \mathcal{F}_t^L, \pi) = P(\theta = l \mid L(t), t_v, v_1, \dots, v_{L(t)}; t; \pi), \ l \in S, \ t \in [0, \infty),$$

where \mathcal{F}_t^L is the sigma-algebra generated by $(L(s))_{0 \le s \le t}$. This is the posterior distribution process at time t that H_l is true given past observations of interarrivals times from the system $v_1, \ldots, v_{L(t)}$, and the absence of arrivals during the time interval $\left(\sum_{i=1}^{L(t)} v_i, t\right]$. That is, the DM updates his belief using all the available information he has from the observed process up to time t.

4.3. The *n*-th System. In this section we define a sequence of random parameter systems indexed by a parameter n, which can be any natural number. All the notation established in Section 4.2 is carried forward, except that we append a superscript n to denote a quantity which depends on n. We assume that the random variables v and θ are independent of n.

For every $n \in \mathbb{N}$, let

$$\mathcal{RS}^n_{\pi}(\theta) = (t^n_v, v, \mu^n_{\theta}, \{v^n_i\}_{i \ge 1}, \pi)$$

be a sequence of random parameter systems with the corresponding counting process

$$L^{n}(t) = L^{n}_{\theta}(t) = \sum_{l \in S} \mathbb{I}_{\{\theta=l\}} L^{n}_{l}(t), \quad t \in [0, \infty).$$

In order to define the diffusion approximation, we investigate the *n*-th system at time nt. Without loss of generality we assume that $0 \in S$, since by taking $m \in S$ one can

⁶Note that the sequence $\{v_i\}_{i\geq 1}$ depends on the random variable θ ; indeed, for every $i\geq 1$, one has $v_i = v_{i,\theta} = \sum_{l\in S} \mathbb{I}_{\{\theta=l\}} v_{i,l}$. To avoid cumbersome notation, we omit the dependence on θ .

look at the random variable $\theta - m$, and 0 belongs to its support. For every $n \in \mathbb{N}$ define the scaled posterior distribution process

(4.3)
$$\tilde{\boldsymbol{\pi}}^{\boldsymbol{n}}(l,t) := \boldsymbol{\pi}^{\boldsymbol{n}}(l,nt), \ l \in S, \ t \in [0,\infty)$$

4.4. The Posterior Distribution Process of the Limit of the Counting Processes. In this section we find a diffusion approximation related to the sequence of processes $\{L^n\}_{n\in\mathbb{N}}$. To this end, we require that the rates under the different types are relatively close, up to order of $\sqrt[7]{\frac{1}{\sqrt{n}}}$. Loosely speaking, it states that $\mu_{\theta}^n \approx \alpha + \frac{1}{\sqrt{n}}\theta$. It reminds the heavy traffic condition, which asserts that the difference between the arrival rate and the departure rate in a G/G/1 queue is by order of $\frac{1}{\sqrt{n}}$

For every $n \in \mathbb{N}$, let $h^n : S \to \mathbb{R}$ be the function

(4.4)
$$h^{n}(l) := \sqrt{n}(\mu_{l}^{n} - \mu_{0}^{n})$$

Assumption 4.3.

4.3.1. $\limsup |h^n(l) - l| = 0.$ $n \to \infty l \in S$ 4.3.2. $\lim_{n\to\infty}\mu_0^n = \alpha$, where α is a positive constant.

In Remark 4.7 below we discuss about the necessity of this assumption. Assumption 4.3.1 relates⁸ to the difference between the arrival rates under the different types. It states that every two possible arrival rates are distinguished by an order of $\frac{1}{\sqrt{n}}$. Assumption 4.3.2 states that the limit of the sequence of rates $\{\mu_0^n\}_{n\in\mathbb{N}}$ is positive. Together with Assumption 4.3.1 it implies that

(4.5)
$$\lim_{n \to \infty} \sup_{l \in S} |\mu_l^n - \alpha| = \lim_{n \to \infty} \sup_{l \in S} |\mu_l^n - \mu_0^n| = 0.$$

This assumption is also fundamental for the diffusion approximation of the sequence of processes $\tilde{\pi}^n$.

For every $n \in \mathbb{N}$, denote

$$\check{L}^n(t) := \frac{L^n(nt) - \mu_\theta^n nt}{\sqrt{n}}, \ t \in [0, \infty).$$

The following result was proved, e.g., in Billingsley (1999, Theorem 14.6) [4].

Proposition 4.4. Under Assumptions 4.1, 4.2, and 4.3.2, there exists a standard Brownian motion process (W(t)), independent of θ , such that

(4.6)
$$\lim_{n \to \infty} \check{L}^n \stackrel{\mathrm{d}}{=} \sigma_v \sqrt{\alpha} W.$$

Although the DM observes the process $(L^n(nt))$, he does not observe θ , and therefore does not observe μ_{θ} . That is, the parameter μ_{θ} is not known by the DM. Hence, the sigma-algebra that is generated by the relative process

$$\frac{L^n(nt) - \mu_{\theta}^n nt}{\sqrt{n}}, \ t \in [0, \infty),$$

⁷In Remarks 4.7 and 8.5 we explain why we require an order of $\frac{1}{\sqrt{n}}$ and detail the differences in the analysis in case that the order is higher or smaller than $\frac{1}{\sqrt{n}}$.

⁸Assumption 4.3.1 can be written also as $\lim_{n \to \infty} e_{\infty}(h^n, I_S) = 0$, where I_S is the identity function on S.

is different from the sigma-algebra $\mathcal{F}_{nt}^{L^n}$. We therefore define the relative process⁹

$$\tilde{L}^n(t) := \frac{L^n(nt) - \mu_0^n nt}{\sqrt{n}}, \ t \in [0,\infty).$$

For every $n \in \mathbb{N}$ the process $(\tilde{L}^n(t))$ can be calculated by the DM, since the sigmaalgebra that is generated by $(\tilde{L}^n(t))$ is $\mathcal{F}_t^{\tilde{L}^n} := \mathcal{F}_{nt}^{L^n}$, which is observed by the DM. The process $(\tilde{L}^n(t))$ can be expressed as

(4.7)
$$\tilde{L}^{n}(t) = \frac{L^{n}(nt) - \mu_{\theta}^{n}nt}{\sqrt{n}} + \sqrt{n}(\mu_{\theta}^{n} - \mu_{0}^{n})t, \ t \in [0, \infty).$$

From Eq. (4.6) the limit of the first term is a Brownian motion with a standard deviation $\sqrt{\alpha}\sigma_v$ and without drift, and from Assumption 4.3.1 the limit of the second term is (θt) . Therefore, the limit process:

$$\tilde{L}(t) := \lim_{n \to \infty} \tilde{L}^n(t), \ t \in [0, \infty),$$

exists and it is a Brownian motion with an unknown drift. This is summarized in the following proposition.

Proposition 4.5. Under Assumptions 4.1, 4.2, and 4.3, there exists a standard Brownian motion process (W(t)), independent of θ , such that

(4.8)
$$\tilde{L}(t) = \theta t + \sqrt{\alpha} \sigma_v W(t), \quad t \in [0, \infty).$$

From the definitions of φ_{σ} and π_{σ} (recall Eqs. (3.1) and (3.2) and the notation given after Remark 3.2) we deduce the following corollary.

Corollary 4.6. Let $\tilde{\varphi}$ and $\tilde{\pi}$ be the Radon–Nikodým derivative process and the posterior distribution process, respectively, under $\mathcal{F}_t^{\tilde{L}}$. Under Assumptions 4.1, 4.2, and 4.3, the process $\tilde{\varphi}$ is distributed as $\varphi_{\sqrt{\alpha}\sigma_v}$ and the process $\tilde{\pi}$ is distributed as $\pi_{\sqrt{\alpha}\sigma_v}$.

The following remark explains the requirement that the proper rates under the different types are relatively close, up to order of $\frac{1}{\sqrt{n}}$ (Assumption 4.3.1).

Remark 4.7. If there exists a parameter value $l^* \in S$ such that the difference between the rates $\mu_{l^*}^n$ and μ_0^n satisfies¹⁰ $|\mu_{l^*}^n - \mu_0^n| >> \frac{1}{\sqrt{n}}$, then under $\theta = l^*$ the second term in Eq. (4.7) converges to $\pm \infty$ and the DM would be able to distinguish between them. On the other hand, if there is a parameter value $l^* \in S$ such that the difference between the rates $\mu_{l^*}^n$ and μ_0^n satisfies $|\mu_{l^*}^n - \mu_0^n| << \frac{1}{\sqrt{n}}$, then under $\theta = l^*$ the second term in Eq. (4.7) converges to 0 and the DM would not be able to distinguish between them. The analysis without Assumption 4.3.1 would be similar, but with more complex notation.

5. The Limit of the Posterior Distribution Processes

In this section we find the limit of the posterior distribution processes, $\lim_{n\to\infty} \tilde{\pi}^n$, and study the relation between this limit and the posterior distribution process $\tilde{\pi}$. In Section 5.1 we formulate assumptions on the density of the random variable v. In Section 5.2 we provide examples of densities that satisfy these assumptions and an example of a

⁹Notice that μ_{θ}^n is replaced by μ_0^n .

¹⁰Hereafter, the notation $|f_1(n) - f_2(n)| >> f_3(n)$ (resp. <<) means that $\lim_{n \to \infty} |f_1(n) - f_2(n)| / f_3(n) = \infty$ (resp. 0).

density that does not. Section 5.3 gives the main theorems in the paper. We find the limit of the posterior distribution processes and discuss its properties. In Section 5.4 we discuss about some generalizations.

5.1. Assumptions on Densities. In order to find the diffusion approximation for the sequence of processes $\{\tilde{\pi}^n\}_{n\in\mathbb{N}}$, we need several assumptions on the distribution of the random variable v. If no such assumptions are made, then it may happen that for some $l \in S$ and some $t \geq 0$, the posterior probability $\tilde{\pi}^n(l,t)$ will vanish with a positive probability for some $n \in \mathbb{N}$. Such cases differ from each other in the form of their analysis and require different tools than the ones that we are using in this paper. The following assumption states that the support of the interarrival times is the positive part of the axis. This assumption rules out a situation where a single arrival can reveal a lot of information.

Assumption 5.1. The random variable v has the probability density function¹¹ (pdf) $f \in C^3$, with the support $(0, \infty)$.

Remark 5.2. For every $l \in S$, denote by F_l^n the cumulative distribution function (cdf) of $v_{1,l}^n$ and by f_l^n the pdf of $v_{1,l}^n$. By Assumption 4.1, for every s > 0, $F_l^n(s) = F(s\mu_l^n)$, where F(s) is the cdf of v. Moreover, for every s > 0, $f_l^n(s) = \mu_l^n f(s\mu_l^n)$, while $f_l^n(s) = 0$ for $s \leq 0$. In particular, for $l \in S$ the support of $v_{1,l}^n$ is $(0, \infty)$.

In the analysis of the posterior distribution process $(\tilde{\pi}^n(l,t))$, the following loglikelihood terms will appear:

$$\ln\left(\frac{f_l(s)}{f_0(s)}\right), \ \ln\left(\frac{1-F_l(s)}{1-F_0(s)}\right)$$

When we will use the representations of the cdfs that were introduced in Remark 5.2 and the Taylor approximation for the log-likelihood ratios above, we will encounter the terms

$$\frac{f'(s)}{f(s)}, \frac{f(s)}{1 - F(s)}$$

The following assumptions state that these functions are "sufficiently" bounded.

Assumption 5.3.

5.3.1. The random variable $\frac{f'(v)}{f(v)}v$ has a finite standard deviation, denoted by σ_f . 5.3.2. There exist a monotone nondecreasing function M(x) and a positive parameter $\epsilon_M > 0$, such that for every $x \in (0, \infty)$

$$\left| \left(\frac{f'(x)}{f(x)} \right)'' x^3 \right| \le M(x)$$

and

$$E\left[M\left(\left(1+\epsilon_M\right)v\right)\right]<\infty.$$

5.3.3. There exist a monotone nondecreasing function N(x) and a positive parameter $\epsilon_N > 0$, such that for every $x \in (0, \infty)$

$$\left|\frac{xf(x)}{1-F(x)}\right| \le N(x)$$

 $^{^{11}\}mathcal{C}^3$ is the class of real-valued functions with continuous third derivative.

and

$$E\left[N\left(\left(1+\epsilon_N\right)v\right)\right]^2 < \infty.$$

5.2. Examples. In this section we provide an example of a family of distributions that satisfy Assumptions 4.2, 5.1, and 5.3. In fact, most of the frequently used continuous distributions that satisfy Assumptions 4.2 and 5.1 also satisfy Assumption 5.3. We then present an example of a distribution that satisfies Assumptions 4.2 and 5.1, but not Assumption 5.3.1. This example illustrates that Assumption 5.3.1 is independent of Assumptions 4.2 and 5.1. As for Assumptions 5.3.2 and 5.3.3, we do not know whether they follow from previous assumptions or they are independent of them.

5.2.1. An Example that satisfies the Assumptions. Let v be a random variable with the support $(0, \infty)$ whose pdf is given by

$$f(x) = w_1(x)e^{w_2(x)},$$

where $w_1(x)$ and $w_2(x)$ are sums of power functions and the powers in $w_2(x)$ are positive. Denote by d the highest power in $w_2(x)$. Since f(x) is a pdf with the support $(0, \infty)$, $w_1(x) > 0$ and the smallest power in $w_1(x)$ is higher than -1. Clearly $f \in C^3$. By simple computations one can verify that there exists a constant C, such that for every $x \in (0, \infty)$ the following holds

$$\left|\frac{f'(x)}{f(x)}x\right|, \left|\left(\frac{f'(x)}{f(x)}\right)''x^3\right|, \left|\frac{xf(x)}{1-F(x)}\right| \le C\max\{1, x^d\}.$$

Assumption 5.3.1 follows since $E[\max\{1, x^d\}]^2 < \infty$. Since for every $\epsilon > 0$,

 $E[\max\{1,((1+\epsilon)x)^d\}], E[\max\{1,((1+\epsilon)x)^d\}]^2 < \infty,$

and the functions $C \max\{1, x^d\}$ and $(C \max\{1, x^d\})^2$ are monotone nondecreasing, it follows that Assumptions 5.3.2 and 5.3.3 hold by choosing $M(x) := C \max\{1, x^d\}$, $N(x) := C^2 \max\{1, x^d\}^2$, and ϵ_N, ϵ_M to be arbitrary positive constants.

This family of distributions contains the gamma, Weibull, Maxwell–Boltzmann, and Rayleigh distributions.

5.2.2. An Example that does not Satisfy Assumption 5.3.1. We show that there exists a random variable that satisfies Assumptions 4.2 and 5.1 and fails to satisfy Assumption 5.3.1. For every $d \in \mathbb{N}$, define $x_d := 1 + \frac{1}{2} + \cdots + \frac{1}{d}$. Let g(x) be the following function: for every $d \in \mathbb{N}$ and every $x_d < x \leq x_{d+1}$ define $g(x) := \frac{1}{d+1}(x-x_d)$. Then g is not a pdf of a random variable and it is not differentiable. However, by smoothing g and changing its values on a bounded interval, one can construct a random variable with a pdf function f that satisfies Assumptions 4.2 and 5.1. Since Assumption 4.2.1 follows from Assumption 4.2.2 by normalization, and Assumption 5.1 is only a matter of smoothing, it is sufficient to show that Assumption 4.2.2 can be satisfied. This follows from the following series of equalities and inequalities:

$$\begin{split} \int_0^\infty x^2 g(x) dx &= \sum_{d=1}^\infty \int_{x_d}^{x_{d+1}} x^2 g(x) dx \le \sum_{d=1}^\infty x_{d+1}^2 \int_{x_d}^{x_{d+1}} g(x) dx = \frac{1}{2} \sum_{d=1}^\infty x_{d+1}^2 \frac{1}{(d+1)^3} \\ &\approx \frac{1}{2} \sum_{d=1}^\infty \ln^2 (d+1) \frac{1}{(d+1)^3} < \infty. \end{split}$$

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Assumption 5.3.1, however, does not hold, since

$$\begin{split} \int_0^\infty \left(\frac{g'(x)}{g(x)}x\right)^2 g(x)dx &= \sum_{d=1}^\infty \int_{x_d}^{x_{d+1}} \left(\frac{g'(x)}{g(x)}x\right)^2 g(x)dx \ge \sum_{d=1}^\infty (d+1)^2 x_d^2 \int_{x_d}^{x_{d+1}} g(x)dx \\ &= \frac{1}{2} \sum_{d=1}^\infty (d+1)^2 x_d^2 \frac{1}{(d+1)^3} \approx \frac{1}{2} \sum_{d=1}^\infty \ln^2 (d+1) \frac{1}{d+1} = \infty. \end{split}$$

5.3. The Limit of the Posterior Distribution Processes. Similar to the constructions of the Radon–Nikodým process and the posterior distribution process in the model of the Brownian motion with an unknown drift (see Eqs. (3.2) and (3.3)), one can show that for every $l \in S$ and every $t \in [0, \infty)$,

(5.1)
$$\tilde{\boldsymbol{\pi}}^{\boldsymbol{n}}(l,t) = \frac{\pi_l \tilde{\boldsymbol{\varphi}}^{\boldsymbol{n}}(l,t)}{\sum_{k \in S} \pi_k \tilde{\boldsymbol{\varphi}}^{\boldsymbol{n}}(k,t)}, \quad l \in S, \ t \in [0,\infty),$$

where

(5.2)
$$\tilde{\boldsymbol{\varphi}}^{\boldsymbol{n}}(l,t) := \frac{d(P_l^n \mid \mathcal{F}_{nt}^{L^n})}{d(P_0^n \mid \mathcal{F}_{nt}^{L^n})}, \ l \in S, \ t \in [0,\infty)$$

is the Radon–Nikodým process. In fact, for every $l \in S$, $(\tilde{\varphi}^n(l, t))$ is the likelihood ratio process w.r.t. t and it satisfies

(5.3)
$$\tilde{\boldsymbol{\varphi}}^{\boldsymbol{n}}(l,t) = \frac{P^n(L^n(nt), v_1^n, \dots, v_{L^n(nt)}^n; nt \mid \theta = l, t_v^n)}{P^n(L^n(nt), v_1^n, \dots, v_{L^n(nt)}^n; nt \mid \theta = 0, t_v^n)}, \quad t \in [0, \infty)$$

The next theorem shows that the Radon–Nikodým process and the posterior distribution process of the *n*-th system converge to φ_{σ} and π_{σ} , respectively, for a properly chosen σ .

Theorem 5.4 (Main Theorem). Under Assumptions 4.1, 4.2, 4.3, 5.1, and 5.3, the following hold:

(5.4)
$$\lim_{n \to \infty} \tilde{\varphi}^n \stackrel{\mathrm{d}}{=} \varphi_{\frac{\sqrt{\alpha}}{\alpha}},$$

(5.5)
$$\lim_{n \to \infty} \tilde{\pi}^n \stackrel{\mathrm{d}}{=} \pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$$

Remark 5.5. The quantity σ_f in Eqs. (5.4) and (5.5) depends on the pdf of the random variable v. That is, the limit of the posterior distribution process depends on the structure of the density of v and not only on its moments. Notice also that $\frac{\sqrt{\alpha}}{\sigma_f}$ is not the parameter that is associated with $(\tilde{L}(t))$ (see Eq. (4.8)). The relation between the parameters σ_v and $\frac{1}{\sigma_f}$ is studied in Theorems 5.6 and 5.9 below.

The proof of Theorem 5.4 is given in the Appendix. We now outline the main ideas of the proof. We start with the first part of the theorem. First, we show that (a) the fact that the system was activated before time t = 0, and (b) the lack of arrivals during the time interval $\left(\sum_{i=1}^{L^n(nt)} v_i^n, t\right]$, have almost-surely an effect of order o(1) on the posterior distribution process as n goes to infinity (Lemma 8.2). Therefore, there is no significant

difference if the DM updates his belief only at arrival times. That is,¹²

(5.6)
$$\tilde{\boldsymbol{\varphi}}^{\boldsymbol{n}}(l,t) = \exp\left\{\sum_{i=1}^{L^n(nt)} \ln\left(\frac{f_l^n(v_i^n)}{f_0^n(v_i^n)}\right) + o(1)\right\} \quad \text{a.s.},$$

where $nt = \sum_{i=1}^{L^n(nt)} v_i^n$ (this means that time nt is an arrival time), see Eqs. (8.9), (8.14), and (8.15). Second, we find the distribution of $\sum_{i=1}^{L^n(nt)} \ln\left(\frac{f_i^n(v_i^n)}{f_0^n(v_i^n)}\right)$. We show that for every $n \in \mathbb{N}$ there exists a process $(\tilde{W}^n(t))$ with the following properties: it is independent of θ ; the limit $\tilde{W} := \lim_{n \to \infty} \tilde{W}^n$ exists and the process $(\tilde{W}(t))$ is a standard Brownian motion such that

$$\sum_{i=1}^{L^n(nt)} \ln\left(\frac{f_l^n(v_i^n)}{f_0^n(v_i^n)}\right) = \sqrt{n}\sigma_f \frac{\mu_l^n - \mu_0^n}{\mu_k^n} \tilde{W}^n\left(\frac{L^n(nt)}{n}\right) \\ - \frac{1}{2}\left(\frac{l\sigma_f}{\alpha}\right)^2 \frac{L^n(nt)}{n} + \left(\frac{l\theta\sigma_f^2}{\alpha^2}\right) \frac{L^n(nt)}{n} + o(1) \text{ a.s.},$$

see Eqs. (8.12) and (8.24). By taking the limit $n \to \infty$ and using the random time-change theorem (see Chen and Yao (2001, Theorem 5.3) [7]) for the composition $\tilde{W}^n\left(\frac{L^n(nt)}{n}\right)$ one gets the desired result (see Proposition 8.4).¹³

We now turn to the second part of the theorem. Notice that if we prove Eq. (5.4), then Eq. (5.5) follows from the definitions of φ_{σ} and π_{σ} and from Eqs. (3.3) and (5.1), because the mapping $\varphi(\cdot, \cdot) \mapsto \frac{\pi_l \varphi(\cdot, \cdot)}{\sum_{k \in S} \pi_k \varphi(k, \cdot)}$ is continuous.

From Eq. (5.5) and the definition of π_{σ} , it follows that $\lim_{n \to \infty} \tilde{\pi}^n$ is distributed as a posterior distribution process of a Brownian motion with an unknown drift. The following theorem summarizes this observation.

Theorem 5.6. Under Assumptions 4.1, 4.2, 4.3, 5.1, and 5.3, the process $\lim_{n\to\infty} \tilde{\pi}^n$ can be expressed as the posterior distribution process of the process

(5.7)
$$\hat{M}(t) = \hat{M}_{\theta}(t) := \theta t + \sqrt{\alpha} \frac{1}{\sigma_f} W'(t), \quad t \in [0, \infty),$$

where (W'(t)) is a standard Brownian motion independent of θ . Moreover, $\frac{1}{\sigma_f} \leq \sigma_v$ where equality holds if and only if the random variable v has a gamma distribution (with expectation 1).

Since $\frac{1}{\sigma_f} \leq \sigma_v$, the paths of the process $(\hat{M}(t))$ will be more concentrated around the path of the linear drift, (θt) , than the paths of the process $(\tilde{L}(t))$. In other words, the process $(\hat{M}(t))$ is less noisy than $(\tilde{L}(t))$. Therefore, it is easier to estimate the parameter θ given $(\hat{M}(t))$ than given $(\tilde{L}(t))$. That is, $\lim_{n\to\infty} \tilde{\pi}^n$ is more informative than $\tilde{\pi}$.

Remark 5.7. If v has a gamma distribution with expectation 1, then its density is of the form $f(s) = \frac{\beta^{\beta}}{\Gamma(\beta)} s^{\beta-1} e^{-\beta s}$, where β is a positive constant. From Remark 5.2 and

¹²The a.s. convergence is with respect to the metric e_{∞} .

¹³see Eq. (3.4) for the structure of φ_{σ} .

Eq. (5.6) it follows that

(5.8)
$$\ln(\tilde{\varphi}^n)(l,s) = L^n(ns) \ln\left(\frac{\mu_l^n}{\mu_0^n}\right)^\beta - ns\beta(\mu_l^n - \mu_0^n) + o(1).$$

That is, for sufficiently large $n \in \mathbb{N}$, the Radon–Nikodým density, and therefore also the posterior distribution process at time nt, depend on the process $(L^n(ns))_{0\leq s\leq t}$ only through $L^n(nt)$, up to order o(1). Loosely speaking, for sufficiently large n's the parameter θ has sufficient statistics (based on $(\tilde{L}^n(s))_{s\leq t}$) that are 'approximately independent of the past'. This is the same property that holds in the Brownian motion with an unknown drift model (see Remark 3.2). Therefore, we expect that indeed this case the processes $\lim_{n\to\infty} \tilde{\pi}^n$ and $\tilde{\pi}$ will be identically distributed, because no information is lost by looking at the present rather than at the past.

Before proving Theorem 5.6, we state a lemma that provides insights about the parameter σ_f , which is then used in the proof.

Lemma 5.8. Under Assumptions 4.2, 5.1, and 5.3.1, the following equalities hold:

(5.9)
$$E\left[\frac{f'(v)}{f(v)}v\right] = -1$$

and

(5.10)
$$E\left[\left(\frac{f'(v)}{f(v)}\right)'v^2\right] = 1 - \sigma_f^2.$$

Proof.

$$E\left[\frac{f'(v)}{f(v)}v\right] = \int_0^\infty f'(v)v \, dv = f(v)v \Big|_0^\infty - \int_0^\infty f(v)dv = -1,$$

where the last equality holds since

$$\int_0^\infty f(v)v\,dv, \int_0^1 f(v)dv < \infty$$

and therefore

$$\lim_{u \to \infty} f(v)v = \lim_{u \to 0+} f(v)v = 0.$$

From Assumption 5.3.1, and by using similar arguments as above, it follows that

$$E\left[\left(\frac{f'(v)}{f(v)}\right)'v^2\right] = 1 - \sigma_f^2.$$

Proof of Theorem 5.6. From the definition of π_{σ} it follows that the posterior distribution process of the process $(\hat{M}(t))$ is given by $\pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$. From Eq. (5.5) it follows that $\lim_{n\to\infty} \tilde{\pi}^n$ is distributed as $\pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$. We now show that

$$\sigma_f \sigma_v \ge 1$$

and that equality holds if and only if v has a gamma distribution with expectation 1. The inequality follows from the following relations:

$$\sigma_f \sigma_v = \sqrt{E\left[\frac{f'(v)}{f(v)}v + 1\right]^2 E\left[v - 1\right]^2} \ge \left|E\left[\left(\frac{f'(v)}{f(v)}v + 1\right)(v - 1)\right]\right| = 1.$$

The first equality holds by the definitions of σ_f and σ_v , Assumption 4.2.1 and by Lemma 5.8 (Eq. (5.9)). The inequality is the Cauchy–Schwartz inequality. The second equality follows from Lemma 5.8 (Eq. (5.10)) and from the equation

$$E\left[\frac{f'(v)}{f(v)}v^2\right] = -2,$$

which is obtained via integration by parts. Notice that the inequality turns into equality if and only if $\frac{f'(v)}{f(v)}v + 1$ and v - 1 are linearly dependent. One can verify that under Assumptions 4.2 and 5.1 this happens if and only if v has a gamma distribution with expectation 1.

The next theorem states that the difference between σ_v and $\frac{1}{\sigma_f}$ can be arbitrarily large. Hence, the distributions of $\pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$ (and by Theorem 5.4 also $\tilde{\pi}^n$) and $\pi_{\sqrt{\alpha}\sigma_v}$ can be very different.

Theorem 5.9. The difference $\sigma_v - \frac{1}{\sigma_f}$ can be arbitrarily large.

The proof of Theorem 5.9 is given in the Appendix. To show that $\sigma_v - \frac{1}{\sigma_f}$ can be arbitrarily large we construct a family of random variables that satisfy Assumptions 4.2, 5.1, and 5.3 and for which the variances σ_v^2 's can be arbitrarily large and the parameters $\frac{1}{\sigma_f^2}$'s are uniformly bounded from above.

In Sections 6.3.1 and 6.4.1 below we show how to use the distribution of $\pi_{\sqrt{\alpha}}$ in order

to solve optimal stopping problems w.r.t. the observed process $(\tilde{L}^n(t))$. We show there that if one calculates his strategy based on the distribution of $\pi_{\sqrt{\alpha}\sigma_v}$ instead of the distribution of $\pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$, then his payoff will be suboptimal. By Theorem 5.9 it turns out that the strategies and the payoffs that follow by the distributions of $\pi_{\sqrt{\alpha}\sigma_v}$ and $\pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$ can be very different and therefore by taking the wrong approximation, the performance can be relatively bad (see Remark 6.11 below).

5.4. Generalizations.

5.4.1. Intermittent System. There are cases where the system operates intermittently. For example, the departure process from a G/G/1 queue with an unknown service rate can be modeled as the system described above that operates only when the queue is not empty (with 'departures' instead of 'arrivals'). In this section we study systems that operate intermittently, and let $(B^n(t))$ be the process that represents the cumulative time that the *n*-th system works during the time interval [0, t]. Let

$$\tilde{\boldsymbol{\pi}}_{\boldsymbol{B}}^{\boldsymbol{n}}(l,t) := \tilde{\boldsymbol{\pi}}^{\boldsymbol{n}}(l,B^{\boldsymbol{n}}(t)) = \boldsymbol{\pi}^{\boldsymbol{n}}(l,B^{\boldsymbol{n}}(nt)), \ l \in S, t \in [0,\infty),$$

be the posterior distribution process for the observed process $(L^n(B^n(nt)))$. The following theorem describes the distribution of $\lim \tilde{\pi}^n_B$. **Theorem 5.10.** Suppose that there is a constant $0 \le \rho \le 1$ such that $\lim_{n\to\infty} \frac{B^n(nt)}{n} = \rho t$ u.o.c. Under Assumptions 4.1, 4.2, 4.3, 5.1, and 5.3, the following holds:

(5.11)
$$\lim_{n \to \infty} \tilde{\pi}^n_B \stackrel{\mathrm{d}}{=} \pi_{\frac{\sqrt{\alpha}}{\sigma_f \sqrt{\rho}}}$$

The proof follows from the random time-change theorem (Chen and Yao (2001, Theorem 5.3) [7]) in a similar way to the proof of Theorem 5.4, and is therefore omitted.

5.4.2. Continuous Distribution over θ . Theorems 5.4, 5.6, and 5.10 also hold in case that θ is a continuous random variable with the density π_l , $l \in S$. In this case, the term $\sum_{k \in S}$ in Eqs. (3.3) and (5.1) is replaced by $\int_{k \in S}$.

6. Optimal Stopping Problems

The problem of finding closed-form solutions for optimal stopping problems w.r.t. $(L^n(t))$ in the general case suffers from high complexity. Buonaguidi and Muliere (2013) [6] and Cohen and Solan (2013) [8] solved such optimal stopping problems in case that, given θ , the process $(L^n(t))$ is a Lévy process. We do not make that assumption and rather find an asymptotically optimal solution by using the limit process $\lim_{n\to\infty} \tilde{\pi}^n$. As mentioned in Section 1, there are several optimal stopping problems that have been studied in the literature with respect to a Brownian motion with an unknown drift. The purpose of this section is to show that optimal stopping problems such as the Bayesian Brownian bandit problem (Berry and Friestedt (1985) [3], Bolton and Harris (1999) [5], Cohen and Solan (2013) [8]) and the sequential testing problem¹⁴ (Shiryaev (1978) [24]), are relevant for a process that is close in distribution to a Brownian motion with an unknown drift. These papers considered a Brownian motion with an unknown drift where there are only two hypotheses about the drift, and therefore we limit the discussion on this section to the case of two available hypotheses H_l and H_0 , where $0 \neq l \in \mathbb{R}$. The optimal stopping problems consist of (a) an observed process, (b) a stopping time adapted to the observed process, and (c) a payoff function that is a function of the observed process. Although the optimal stopping problems are formulated with the observed process, which is a Brownian motion with an unknown drift, it is possible to formulate the problems and their solutions in terms of the posterior distribution process. We present a sequence of random parameter systems that converges to a Brownian motion with an unknown drift. Under modest assumptions we formulate a stopping time problem with respect to the posterior distribution process $(\tilde{\pi}^n(l,t), \tilde{\pi}^n(0,t))$. We solve these problems by using Theorem 5.4, and we deduce from Theorem 5.9 that by using the approximation $\tilde{\pi}$ instead of $\lim \tilde{\pi}^n$, the performance can be relatively bad.

In Section 6.1 we define the cost function and the optimal stopping problems with respect to the posterior distribution process $(\tilde{\pi}^n(l,t), \tilde{\pi}^n(0,t))$. In Section 6.2 we find an approximate solution by using Theorem 5.4. In Sections 6.3 and 6.4 we show that the Bayesian Brownian bandit problem and the Brownian sequential testing problem are special cases of the general problem that is described here.

Define $\hat{\varphi} \stackrel{\text{d}}{:=} \lim_{n \to \infty} \tilde{\varphi}^n$ and $\hat{\pi} \stackrel{\text{d}}{:=} \lim_{n \to \infty} \tilde{\pi}^n$. From Eq. (5.4) it follows that $\hat{\varphi}$ is distributed as $\varphi_{\frac{\sqrt{\alpha}}{\sigma_f}}$.

¹⁴We consider here discounted optimal stopping problems, whereas Shiryaev considers an undiscounted problem.

Recall that in this section we study the case where the support of θ consists of two states: 0 and l. By knowing the prior/posterior probability of one state, the DM can infer the probability of the other. Therefore, it is sufficient to make the forthcoming analysis w.r.t. the following processes $\hat{\pi}(t) := \hat{\pi}(l,t), \ \tilde{\pi}^n(t) := \tilde{\pi}^n(l,t), \ \hat{\varphi}(t) := \hat{\varphi}(l,t), \ \tilde{\varphi}^n(t) := \tilde{\varphi}^n(l,t), \ t \in [0,\infty)$, and the prior probability $\pi := \pi_l$.

6.1. The Cost Function. Suppose that a DM who operates the *n*-th system, observes the process $(L^n(t))$, and continuously updates his belief about the hypotheses H_l and H_0 . Let $k^n, K^n : [0,1] \to \mathbb{R}$ be two functions that stand for the instantaneous cost and for the terminal cost, respectively; the DM's instantaneous discounted \cot^{15} for operating the system during the time interval [t, t + dt) is $\frac{r}{n}e^{-\frac{r}{n}t}k^n(\boldsymbol{\pi}^n(t))dt$, where $\boldsymbol{\pi}^n(t) := P(\theta = l \mid \mathcal{F}_t^{L^n}; \pi)$. The choice that the DM should make is when to stop operating the system. If the DM stops at time T then he has an additional discounted $\cot \frac{r}{n}e^{-\frac{r}{n}T}K^n(\boldsymbol{\pi}^n(T))$. Formally, the DM chooses a stopping time τ^n for the process $(L^n(t))$; that is, the stopping time is adapted to the filtration $\mathcal{F}_t^{L^n}$, which is the natural filtration generated by $(L^n(t))$. The expected discounted loss of the DM if he chooses the stopping time τ^n is

(6.1)
$$V_{\tau^n}^n(\pi) := E^{\pi} \left[\int_0^{\tau^n} \frac{r}{n} e^{-\frac{r}{n}t} k^n(\pi^n(t)) dt + \frac{r}{n} e^{-\frac{r}{n}\tau^n} K^n(\pi^n(\tau^n)) \right]$$

Set $\tilde{\tau}^n := \frac{1}{n} \tau^n$. The stopping time $\tilde{\tau}^n$ is adapted to the filtration $\mathcal{F}_{nt}^{L^n}$, which is identical to the filtration $\mathcal{F}_t^{\tilde{\pi}^n}$. Eq. (6.1) is equivalent to¹⁶

(6.2)
$$V_{\tilde{\tau}^n}^n(\pi) = E^{\pi} \left[\int_0^{\tilde{\tau}^n} r e^{-rt} k^n (\tilde{\boldsymbol{\pi}}^n(t)) dt + \frac{r}{n} e^{-r\tilde{\tau}^n} K^n (\tilde{\boldsymbol{\pi}}^n(\tilde{\tau}^n)) \right].$$

The goal of the DM is to minimize $V_{\tilde{\tau}^n}^n(p)$ and to find, if exists, the optimal stopping time $\tau^{*,n}$ for which the infimum of (6.2) is attained. Let

(6.3)
$$U^n(\pi) := \inf_{\tilde{\tau}^n} V^n_{\tilde{\tau}^n}(\pi)$$

be the minimal loss that the DM can achieve, and in case that the infimum is attained, let

(6.4)
$$\tilde{\tau}^{*,n}(\pi) \in \operatorname*{arg\,min}_{\tilde{\tau}^n} V^n_{\tilde{\tau}^n}(\pi)$$

be an optimal stopping time given that the prior belief is π .

Assumption 6.1.

6.1.1. The sequence of functions k^n converges uniformly to a function k on [0, 1].

6.1.2. k is continuous on the interval [0, 1].

6.1.3. The sequence of functions K^n/n converges uniformly to a function K on [0,1]. 6.1.4. K is continuous on the interval [0,1].

Remark 6.2. From Assumption 6.1.2 (resp. 6.1.4) it follows that the function k (resp. K) is bounded and uniformly continuous on [0, 1]. From Assumption 6.1.1 (resp. 6.1.3) it follows that there exists a constant $C_k > 0$ (resp. C_K), such that for every $n \in \mathbb{N}$ and every $\pi \in [0, 1]$, one has $|k^n(\pi)|, |k(\pi)| \leq C_k$ (resp. $|K^n(\pi)/n|, |K(\pi)| \leq C_K$).

¹⁵Notice that the discount factor r is scaled by an order of n.

¹⁶Recall that for every t > 0 we defined $\tilde{\pi}^n(t) := \pi^n(nt)$ (see Eq. (4.3)).

We now define the expected cost and the value function with respect to $\mathcal{F}_t^{\hat{\pi}}$. Fix $\pi \in [0, 1]$. Then the expected cost by using the $\mathcal{F}_t^{\hat{\pi}}$ -adapted stopping time τ is

$$V_{\tau}(\pi) := E^{\pi} \left[\int_0^{\tau} r e^{-rt} k(\hat{\boldsymbol{\pi}}(t)) dt + r e^{-r\tau} K(\hat{\boldsymbol{\pi}}(\tau)) \right].$$

Let

(6.5)
$$U(\pi) := \inf V_{\tau}(\pi)$$

be the value function, and in case that the infimum is attained, let

(6.6)
$$\tau^*(\pi) \in \operatorname*{arg\,min}_{\tau} V_{\tau}(\pi)$$

be an optimal stopping time given that the prior belief is π .

6.2. Stopping Times. Since the optimal stopping times (if exist) of the problems (6.3)– (6.4) and (6.5)–(6.6) are stationary Markovian stopping times with respect to the posterior distributions processes ($\tilde{\pi}^{n}(t)$) and ($\hat{\pi}(t)$), respectively (see Cohen and Solan (2013, Remark 4) [8]), it is natural to confine our discussion to the set of stationary Markovian stopping times. We now define a *first exit time strategy*. To this end, we define a subset of [0, 1] such that if the posterior is within this subset, then the DM continues and stops otherwise. Let $D = \bigcup_i (a_i, b_i) \subseteq [0, 1]$ be a finite union of disjoint open intervals such that if $b_j = 1$ (resp. $a_i = 0$), then the open interval (a_j, b_j) (resp. (a_i, b_i)) is replaced by the semi-open interval $(a_j, 1]$ (resp. $[0, b_i)$).

Assumption 6.3. For every i < j one has $b_i < a_j$.

Assumption 6.3 merely says that the intervals do not 'touch each other'. Define¹⁷

(6.7)
$$\tilde{\tau}_D^n(\pi) := \inf\{t \mid \tilde{\boldsymbol{\pi}}_{\pi}^n(t) \notin D\},\$$

(6.8)
$$\tilde{\tau}_D(\pi) := \inf\{t \mid \hat{\boldsymbol{\pi}}_{\pi}(t) \notin D\}.$$

That is, D is the *continuation region* with respect to the stopping times $\{\tilde{\tau}_D^n\}_{n\in\mathbb{N}}$ and $\tilde{\tau}_D$. From Assumption 6.3 it follows that if the DM continues for every prior in a certain punctured neighborhood of a, then he should also continue for the prior a.

The next theorem asserts that by using the same continuation region D for every $n \in \mathbb{N}$, the stopping times $\tilde{\tau}_D^n(\pi)$ converge in distribution to $\tilde{\tau}_D(\pi)$ and the expected cost functions $V_{\tilde{\tau}_D}^n(\pi)$ converge to $V_{\tilde{\tau}_D}(\pi)$.

Theorem 6.4. Under Assumptions 4.1, 4.2, 4.3, 5.1, 5.3, 6.1, and 6.3, we have

(6.9)
$$\lim_{n \to \infty} \tilde{\tau}_D^n(\pi) \stackrel{\mathrm{d}}{=} \tilde{\tau}_D(\pi)$$

and

(6.10)
$$\lim_{n \to \infty} V^n_{\tilde{\tau}^n_D}(\pi) = V_{\tilde{\tau}_D}(\pi).$$

The proof is relegated to the Appendix. In fact, Theorem 6.4 holds even if we replace the D's on the left-hand sides of Eqs. (6.9) and (6.10) by D^n 's, where $D^n \to D$ in the sense that the indicators of D^n converge pointwise to the indicator of D. The proof requires some technical modifications that we wish to avoid in order to ease the notation.

In some models such as the Bayesian Brownian bandit and the Sequential testing (as shown in Sections 6.3.1 and 6.4.1 respectively) the limit problem admits a unique optimal

¹⁷The subscript π indicates the prior probability that $\theta = l$. That is, $\tilde{\pi}_{\pi}^{n}(0) = \pi$ and $\hat{\pi}_{\pi}(0) = \pi$.

stopping time that is associated with a continuation region D^* . That is, $V_{\tilde{\tau}_{D^*}} = U$. Therefore, by Theorem 6.4 it follows that for every $\pi \in [0, 1]$ one has

$$\lim_{n \to \infty} V^n_{\tilde{\tau}^n_{D^*}}(\pi) = V_{\tilde{\tau}_{D^*}}(\pi) = U(\pi),$$

whereas for every $\overline{D} \neq D^*$ and every $\pi \in [0, 1]$ one has

(6.11)
$$\lim_{n \to \infty} V^n_{\tilde{\tau}^n_{\bar{D}}}(\pi) = V_{\tilde{\tau}_{\bar{D}}}(\pi) \ge U(\pi)$$

This is summarized in the following corollary.

Corollary 6.5. Under Assumptions 4.1, 4.2, 4.3, 5.1, 5.3, 6.1, and 6.3, if the limit problem admits an optimal stopping time that is associated with a continuation region D^* , then

$$\lim_{n \to \infty} U^n(\pi) = \lim_{n \to \infty} V^n_{\tilde{\tau}^n_{D^*}}(\pi) = V_{\tilde{\tau}_{D^*}}(\pi) = U(\pi).$$

Remark 6.6. For every n we defined the expected discounted loss in Eq. (6.1) by using the functions k^n and K^n , and found an equivalent representation in Eq. (6.2). By Assumption 6.1, the functions k^n and K^n/n converge uniformly to the functions k and K, respectively. Therefore, it would not make much difference if we defined

$$V^n_{\tilde{\tau}^n_D}(\pi) := E^\pi \left[R(\tilde{\pi}^n) \right],$$

and

$$V_{\tilde{\tau}_D}(\pi) := E^{\pi} \left[R(\hat{\pi}) \right]$$

where¹⁸

$$R(\boldsymbol{\pi}) := \int_0^{\tau_D} r e^{-rt} k(\boldsymbol{\pi}(t)) dt + r e^{-r\tau_D} K(\boldsymbol{\pi}(\tau_D))$$

and

$$\tau_D := \tau_D(\pi) = \inf\{t \mid \boldsymbol{\pi}_{\pi}(t) \notin D\}.$$

That is, for every $n \in \mathbb{N}$ one has $k^n \equiv k$ and $K^n/n \equiv K$. In this case, one may try to use the convergence in distribution $\lim_{n\to\infty} \tilde{\pi}^n \stackrel{\mathrm{d}}{=} \hat{\pi}$ and conclude that $\lim_{n\to\infty} E^{\pi}[R(\tilde{\pi}^n)] = E^{\pi}[R(\hat{\pi})]$. However, the function R is not continuous with respect to the process π , since it is possible to exhibit two processes π_1 and π_2 that are relatively close, but that the stopping times $\tau_D(\pi_1)$ and $\tau_D(\pi_2)$ are relatively far from each other, in which case the difference $|R(\pi_1) - R(\pi_2)|$ may be large. Hence, the inference that $\lim_{n\to\infty} E^{\pi}[R(\tilde{\pi}^n)] = E^{\pi}[R(\hat{\pi})]$ holds is incorrect.

6.3. Bayesian Brownian Bandit Problem. In Sections 6.1 and 6.2 we studied a family of optimal stopping problems w.r.t. a sequence of discrete processes whose weak limit is a Brownian motion with an unknown drift. In this section we provide an example of an optimal stopping problem for which the limit problem is the Bayesian Brownian bandit problem (see Berry and Friestedt (1985) [3], Bolton and Harris (1999) [5], Cohen and Solan (2013) [8]). We provide an asymptotically optimal solution by using Theorem 6.4 and Corollary 6.5. We also infer that if one calculates his strategy based on the distribution of $\tilde{\pi}$ instead of the distribution of $\lim_{n\to\infty} \tilde{\pi}^n$, then his payoff will be suboptimal.

¹⁸Notice that according to Eqs. (6.7) and (6.8), $\tilde{\tau}_D^n$ and $\tilde{\tau}_D^n$ are functions of the processes $\tilde{\pi}^n$ and $\hat{\pi}$ respectively.

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A DM operates a system in continuous time which can be of two types, High (H_l) or Low (H_0) . The DM observes the process $(L^n(t))$ where $n \in \mathbb{N}$ is fixed and updates his belief continuously about the hypotheses H_l and H_0 . For each job arriving to the system, the DM gets 1 dollar. In addition, he pays c^n dollars per time unit for operating the system. The choice that the DM should make is when to stop operating the system. Formally, the DM should choose a stopping time τ^n for the process $(L^n(t))$; that is, the stopping time is adapted to the filtration $\mathcal{F}_t^{L^n}$. The expected discounted loss of the DM if he chooses the stopping time τ^n is

(6.12)
$$V_{\tau^n}^n(\pi) := \sqrt{n} E^{\pi} \left[\int_0^{\tau^n} \frac{r}{n} e^{-\frac{r}{n}t} d(c^n t - L^n(t)) \right].$$

The goal of the DM is to minimize $V_{\tau}^{n}(\pi)$, and to find, if it exists, the optimal stopping time $\tau^{*,n}$ for which the infimum of (6.12) is attained.

We now present the cost function by using $\tilde{\pi}^n$. Since for every $k \in \{0, l\}$ one has $E[c^n t - L^n(t) \mid \theta = k] = (c^n - \mu_k^n)t$, we naturally assume that $\mu_0^n < c^n < \mu_l^n$. That is, the arrival rate is higher (resp. lower) in the High (resp. Low) type than the cost per time unit for operating the system; otherwise, the problem would be degenerate: if $\mu_0^n < \mu_l^n < c^n$ the DM will stop operating the system at time 0, while if $c^n < \mu_0^n < \mu_l^n$ he will operate it indefinitely. By standard arguments (see Cohen and Solan (2013, Lemma 4) [8]), one can represent the function $V_{\tau^n}^n(\pi)$ as follows:

(6.13)
$$V_{\tau^n}^n(\pi) = \sqrt{n} E^{\pi} \left[\int_0^{\tau^n} \frac{r}{n} e^{-\frac{r}{n}t} [(c^n - \mu_l^n) \boldsymbol{\pi}^n(t) + (c^n - \mu_0^n)(1 - \boldsymbol{\pi}^n(t))] dt \right],$$

which by Eq. (6.2) equals

(6.14)
$$V_{\tilde{\tau}^n}^n(\pi) = \sqrt{n} E^{\pi} \left[\int_0^{\tilde{\tau}^n} r e^{-rt} [(c^n - \mu_l^n) \tilde{\pi}^n(t) + (c^n - \mu_0^n) (1 - \tilde{\pi}^n(t))] dt \right].$$

That is, the cost functions k^n and K^n of the *n*-th system can be represented as follows:

(6.15)
$$k^{n}(\pi) = \sqrt{n}(c^{n} - \mu_{l}^{n})\pi + \sqrt{n}(c^{n} - \mu_{0}^{n})(1 - \pi), \ \pi \in [0, 1]$$

and

(6.16)
$$K^n(\pi) \equiv 0, \ \pi \in [0,1].$$

Suppose that for every $n \in \mathbb{N}$, $\mu_0^n < c^n < \mu_l^n$. Moreover, we need the following assumption that states that the High type is better than the Low type by an " $\frac{1}{\sqrt{n}}$ order style".

Assumption 6.7.

$$c_0 := \lim_{n \to \infty} \sqrt{n} (c^n - \mu_0^n) > 0 > \lim_{n \to \infty} \sqrt{n} (c^n - \mu_l^n) =: c_l$$

Assumption 6.7 says that the scaled limit of the difference between the operation cost and the arrival rate in the High (resp. Low) type yields a negative (resp. positive) expected loss. Under Assumption 6.7 it follows that k^n converges uniformly on [0, 1] to

(6.17)
$$k(\pi) := c_l \pi + c_0 (1 - \pi), \ \pi \in [0, 1].$$

6.3.1. Asymptotic Optimality. In this section we define cut-off strategies by using the notion of first exit time strategies of the posterior processes from an interval of the form $(\bar{p}, 1]$. We call \bar{p} the cut-off point. We prove that the *n*-th system admits a unique optimal stopping time and that it is a cut-off strategy. We will therefore restrict the class of stopping times to the class of cut-off strategies. We also show that for every cut-off point \bar{p} , the first exit time of the process $(\tilde{\pi}^n(t))$ from the interval $(\bar{p}, 1]$ and the payoff that is associated with this strategy, converge to the first exit time of the process $(\hat{\pi}(t))$ from that interval $(\bar{p}, 1]$ and the payoff that is associated with this strategy, respectively. We conclude this section by finding asymptotically optimal stopping time and the asymptotic value function.

We start with a few properties of the value function $U^n(\pi)$ and deduce that the optimal strategy in the *n*-th system is a cut-off strategy. The proof is similar to the proof of Proposition 2 in Cohen and Solan (2013) [8] and is therefore omitted.

Proposition 6.8. For every fixed $n \in \mathbb{N}$, the function $\pi \mapsto U^n(\pi)$ is monotone, nonincreasing, bounded from above by 0, concave, and continuous.

Remark 6.9. From Proposition 6.8 it follows that there is a cut-off point $p^{*,n}$ in (0, 1], such that $U^n(\pi) = 0$ if $\pi \leq p^{*,n}$, and $U^n(\pi) < 0$ otherwise. That is, the optimal strategy is to continue while the posterior lies in the interval $(p^{*,n}, 1]$, and to stop otherwise. We call this strategy a cut-off strategy with cut-off point $p^{*,n}$.

Berry and Friestedt (1985) [3] showed that the Bayesian Brownian bandit problem admits a unique optimal strategy and that it is a cut-off strategy w.r.t. the posterior process of the Brownian motion with the unknown drift. Denote by p^* the cut-off point that is associated with the optimal cut-off w.r.t. the limit process $\lim_{n\to\infty} \tilde{\pi}^n = \hat{\pi}$ (which is distributed as $\pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$). That is, for every $\pi \in [0, 1]$ one has $U(\pi) = V_{\tilde{\tau}_{(p^*, 1]}}(\pi)$. Recall that $(p^*, 1]$ is the continuation region for the posterior process. For every $n \in \mathbb{N}$ and every $\bar{p} \in [0, 1]$ define the continuation region $(\bar{p}, 1]$. The next result follows from Eqs. (6.15)– (6.17), Theorem 6.4, and Corollary 6.5.

Theorem 6.10. Fix $0 \le \overline{p} \le 1$. Under Assumptions 4.1, 4.2, 4.3, 5.1, 5.3, and 6.7, we have¹⁹

(6.18)
$$\lim_{n \to \infty} \tilde{\tau}^n_{(\bar{p},1]}(\pi) \stackrel{\mathrm{d}}{=} \tilde{\tau}_{(\bar{p},1]}(\pi),$$

(6.19)
$$\lim_{n \to \infty} V^n_{\tilde{\tau}^n_{(\bar{p},1]}}(\pi) = V_{\tilde{\tau}_{(\bar{p},1]}}(\pi)$$

and there exists $p^* \in [0, 1]$ such that

(6.20)
$$\lim_{n \to \infty} U^n(\pi) = \lim_{n \to \infty} V^n_{\tilde{\tau}^n_{(p^*, 1]}}(\pi) = V_{\tilde{\tau}_{(p^*, 1]}}(\pi) = U(\pi).$$

Remark 6.11. From Eq. (6.20) it follows that in order to find the asymptotically optimal cut-off point p^* , the DM must use the cut-off point taken from the optimal solution of the Bayesian Brownian bandit problem w.r.t. the posterior process $\pi_{\sqrt{\alpha}}$ and not w.r.t. the posterior process $\pi_{\sqrt{\alpha}\sigma_v}$. Denote by p_f^* and p_v^* the cut-off points that are

¹⁹The function $V_{\tilde{\tau}_{(\bar{p},1]}}(\pi)$ can be expressed explicitly through the parameters of the problem, but since it has no fundamental contribution, this expression is omitted (see Berry and Friestedt (1985, pp. 171–172) [3]).

associated with the Bayesian Brownian bandit problem w.r.t. the posteriors $\pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$ and $\pi_{\sqrt{\alpha}\sigma_v}$, respectively. Theorem 5.9 states that the difference between σ_v and $\frac{1}{\sigma_f}$ can be arbitrarily large and therefore the distributions of $\pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$ and $\pi_{\sqrt{\alpha}\sigma_v}$ can be relatively different, and so the difference between the optimal cut-off points p_f^* and p_v^* can be arbitrarily large within the interval [0, 1]. By Eq. (6.19) it follows that for every prior $\pi \in [0, 1]$ and for sufficiently large n, the payoff that is associated with the cut-off point p_v^* is approximately $V_{\tilde{\tau}_{(p_v^*,1]}}(\pi)$, which, by Eq. (6.11), is greater than $V_{\tilde{\tau}_{(p_f^*,1]}}(\pi) = U(\pi)$. The difference between these functions can be relatively large, see Berry and Friestedt (1985, pp. 171–172) [3] for closed-form formulas.

6.4. Discounted Sequential Testing. In this section we provide an example of an optimal stopping problem w.r.t. a sequence of discrete processes for which the limit problem is a discounted version of the sequential testing problem (Shiryaev (1978) [24]). We provide an asymptotically optimal solution by using Theorem 6.4 and Corollary 6.5. We also infer that if one calculates his strategy based on the distribution of $\tilde{\pi}$ instead of the distribution of $\lim_{n \to \infty} \tilde{\pi}^n$, then his payoff will be suboptimal.

Fix $n \in \mathbb{N}$. The DM observes the process $(L^n(t))$ and continuously updates his belief on the hypotheses H_l and H_0 . Using the belief process, his goal is to test sequentially these hypotheses with minimal loss. The choice that the DM should make is when to stop operating the system, and at that time to guess which one of the two hypotheses holds. Formally, the DM should choose a decision rule (τ^n, d^n) for $(L^n(t))$, that is, a stopping time τ^n that is adapted to the filtration $\mathcal{F}_t^{L^n}$, and a decision function d^n that is a $\mathcal{F}_{\tau}^{L^n}$ -measurable random variable taking the values 0 and l. The choice $d^n = l$ is interpreted to mean that the DM accepts H_l , while the choice $d^n = 0$ is interpreted to mean that the DM accepts H_0 . The expected loss of the DM under the decision rule (τ^n, d^n) is

(6.21)
$$Y_{(\tau^n,d^n)}^n(\pi) := E^{\pi} \left[\int_0^{\tau^n} \frac{r}{n} e^{-\frac{r}{n}t} c^n dt + \frac{r}{n} e^{-\frac{r}{n}\tau^n} (a^n \mathbb{I}_{(d^n=0,\theta=l)} + b^n \mathbb{I}_{(d^n=l,\theta=0)}) \right],$$

where a^n , b^n , and c^n are given positive constants that represent the cost of type II error, the cost of type I error, and the operation cost per unit of time, respectively. The goal of the DM is to minimize $Y^n_{(\tau^n, d^n)}(\pi)$, and to find, if exists, the optimal stopping rule $(\tau^{*,n}, d^{*,n})$ for which the infimum (6.21) is attained. Formally, let

$$U^{n}(\pi) := \inf_{(\tau^{n}, d^{n})} Y^{n}_{(\tau^{n}, d^{n})}(\pi)$$

be the minimal loss that the DM can achieve and in case that the infimum is attained, let

$$(\tau^{*,n}, d^{*,n})(\pi) \in \underset{(\tau^n, d^n)}{\arg\min} Y^n_{(\tau^n, d^n)}(\pi)$$

be an optimal decision rule, given that the prior belief is π .

We now present the cost function by using $\tilde{\pi}^n$. By standard arguments (see Shiryaev (1978, pp. 166–167)) [24], one can show that the optimal terminal decision $d^{*,n}$ exists

and satisfies $d^{*,n} = l$ if and only if $\tilde{\pi}^n(\tau^{*,n}) \ge \frac{b^n}{a^n + b^n}$. Therefore, we define

(6.22)
$$V_{\tau^{n}}^{n}(\pi) := Y_{(\tau^{n},d^{*,n})}^{n}(\pi)$$
$$= E^{\pi} \left[\int_{0}^{\tau^{n}} \frac{r}{n} e^{-\frac{r}{n}t} c^{n} dt + \frac{r}{n} e^{-\frac{r}{n}\tau^{n}} (a^{n} \pi^{n}(\tau^{n}) \wedge b^{n}(1 - \pi^{n}(\tau^{n}))) \right]$$

which from Eq. (6.2) equals

(6.23)

$$V_{\tilde{\tau}^{n}}^{n}(\pi) := Y_{(\tilde{\tau}^{n}, d^{*, n})}^{n}(\pi) = E^{\pi} \left[\int_{0}^{\tilde{\tau}^{n}} r e^{-rt} c^{n} dt + \frac{r}{n} e^{-r\tilde{\tau}^{n}} (a^{n} \tilde{\pi}^{n}(\tilde{\tau}^{n}) \wedge b^{n}(1 - \tilde{\pi}^{n}(\tilde{\tau}^{n}))) \right].$$

That is, the cost functions k^n and K^n of the *n*-th system can be represented as

(6.24) $k^n(\pi) = c^n, \ \pi \in [0,1]$

and

(6.25)
$$K^{n}(\pi) = a^{n}\pi \wedge b^{n}(1-\pi), \ \pi \in [0,1].$$

Suppose that the limits $\lim_{n\to\infty} a^n/n$, $\lim_{n\to\infty} b^n/n$, and $\lim_{n\to\infty} c^n$ exist and denote them by a, b, and c, respectively. It follows that k^n and K^n/n converge uniformly on [0, 1] to

(6.26)
$$k(\pi) = c, \ \pi \in [0, 1]$$

and

(6.27)
$$K(\pi) = a\pi \wedge b(1-\pi), \ \pi \in [0,1],$$

respectively.

6.4.1. Asymptotic Optimality. In this section we prove that the optimal stopping time in the *n*-th system exists uniquely and that it is the first exit time from an interval. We will therefore restrict the class of the stopping times that we consider to the class of first exit time strategies. We also show that for every interval (q_1, q_2) , the first exit time of the process $(\tilde{\pi}^n(t))$ from that interval and the payoff that is associated with this strategy converge to the first exit time of the process $(\hat{\pi}(t))$ from that interval and the payoff that is associated with this strategy, respectively. We conclude this section by finding the asymptotically optimal stopping time and asymptotic value function.

We start with a few properties of the value function $U^n(\pi)$ and deduce that the optimal strategy in the *n*-th system is a first exit time strategy. The proof is very similar to the proof of Theorem 1 in Shiryaev (1978, Ch. IV) [24] and is therefore omitted.

Proposition 6.12. For every fixed $n \in \mathbb{N}$, the function $\pi \mapsto U^n(\pi)$ is bounded from above by $K^n(\pi)/n$, concave, and continuous. Moreover, $U^n(0) = U^n(1) = 0$.

Remark 6.13. From Proposition 6.12 it follows that there are two points $0 \leq q_1^{*,n} < q_2^{*,n} \leq 1$, such that $U^n(\pi) = (a^n \pi \wedge b^n (1-\pi))/n = K^n(\pi)/n$ if $\pi \notin (q_1^{*,n}, q_2^{*,n})$, and $U^n(\pi) < K^n(\pi)$ otherwise. That is, the optimal strategy is the first exit time from the interval $(q_1^{*,n}, q_2^{*,n})$ (see the discussion in Shiryaev (1978, Ch. IV, pp. 168–169)) [24].

Proposition 6.12 and Remark 6.13 can be formulated for the limit problem as well. Therefore, one can deduce that there exists an optimal stopping time that is associated with the continuation region $D^* = (q_1^*, q_2^*)$. For every $n \in \mathbb{N}$, every $\pi \in [0, 1]$, and every $q_1 < q_2 \in [0, 1]$, define the continuation region (q_1, q_2) . The next theorem follows from Eqs. (6.24)–(6.27), Theorem 6.4, and Corollary 6.5. **Theorem 6.14.** Fix $0 \le q_1 < q_2 \le 1$. Under Assumptions 4.1, 4.2, 4.3, 5.1, and 5.3, the following limits hold:²⁰

(6.28)
$$\lim_{n \to \infty} \tilde{\tau}^n_{(q_1, q_2)}(\pi) \stackrel{\mathrm{d}}{=} \tilde{\tau}_{(q_1, q_2)}(\pi),$$

(6.29)
$$\lim_{n \to \infty} V^n_{\tilde{\tau}^n_{(q_1, q_2)}}(\pi) = V_{\tilde{\tau}_{(q_1, q_2)}}(\pi),$$

and there are two points $0 \le q_1^* < q_2^* \le 1$ such that

$$\lim_{n \to \infty} U^n(\pi) = \lim_{n \to \infty} V^n_{\tilde{\tau}^n_{(q_1^*, q_2^*)}}(\pi) = V_{\tilde{\tau}_{(q_1^*, q_2^*)}}(\pi) = U(\pi).$$

The analog to Remark 6.11 to this model holds.

7. Conclusion

7.1. Summary. In this paper we studied a problem of estimating a parameter θ . We started with a sequence of scaled counting processes $\{(\tilde{L}^n_{\theta}(t))\}_n$ whose distributions depend on an unknown parameter θ , the prior distribution of which is known. Moreover, we assumed that $\{(\tilde{L}^n_{\theta}(t))\}_n$ converges in distribution to a Brownian motion $(\tilde{L}_{\theta}(t))$ with an unknown drift (θt) . We defined by $(\tilde{\pi}^n(t))$ the posterior distribution process of the parameter θ , given the observations $(\tilde{L}^n_{\theta}(s))_{s\leq t}$ and by $(\tilde{\pi}(t))$ the posterior distribution process of the parameter θ , given the observations $(\tilde{L}_{\theta}(s))_{s\leq t}$. We showed that, generally, $\lim_{n\to\infty} \tilde{\pi}^n \neq \tilde{\pi}$, unless the counting processes satisfy a memorylessness property and no information, regarding the posterior processes, is lost by looking at the present of the counting processes rather than at their past and present.

We also proved that the limit process $\lim_{n\to\infty} \tilde{\pi}^n$ equals to a posterior distribution process of the process $(\hat{M}_{\theta}(t))$, which is a Brownian motion with the same unknown drift and a different standard deviation coefficient than the one of $(\tilde{L}_{\theta}(t))$. Apparently, the difference between the standard deviation coefficients of $(\tilde{L}_{\theta}(t))$ and $(\hat{M}_{\theta}(t))$ can be arbitrarily large. Therefore, we concluded that results concerning optimal stopping problems w.r.t. $(\tilde{L}_{\theta}(t))$ cannot be applied to optimal stopping problems w.r.t. $(\tilde{L}_{\theta}^n(t))$, as the difference in the performance can be arbitrarily bad.

7.2. Future Directions.

7.2.1. The Disorder Problem, Diffusion Approximations, and Queues. The Brownian disorder problem was introduced in Shiryaev (1978) [24].²¹ In this problem, the drift of a Brownian motion changes at some unknown and unobservable disorder time. The objective is to detect this change as quickly as possible after it happens. This problem is also studied by using the Bayesian posterior process, that now estimates the probability that the drift has already changed, based on the past information. I managed to show that the Bayesian posterior distribution process of a disorder discrete process that is close in distribution to a disorder Brownian motion, has a similar structure to the posterior

²⁰As in Section 6.3, the function $V_{\tilde{\tau}_{(q_1,q_2)}}(\pi)$ can be expressed explicitly through the parameters of the problem, but since it has no fundamental contribution, this expression is omitted.

²¹This model was generalized in the context of Brownian motion by, e.g., Vellekoop and Clark (2001) [25], Gapeev and Peskir (2006) [13], Dayanik (2010) [9], Sezer (2010) [23], and in the context of other processes different from the Brownian motion, e.g., Peskir and Shiryaev (2002) [21], Gapeev (2005) [11], and Bayraktar, Dayanik, and Karatzas (2006) [1].

distribution process in our paper. I would like to apply this result to optimal stoppingtime problem in the context of a G/G/1 queue under heavy traffic where one of the parameters of the model such as the arrival/service rate changes randomly.

I believe that 'disorder queues' can enrich the classical models, as it often happens in real life situations that the parameters of the system change over time.

7.2.2. Parameter Estimation in General Diffusion Processes. The structure of the limit process $\lim_{n\to\infty} \tilde{\pi}^n$ is surprising and raises further questions about the structure of Bayesian posterior distribution processes of more general diffusion processes with uncertainty. I plan to study an approximation for a model suggested by Zakai (1969) [27]. This model is fundamental in filtering theory and signal processing. Zakai analyzed a model with a diffusion process (X(t)) satisfying the stochastic differential equation

(7.1)
$$X(t) = X(0) + \int_0^t a(X(s))ds + \int_0^t b(X(s))dW_1(s)ds$$

where X(0) is a random variable, $(W_1(t))$ is a Brownian motion, and a and b are realvalued functions such that $b \neq 0$. Let (L(t)) be the observed process which is related to (X(t)) by

(7.2)
$$L(t) = \int_0^t g(X(s))ds + \int_0^t \sigma dW_2(s),$$

where $(W_2(t))$ is a Brownian motion, g is a real-valued function, and σ is a positive constant. Notice that if g is the identity function and if a = b = 0 then $X(t) \equiv X(0)$ and (L(t)) is a Brownian motion with an unknown linear drift (X(0)t). This is the model that we studied in this paper with $\theta = X(0)$. Zakai presented an equation that is satisfied by the unnormalized Bayesian posterior distribution process of the location of (X(t)) given the observation $(L(s))_{0 \leq s \leq t}$, commonly known as the Zakai equation, see Zakai (1969, equation (11)) [27]. I would like to consider a sequence of processes $\{(X^n(t), L^n(t))\}_{n \in \mathbb{N}}$ that converges in distribution to (X(t), L(t)) and to analyze the limit of the Bayesian posterior distribution processes

$$p^{n}(t,l) := P(\theta = l \mid (L^{n}(s))_{0 \le s \le t}), \ t \in [0,\infty), \ l \in S.$$

I would like to see whether the limit of p^n exists, under proper scaling of the parameters, the functions and the processes, and if so, what is its structure and when can it be considered as the Bayesian posterior distribution process of another process (X'(t), L'(t)) that satisfies Eqs. (7.1)–(7.2) with some a', b', g', and σ' .

This research can shed a light on the behavior of Bayesian posterior distribution processes in more general and realistic models, where the process (X(t)) evolves randomly over time.

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8. APPENDIX - PROOFS

For the proofs of Theorems 5.4 and 6.4 it is convenient to present a precise probability space on which the sequence of random parameter systems is defined.

8.1. **Probability Space.** Let θ be a random variable defined on the probability space $(\Omega_{\theta}, \mathcal{F}_{\theta}, P_{\theta})$. Denote the support of θ by $S \subseteq \mathbb{R}$ and suppose that S is bounded and countable. For every $l \in S$, let $\pi_l := P_{\theta}(\theta = l)$. Let $(\Omega_V, \mathcal{F}_V, P_V)$ be a probability space on which a sequence of i.i.d. random variables $\{u_i\}_{i\geq 1}$ is defined such that for every $i \geq 2$, u_i is distributed as the random variable v that satisfies Assumption 4.2. Define the probability space $(\Omega_{\theta} \times \Omega_V, \sigma(\mathcal{F}_{\theta} \times \mathcal{F}_V), P)$ such that for every $A_1 \times A_2 \subseteq S \times \Omega_V$ one has $P(A_1 \times A_2) = P_{\theta}(A_1)P_V(A_2)$. Therefore, $\{u_i\}_{i\geq 1}$ and θ are independent with respect to the probability function P. Denote by P_l the probability measure over $\Omega_{\theta} \times \Omega_V$ given $\theta = l$. That is, for every $l \in S$, and every $C \subseteq S \times \Omega_V$, set $P_l(C) := P(C \mid \theta = l)$.

8.2. The *n*-th System. Let t_u be a parameter and u_1 be the random variable defined in the previous paragraph, such that for every $t \in [0, \infty)$ one has

(8.1)
$$P(u_1 + t_u \ge t) = P(v \ge t \mid v \ge t_u)$$

For every $l \in S$ and every $n \in \mathbb{N}$, define the parameter μ_l^n . For every $i \ge 1$, every $l \in S$, and every $n \in \mathbb{N}$ define $v_{i,l}^n := \frac{u_i}{\mu_l^n}$ and $t_v^n := \frac{t_u}{\mu_l^n}$. For every $n \in \mathbb{N}$, let

$$\mathcal{RS}^{n}_{\pi}(\theta) = (t^{n}_{v}, v, \mu^{n}, \{v^{n}_{i}\}_{i \geq 1}, \pi) = \sum_{l \in S} \mathbb{I}_{\{\theta = l\}} \left(t^{n}_{v}, v, \mu^{n}_{l}, \{v^{n}_{i,l}\}_{i \geq 1}\right)$$

be a sequence of random parameter systems. For every $n \in \mathbb{N}$ this construction generates the random parameter system that was defined in Section 4.3. Notice that for every $l \in S$ and every $t \in [0, \infty)$ one has

$$P_l\left(v_{1,l}^n + t_v^n \ge t\right) = P_l\left(\frac{u_1}{\mu_l^n} + \frac{t_v}{\mu_l^n} \ge t\right) = P_l\left(u_1 + t_v \ge t\mu_l^n\right) = P\left(v \ge t\mu_l^n | v \ge t_u\right)$$
$$= P\left(\frac{v}{\mu_l^n} \ge t \middle| \frac{v}{\mu_l^n} \ge \frac{t_u}{\mu_l^n}\right) = P\left(\frac{v}{\mu_l^n} \ge t \middle| \frac{v}{\mu_l^n} \ge t_v^n\right),$$

where the third equality follows from Eq. (8.1). Therefore, for every $n \in \mathbb{N}$ Assumption 4.1 is satisfied.

8.3. **Proof of Theorem 5.4.** We divide the proof into two parts. We first prove Eq. (5.4) and thereafter conclude Eq. (5.5).

8.3.1. **Proof of Eq. (5.4)**. Recall that by the definition of φ_{σ} , the process $\varphi_{\frac{\sqrt{\alpha}}{\sigma_f}}$ satisfies

(8.2)
$$\varphi_{\frac{\sqrt{\alpha}}{\sigma_f}}(l,t) = \exp\left\{\frac{l\sigma_f}{\sqrt{\alpha}}W'(t) - \frac{1}{2}\left(\frac{l\sigma_f}{\sqrt{\alpha}}\right)^2 t + \frac{\theta\sigma_f}{\sqrt{\alpha}} \cdot \frac{l\sigma_f}{\sqrt{\alpha}}t\right\}, \ l \in S, \ t \in [0,\infty),$$

where (W'(t)) is a standard Brownian motion independent of θ .

In order to prove Eq. (5.4) it suffices to prove that

(8.3)
$$\lim_{n \to \infty} \ln(\tilde{\varphi}^n) \stackrel{\mathrm{d}}{=} \ln(\varphi_{\frac{\sqrt{\alpha}}{\sigma_f}}).$$

Denote

$$\tilde{W}^n(t) := \frac{\sum_{i=1}^{\lfloor nt \rfloor} \left[-\frac{f'(u_i)}{f(u_i)} u_i \right] - nt}{\sigma_f \sqrt{n}}, \quad t \in [0, \infty),$$
$$\bar{L}^n(t) := \frac{L^n(nt)}{n}, \quad t \in [0, \infty),$$

and

(8.4)
$$\tilde{\boldsymbol{\zeta}}^{\boldsymbol{n}}(l,t) := \sigma_f \sqrt{n} \frac{\mu_l^n - \mu_0^n}{\mu_\theta^n} \tilde{W}^n(\bar{L}^n(t)) - \frac{1}{2} \left(\frac{l\sigma_f}{\alpha}\right)^2 \bar{L}^n(t) + \frac{\theta\sigma_f}{\alpha} \cdot \frac{l\sigma_f}{\alpha} \bar{L}^n(t), \ l \in S, \ t \in [0,\infty).$$

From Eq. (8.3) and Theorem 3.1 in Billingsley (1999) [4] it follows that in order to prove Eq. (5.4) it suffices to prove that

(8.5)
$$\lim_{n \to \infty} (\ln(\tilde{\varphi}^n) - \tilde{\zeta}^n) = 0 \text{ u.o.c. (Proposition 8.1)}$$

and

(8.6)
$$\lim_{n \to \infty} \tilde{\boldsymbol{\zeta}}^n \stackrel{\mathrm{d}}{=} \ln(\boldsymbol{\varphi}_{\frac{\sqrt{\alpha}}{\sigma_f}}) \quad (\text{Proposition 8.4}).$$

Proposition 8.1 (Proving Eq. (8.5)). Under Assumptions 4.1, 4.2, 4.3, 5.1, and 5.3, the following holds:

(8.7)
$$\lim_{n \to \infty} (\ln(\tilde{\boldsymbol{\varphi}}^n) - \tilde{\boldsymbol{\zeta}}^n) = 0 \text{ u.o.c.}$$

Proof. The following series of equations presents the Radon–Nikodým derivative $(\tilde{\varphi}^n(l,t))$ in a more convenient form. For every $l \in S$ and every $t \in [0,\infty)$ one has

$$(8.8) \qquad \tilde{\varphi}^{n}(l,t) = \frac{f_{l}^{n}(v_{1}^{n}|v_{1}^{n} > t_{v}^{n})}{f_{0}^{n}(v_{1}^{n}|v_{1}^{n} > t_{v}^{n})} \cdot \frac{\prod_{i=2}^{L^{n}(nt)} f_{l}^{n}(v_{i}^{n})}{\prod_{i=2}^{L^{n}(nt)} f_{0}^{n}(v_{i}^{n})} \\ \cdot \frac{P_{l}\left(v_{L^{n}(nt)+1}^{n} > nt - \sum_{i=1}^{L^{n}(nt)} v_{i}^{n} \mid \sum_{i=1}^{L^{n}(nt)} v_{i}^{n}\right)}{P_{0}\left(v_{L^{n}(nt)+1}^{n} > nt - \sum_{i=1}^{L^{n}(nt)} v_{i}^{n} \mid \sum_{i=1}^{L^{n}(nt)} v_{i}^{n}\right)} \\ = \frac{P_{0}^{n}(v_{1}^{n} > t_{v}^{n})}{P_{l}^{n}(v_{1}^{n} > t_{v}^{n})} \cdot \frac{\prod_{i=1}^{L^{n}(nt)} f_{l}^{n}(v_{i}^{n})}{\prod_{i=1}^{L^{n}(nt)} f_{0}^{n}(v_{i}^{n})} \\ \cdot \frac{P_{l}\left(v_{L^{n}(nt)+1}^{n} > nt - \sum_{i=1}^{L^{n}(nt)} v_{i}^{n} \mid \sum_{i=1}^{L^{n}(nt)} v_{i}^{n}\right)}{P_{0}\left(v_{L^{n}(nt)+1}^{n} > nt - \sum_{i=1}^{L^{n}(nt)} v_{i}^{n} \mid \sum_{i=1}^{L^{n}(nt)} v_{i}^{n}\right)} \\ = \exp\left\{\sum_{i=1}^{L^{n}(nt)} \ln\left(\frac{f_{l}^{n}(v_{i}^{n})}{f_{0}^{n}(v_{i}^{n})}\right) + \ln\left(\frac{1 - F_{0}^{n}(t_{v})}{1 - F_{l}^{n}(t_{v})}\right) \\ + \ln\left(\frac{1 - F_{l}^{n}\left(nt - \sum_{i=1}^{L^{n}(nt)} v_{i}^{n} \mid \sum_{i=1}^{L^{n}(nt)} v_{i}^{n}\right)}{1 - F_{0}^{n}\left(nt - \sum_{i=1}^{L^{n}(nt)} v_{i}^{n} \mid \sum_{i=1}^{L^{n}(nt)} v_{i}^{n}\right)}\right)\right\}.$$

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From Eq. (8.9) and the triangle inequality it follows that, for every $l \in S$ and $t \in [0, \infty)$, (8.10)

$$\begin{split} &|\ln(\tilde{\varphi}^{n})(l,t) - \tilde{\zeta}^{n}(l,t)| \\ \leq \left| \sum_{i=1}^{L^{n}(nt)} \ln\left(\frac{f_{\theta}^{n}(v_{i}^{n})}{f_{0}^{n}(v_{i}^{n})}\right) + \ln\left(\frac{1 - F_{0}^{n}(t_{v}^{n})}{1 - F_{\theta}^{n}(t_{v}^{n})}\right) + \ln\left(\frac{1 - F_{\theta}^{n}\left(nt - \sum_{i=1}^{L^{n}(nt)} v_{i}^{n} \mid \sum_{i=1}^{L^{n}(nt)} v_{i}^{n}\right)}{1 - F_{0}^{n}\left(nt - \sum_{i=1}^{L^{n}(nt)} v_{i}^{n} \mid \sum_{i=1}^{L^{n}(nt)} v_{i}^{n}\right)} \right) \\ &- \sigma_{f}\sqrt{n}\frac{\mu_{0}^{n} - \mu_{\theta}^{n}}{\mu_{\theta}^{n}}\tilde{W}^{n}(\bar{L}^{n}(t)) - \frac{1}{2}\left(\frac{(0 - \theta)\sigma_{f}}{\alpha}\right)^{2}\bar{L}^{n}(t) \right| \\ &+ \left| \sum_{i=1}^{L^{n}(nt)} \ln\left(\frac{f_{\theta}^{n}(v_{i}^{n})}{f_{l}^{n}(v_{i}^{n})}\right) - \ln\left(\frac{1 - F_{l}^{n}(t_{v}^{n})}{1 - F_{\theta}^{n}(t_{v}^{n})}\right) - \ln\left(\frac{1 - F_{l}^{n}(t_{v}^{n})}{1 - F_{\theta}^{n}(t_{v}^{n})}\right) - \ln\left(\frac{1 - F_{l}^{n}(nt - \sum_{i=1}^{L^{n}(nt)} v_{i}^{n} \mid \sum_{i=1}^{L^{n}(nt)} v_{i}^{n}\right) \right) \\ &- \sigma_{f}\sqrt{n}\frac{\mu_{l}^{n} - \mu_{\theta}^{n}}{\mu_{\theta}^{n}}\tilde{W}^{n}(\bar{L}^{n}(t)) - \frac{1}{2}\left(\frac{(l - \theta)\sigma_{f}}{\alpha}\right)^{2}\bar{L}^{n}(t) \right|. \end{split}$$

We prove that the second term on the right-hand side of Eq. (8.10) converges to zero u.o.c. The proof for the first term is similar and is therefore omitted. From the triangle inequality it follows it is sufficient to verify that the following two processes converge to zero u.o.c.:

(8.11)
$$\tilde{\boldsymbol{\xi}}^{\boldsymbol{n}}(l,t) := \ln\left(\frac{1 - F_l^{\boldsymbol{n}}(t_v^{\boldsymbol{n}})}{1 - F_{\theta}^{\boldsymbol{n}}(t_v^{\boldsymbol{n}})}\right) + \ln\left(\frac{1 - F_{\theta}^{\boldsymbol{n}}\left(nt - \sum_{i=1}^{L^n(nt)} v_i^{\boldsymbol{n}} \mid \sum_{i=1}^{L^n(nt)} v_i^{\boldsymbol{n}}\right)}{1 - F_l^{\boldsymbol{n}}\left(nt - \sum_{i=1}^{L^n(nt)} v_i^{\boldsymbol{n}} \mid \sum_{i=1}^{L^n(nt)} v_i^{\boldsymbol{n}}\right)}\right),$$

 $l \in S, \ t \in [0, \infty)$, and (8.12)

$$\tilde{\boldsymbol{\chi}}^{\boldsymbol{n}}(l,t) := \sum_{i=1}^{n\bar{L}^{n}(t)} \ln\left(\frac{f_{\theta}^{n}(v_{i}^{n})}{f_{l}^{n}(v_{i}^{n})}\right) - \sigma_{f}\sqrt{n}\frac{\mu_{l}^{n}-\mu_{\theta}^{n}}{\mu_{\theta}^{n}}\tilde{W}^{n}(\bar{L}^{n}(t)) - \frac{1}{2}\left(\frac{(l-\theta)\sigma_{f}}{\alpha}\right)^{2}\bar{L}^{n}(t),$$

 $l \in S, \ t \in [0,\infty).$ We prove these convergence in Lemma 8.2 and Lemma 8.3, respectively.

Lemma 8.2. Under Assumptions 4.1, 4.2, 4.3, and 5.3.3,

(8.13)
$$\lim_{n \to \infty} \tilde{\boldsymbol{\xi}}^n = 0 \quad \text{u.o.c.}$$

Proof. To prove Eq. (8.13) it suffices to show that for every T > 0 the following two equalities hold:

(8.14)
$$P\left(\lim_{n \to \infty} \sup_{S \times [0,T]} \left| \ln\left(\frac{1 - F_l^n(t_v^n)}{1 - F_{\theta}^n(t_v^n)}\right) \right| = 0 \right) = 1$$

and

(8.15)
$$P\left(\lim_{n \to \infty} \sup_{S \times [0,T]} \left| \ln\left(\frac{1 - F_l^n\left(nt - \sum_{i=1}^{L^n(nt)} v_i^n \mid \sum_{i=1}^{L^n(nt)} v_i^n\right)}{1 - F_{\theta}^n\left(nt - \sum_{i=1}^{L^n(nt)} v_i^n \mid \sum_{i=1}^{L^n(nt)} v_i^n\right)}\right) \right| = 0\right) = 1.$$

We prove only Eq. (8.15). The proof of Eq. (8.14) is similar and is therefore omitted. The following series of equations, which holds for sufficiently large $n \in \mathbb{N}$, yields an upper bound for the expression $\sup_{S \times [0,T]} \left| \ln \left(\frac{1 - F_l^n \left(nt - \sum_{i=1}^{L^n(nt)} v_i^n | \sum_{i=1}^{L^n(nt)} v_i^n \right)}{1 - F_{\theta}^n \left(nt - \sum_{i=1}^{L^n(nt)} v_i^n | \sum_{i=1}^{L^n(nt)} v_i^n \right)} \right) \right|$ (8.16) $\sup_{S \times [0,T]} \left| \ln \left(1 - F \left(\mu_l^n \left(nt - \sum_{i=1}^{L^n(nt)} v_i^n | \sum_{i=1}^{L^n(nt)} v_i^n \right) \right) \right|$ $- \ln \left(1 - F \left(\mu_{\theta}^n \left(nt - \sum_{i=1}^{L^n(nt)} v_i^n \right) \right) \left| \sum_{i=1}^{L^n(nt)} v_i^n \right) \right) \right|$ (8.17) $= \sup_{S \times [0,T]} \left| \mu_{\theta}^n - \mu_l^n \right| \left(nt - \sum_{i=1}^{L^n(nt)} v_i^n \right) \right| \frac{f \left(d_l^n \left(nt - \sum_{i=1}^{L^n(nt)} v_i^n \right) \right)}{1 - F \left(d_l^n \left(nt - \sum_{i=1}^{L^n(nt)} v_i^n \right) \right)} \right|$ (8.18) $\leq \sup_{S \times [0,T]} \frac{\sqrt{n} |\mu_{\theta}^n - \mu_l^n|}{d_l^n} \frac{1}{\sqrt{n}} N \left(d_l^n \left(nt - \sum_{i=1}^{L^n(nt)} v_i^n \right) \right)$

(8.19)
$$\leq \sup_{S} \frac{\sqrt{n}|\mu_{\theta}^{n} - \mu_{l}^{n}|}{d_{l}^{n}} \cdot \sup_{[0,T]} \frac{1}{\sqrt{n}} N\left((1+\epsilon_{N})\mu_{\theta}^{n} v_{L^{n}(nt)+1}^{n}\right)$$

(8.20)
$$= \sup_{S} \frac{\sqrt{n} |\mu_{\theta}^{n} - \mu_{l}^{n}|}{d_{l}^{n}} \cdot \sup_{[0,T]} \frac{1}{\sqrt{n}} N\left((1+\epsilon_{N}) u_{L^{n}(nt)+1}\right),$$

where $d_l^n \in (\mu_{\theta}^n, \mu_l^n)$ or $d_l^n \in (\mu_l^n, \mu_{\theta}^n)$. Eq. (8.16) follows from Remark 5.2, while Eq. (8.17) follows from the Lagrange mean value theorem. Inequality (8.18) follows from Assumption 5.3.3 and the fact that N(x) is monotone nondecreasing. Inequality (8.19) follows since, by Eq. (4.5),

(8.21)
$$\lim_{n \to \infty} \sup_{S} |d_l^n - \mu_{\theta}^n| \le \limsup_{n \to \infty} \sup_{S} |\mu_l^n - \mu_{\theta}^n| = 0,$$

and since N(x) is monotone nondecreasing. Eq. (8.20) follows since for every $i \ge 1$ and every $n \in \mathbb{N}$ one has $\mu_{\theta}^n v_i^n = u_i$ (see Section 8.1). Assumption 5.3.3 implies that $E[N((1 + \epsilon_N)v)]^2 < \infty$ and therefore

(8.22)
$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} N\left((1+\epsilon_N) u_{L^n(nt)+1} \right) = \lim_{n \to \infty} \sqrt{\frac{1}{n} N^2 \left((1+\epsilon_N) u_{L^n(nt)+1} \right)} = 0 \text{ u.o.c.}$$

Finally, from Assumption 4.3.1 it follows that

$$(8.23) \quad \lim_{n \to \infty} \sup_{S \times [0,T]} \sqrt{n} |\mu_l^n - \mu_\theta^n| \leq \lim_{n \to \infty} \sup_{S} |\sqrt{n}(\mu_l^n - \mu_0^n) - l| + \sup_{S} |\sqrt{n}(\mu_\theta^n - \mu_0^n) - \theta| + \sup_{S} |l| + |\theta| < \infty,$$

where the last inequality follows since S is bounded. Eqs. (8.22)-(8.23) imply that the right-hand side of Eq. (8.20) converges to 0 u.o.c.

Lemma 8.3. Under Assumptions 4.2, 4.3, 5.1, 5.3.1, and 5.3.2, (8.24) $\lim_{n \to \infty} \tilde{\chi}^n(l,t) = 0 \text{ u.o.c.}$

Proof. Eq. (8.24) is equivalent to the requirement that

(8.25)
$$P\left(\lim_{n \to \infty} \sup_{S \times [0,T]} |\tilde{\boldsymbol{\chi}}^{\boldsymbol{n}}(l,t)| = 0\right) = 1$$

for every T > 0. The first term in Eq. (8.12) is $\sum_{i=1}^{\bar{L}^n(t)} \ln\left(\frac{f_{\theta}^n(v_i^n)}{f_l^n(v_i^n)}\right)$ which is a composition of

$$\sum_{i=1}^{\lfloor nt \rfloor} \ln \left(\frac{f_{\theta}^n(v_i^n)}{f_l^n(v_i^n)} \right)$$

and $\overline{L}^n(t)$. For every $l \in S$ denote

(8.26)
$$\hat{\mu}_l^n := \frac{\mu_l^n - \mu_\theta^n}{\mu_\theta^n}$$

The following series of equations presents $\sum_{i=1}^{\lfloor nt \rfloor} \ln \left(\frac{f_{\theta}^n(v_i^n)}{f_l^n(v_i^n)} \right)$ in a more convenient form: (8.27)

$$\sum_{i=1}^{\lfloor nt \rfloor} \ln\left(\frac{f_{\theta}^{n}(v_{i}^{n})}{f_{l}^{n}(v_{i}^{n})}\right) = \sum_{i=1}^{\lfloor nt \rfloor} \ln\left(\frac{\mu_{\theta}^{n}f(\mu_{\theta}^{n}v_{i}^{n})}{\mu_{l}^{n}f(\mu_{l}^{n}v_{i}^{n})}\right)$$
(8.28)
$$\lfloor nt \rfloor$$

$$= -nt\ln(1+\hat{\mu}_{l}^{n}) - \sum_{i=1}^{\lfloor nn \rfloor} \left[\ln\left(f(\mu_{l}^{n}v_{i}^{n})\right) - \ln\left(f(\mu_{\theta}^{n}v_{i}^{n})\right)\right]$$

(8.29)

$$= -nt \ln(1 + \hat{\mu}_l^n) - \sum_{i=1}^{\lfloor nt \rfloor} \left[\ln \left(f(u_i + u_i \hat{\mu}_l^n) \right) - \ln \left(f(u_i) \right) \right]$$

(8.30)

(

$$= -nt\left(\hat{\mu}_{l}^{n} - \frac{1}{2}(\hat{\mu}_{l}^{n})^{2}\right) - nt\left(\ln(1+\hat{\mu}_{l}^{n}) - \hat{\mu}_{l}^{n} + \frac{1}{2}(\hat{\mu}_{l}^{n})^{2}\right) \\ - \sum_{i=1}^{\lfloor nt \rfloor} \left[\frac{f'(u_{i})}{f(u_{i})}u_{i}\hat{\mu}_{l}^{n} + \frac{1}{2!}\left(\frac{f'(u_{i})}{f(u_{i})}\right)'u_{i}^{2}(\hat{\mu}_{l}^{n})^{2} + \frac{1}{3!}\left(\frac{f'(c_{i,l}^{n})}{f(c_{i,l}^{n})}\right)''u_{i}^{3}(\hat{\mu}_{l}^{n})^{3}\right]$$

$$(8.31)$$

$$=\sigma_{f}\sqrt{n}\hat{\mu}_{l}^{n}\tilde{W}^{n}(t) + \frac{1}{2}(\sqrt{n}\hat{\mu}_{l}^{n})^{2}\left(t - \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{f'(u_{i})}{f(u_{i})}\right)'u_{i}^{2}\right) + \frac{(\sqrt{n}\hat{\mu}_{l}^{n})^{3}}{3!} \cdot \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{n^{1.5}}\left(\frac{f'(c_{i,l}^{n})}{f(c_{i,l}^{n})}\right)''u_{i}^{3} + (\sqrt{n}\hat{\mu}_{l}^{n})^{2}\left(\frac{\ln(1+\hat{\mu}_{l}^{n})-\hat{\mu}_{l}^{n}+\frac{1}{2}(\hat{\mu}_{l}^{n})^{2}}{(\hat{\mu}_{l}^{n})^{2}}\right)t,$$

where $c_{i,l}^n \in (u_i, u_i + u_i \hat{\mu}_l^n)$ or $c_{i,l}^n \in (u_i + u_i \hat{\mu}_l^n, u_i)$. Eq. (8.28) follows from Remark 5.2 and the definition of $\hat{\mu}_l^n$. Eq. (8.29) follows by the definition of u_i . Since $f \in \mathcal{C}^3$, Eq. (8.30)

follows from the Taylor expansion of the function $\ln(f(x))$ with Lagrange remainder of order 3. Eq. (8.31) is merely a rearrangement of the terms. From Eqs. (8.12) and (8.31) it follows that for every $l \in S$ and $t \in [0, \infty)$ one has

(8.32)

$$\begin{split} \tilde{\boldsymbol{\chi}}^{n}(l,t) &= \sum_{i=1}^{n\bar{L}^{n}(t)} \ln\left(\frac{f_{\theta}^{n}(v_{i}^{n})}{f_{l}^{n}(v_{i}^{n})}\right) - \sigma_{f}\sqrt{n}\hat{\mu}_{l}^{n}\tilde{W}^{n}(\bar{L}^{n}(t)) - \frac{1}{2}\left(\frac{(l-\theta)\sigma_{f}}{\alpha}\right)^{2}\bar{L}^{n}(t) \\ (8.33) &= \left[-\frac{1}{2}\left(\frac{(l-\theta)\sigma_{f}}{\alpha}\right)^{2}\bar{L}^{n}(t) + \frac{1}{2}(\sqrt{n}\hat{\mu}_{l}^{n})^{2}\left(\bar{L}^{n}(t) - \frac{1}{n}\sum_{i=1}^{n\bar{L}^{n}(t)}\left(\frac{f'(u_{i})}{f(u_{i})}\right)'u_{i}^{2}\right)\right] \\ &+ \frac{(\sqrt{n}\hat{\mu}_{l}^{n})^{3}}{3!} \cdot \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{n^{1.5}}\left(\frac{f'(c_{i,l}^{n})}{f(c_{i,l}^{n})}\right)''u_{i}^{3} \\ &+ (\sqrt{n}\hat{\mu}_{l}^{n})^{2}\left(\frac{\ln(1+\hat{\mu}_{l}^{n}) - \hat{\mu}_{l}^{n} + \frac{1}{2}(\hat{\mu}_{l}^{n})^{2}}{(\hat{\mu}_{l}^{n})^{2}}\right)\bar{L}^{n}(t). \end{split}$$

We are now ready to prove Eq. (8.24). We show that each of the three terms on the right-hand side of Eq. (8.33) converges to zero u.o.c.

Part I: First term. Define the following functions and processes:

(8.34)
$$g_1(l) := \frac{l-\theta}{\alpha}, \ l \in S,$$

(8.35)
$$g_1^n(l) := \sqrt{n}\hat{\mu}_l^n = \sqrt{n}\frac{\mu_l^n - \mu_\theta^n}{\mu_\theta^n}, \ l \in S,$$

(8.36)
$$G_1(l) := (1 - \sigma_f^2)t, \ t \in [0, \infty),$$

(8.37)
$$G_1^n(l) := \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{f'(u_i)}{f(u_i)} \right)' u_i^2, \ t \in [0, \infty),$$

and

(8.38)
$$\bar{L}(t) = \alpha t, \ t \in [0, \infty).$$

Therefore, the first term in Eq. (8.32) can be expressed as

(8.39)
$$-\frac{1}{2} (g_1(l))^2 \sigma_f^2 \bar{L}^n(t) + \frac{1}{2} (g_1^n(l))^2 \left(\bar{L}^n(t) - G_1^n(\bar{L}^n(t)) \right), \quad l \in S, \ t \in [0, \infty).$$

From the definition of $\hat{\mu}_l^n$ it follows that

(8.40)
$$(\sqrt{n}\hat{\mu}_l^n) = \left(\frac{\sqrt{n}(\mu_l^n - \mu_\theta^n)}{\mu_\theta^n}\right) = \left(\frac{\sqrt{n}(\mu_l^n - \mu_0^n)}{\mu_\theta^n} + \frac{\sqrt{n}(\mu_\theta^n - \mu_0^n)}{\mu_\theta^n}\right).$$

Assumption 4.3 and Eq. (8.40) implies that

(8.41)
$$\lim_{n \to \infty} g_1^n = g_1 \quad \text{u.o.c.}$$

From Lemma 5.8 (Eq. (5.10)) and the Functional Strong Law of Large Numbers (FSLLN, see Chen and Yao (2001, Theorem 5.10) [7]) it follows that

(8.42)
$$\lim_{n \to \infty} G_1^n = G_1 \quad \text{u.o.c.}$$

Next, Assumption 4.3.2 and the FSLLN imply that

(8.43)
$$\lim_{n \to \infty} \bar{L}^n = \bar{L} \text{ u.o.c.}$$

and therefore, by Eqs. (8.42) and (8.43) and the random time-change theorem (Chen and Yao (2001, Theorem 5.3) [7]),

(8.44)
$$\lim_{n \to \infty} G_1^n(\bar{L}^n) = G_1(\bar{L}) \quad \text{u.o.c.}$$

Therefore, from Eqs. (8.41), (8.43), and (8.44) it follows that

(8.45)
$$\lim_{n \to \infty} \left[-\frac{1}{2} (g_1)^2 \sigma_f^2 \bar{L}^n + \frac{1}{2} (g_1^n)^2 \left(\bar{L}^n - G_1^n(\bar{L}^n) \right) \right] = 0 \text{ u.o.c.}$$

Part II: Second term. Define the process

(8.46)
$$G_2^n(t) := \frac{1}{n^{1.5}} \sum_{i=1}^{\lfloor nt \rfloor} \left[\left(\frac{f'(c_{i,l}^n)}{f(c_{i,l}^n)} \right)'' u_i^3(\hat{\mu}_l^n)^3 \right], \ t \in [0,\infty).$$

Therefore, the second term can be expressed as

(8.47)
$$\frac{(g_1^n(l))^3}{3!}G_2^n(\bar{L}^n(t)), \ l \in S, \ t \in [0,\infty).$$

The following equations hold for sufficiently large n:

(8.48)
$$G_2^n(t) = \frac{1}{n^{1.5}} \sum_{i=1}^{\lfloor nt \rfloor} \left| \left(\frac{f'(c_{i,l}^n)}{f(c_{i,l}^n)} \right)'' u_i^3 \right| \le \frac{1}{(1-\epsilon_M)^3} \frac{1}{n^{1.5}} \sum_{i=1}^{\lfloor nt \rfloor} M((1+\epsilon_M)u_i)$$

The inequality in Eq. (8.48) follows from Assumption 5.3.2 since $c_{i,l}^n \in (\mu_k^n v_i^n, \mu_l^n v_i^n)$ or $c_{i,l}^n \in (\mu_l^n v_i^n, \mu_k^n v_i^n)$. Eq. (4.5) implies that for sufficiently large $n \in \mathbb{N}$ and every $l \in S$ one has $(1 - \epsilon_M)u_i \leq c_{i,l}^n \leq (1 + \epsilon_M)u_i$. From the FSLLN and Eq. (8.48) it follows that

(8.49)
$$\lim G_2^n(t) = 0$$
 u.o.c

Now Eqs. (8.41), (8.43), and (8.49) and the random time-change theorem (Chen and Yao (2001, Theorem 5.3) [7]) yield that

(8.50)
$$\lim_{n \to \infty} \frac{g_1^n}{3!} G_2^n(\bar{L}^n) = 0, \text{ u.o.c.}$$

Part III: Third term. Define the function

(8.51)
$$g_2^n(l) = \frac{\ln(1+\hat{\mu}_l^n) - \hat{\mu}_l^n + \frac{1}{2}(\hat{\mu}_l^n)^2}{(\hat{\mu}_l^n)^2}, \ l \in S.$$

Therefore, the third term can be expressed as

(8.52)
$$(g_1^n(l))^2 g_2^n(l) \bar{L}^n(t), \ l \in S, \ t \in [0, \infty)$$

From the Taylor expansion of $\ln(1+x)$ and Eq. (8.41) and (8.43) it follows that

$$\lim_{n \to \infty} (g_1^n)^2 g_2^n \bar{L}^n = 0 \quad \text{u.o.c.}$$

This completes the proof of Lemma 8.3. \blacksquare

This completes the proof of Proposition 8.1. \blacksquare

Proposition 8.4 (Proving Eq. (8.6)). Under Assumptions 4.3 and 5.3.1,

$$\lim_{n\to\infty} \tilde{\boldsymbol{\zeta}}^n \stackrel{\mathrm{d}}{=} \ln(\boldsymbol{\varphi}_{\frac{\sqrt{\alpha}}{\sigma_f}}).$$

Proof. From Eq. (4.4) it follows that for every $n \in \mathbb{N}$ the process $\tilde{\zeta}^n$ can be expressed as

$$\tilde{\boldsymbol{\zeta}}^{\boldsymbol{n}}(l,t) = \sigma_f \sqrt{n} h^n(l) \tilde{W}^n(\bar{L}^n(t)) - \frac{1}{2} \left(\frac{I_S(l)\sigma_f}{\alpha}\right)^2 \bar{L}^n(t) + \frac{\theta\sigma_f}{\alpha} \cdot \frac{I_S(l)\sigma_f}{\alpha} \bar{L}^n(t), \ l \in S, \ t \in [0,\infty).$$

We prove that there exists a probability space Ω_W such that

(8.53)
$$\lim_{n \to \infty} \tilde{\boldsymbol{\zeta}}^n = \ln(\boldsymbol{\varphi}_{\frac{\sqrt{\alpha}}{\sigma_f}}) \quad \text{u.o.c}$$

in the probability space $\Omega_{\theta} \times \Omega_{W}$. For this, we investigate separately the parts of the process $\tilde{\boldsymbol{\zeta}}^{n}$ that depend on θ and the parts that depend on $\{u_i\}_{i\geq 1}$. From Assumption 4.3 it follows that

(8.54)
$$\lim_{n \to \infty} \sigma_f \sqrt{n} h^n = \sigma_f \frac{I_S}{\alpha} \text{ u.o.c.}$$

The processes $(\bar{L}^n(t))$ and $(\tilde{W}^n(\bar{L}^n(t)))$ depend on $\{u_i\}_{i\geq 1}$, which is independent of θ . From the Skorokhod Representation Theorem and the random time-change theorem (see Chen and Yao (2001, Theorems 5.1 and 5.3) [7]) it follows that there exist a probability space Ω_W and a standard Brownian motion $(\tilde{W}(t))$ defined on Ω_W , such that

(8.55)
$$\lim_{n \to \infty} (\bar{L}^n, \tilde{W}^n(\bar{L}^n)) = (\bar{L}, \tilde{W}(\bar{L})) \text{ u.o.c.}$$

From Eqs. (8.54) and (8.55) it follows that in the probability space $\Omega_{\theta} \times \Omega_{W}$

(8.56)
$$\lim_{n \to \infty} \tilde{\boldsymbol{\zeta}}^{\boldsymbol{n}} = \lim_{n \to \infty} \sigma_f \sqrt{n} h^n \tilde{W}^n(\bar{L}^n) - \frac{1}{2} \left(\frac{I_S \sigma_f}{\alpha}\right)^2 \bar{L}^n + \frac{\theta \sigma_f}{\alpha} \cdot \frac{I_S \sigma_f}{\alpha} \bar{L}^n$$
$$= \sigma_f \frac{I_S}{\alpha} \tilde{W}(\bar{L}) - \frac{1}{2} \left(\frac{I_S \sigma_f}{\alpha}\right)^2 \bar{L} + \frac{\theta \sigma_f}{\alpha} \cdot \frac{I_S \sigma_f}{\alpha} \bar{L} \quad \text{u.o.c.}$$

and since convergence u.o.c. implies convergence in distribution,

(8.57)
$$\lim_{n \to \infty} \tilde{\boldsymbol{\zeta}}^{\boldsymbol{n}} \stackrel{\mathrm{d}}{=} \sigma_f \frac{I_S}{\alpha} \tilde{W}(\bar{L}) - \frac{1}{2} \left(\frac{I_S \sigma_f}{\alpha}\right)^2 \bar{L} + \frac{\theta \sigma_f}{\alpha} \cdot \frac{I_S \sigma_f}{\alpha} \bar{L}$$

The scaling of the standard Brownian motion implies that (W(L(t))) is distributed as $(\sqrt{\alpha}\tilde{W}(t))$ and the result follows.

This completes the proof of Eq. (5.4).

The following remark explains the requirement that the appropriate rates under the different types are relatively close, up to order $\frac{1}{\sqrt{n}}$ (Assumption 4.3.1).

Remark 8.5. If there exists a parameter value $l^* \in S$ such that the difference between the rates $\mu_{l^*}^n$ and μ_0^n satisfies $|\mu_{l^*}^n - \mu_0^n| >> \frac{1}{\sqrt{n}}$, then for every t > 0 the following limit holds: $\lim_{n \to \infty} \sigma_f \sqrt{n} h^n(t, l^*) = \pm \infty$, and there will be no convergence of $\tilde{\zeta}^n(t, l^*)$. On the other hand, if there is a parameter value $l^* \in S$ such that the difference between the rates μ_l^n and μ_0^n satisfies $|\mu_{l^*}^n - \mu_0^n| \ll \frac{1}{\sqrt{n}}$, then for every t > 0 the following limit holds: $\lim_{n \to \infty} \sigma_f \sqrt{n} h^n(t, l^*) = 0$, and the DM will not be able to distinguish between them.

8.3.2. *Proof of Formula (5.5)*. From Eq. (3.3) we have

$$\boldsymbol{\pi}_{\frac{\sqrt{\alpha}}{\sigma_{f}}}(l,t) := \frac{\pi_{l}\boldsymbol{\varphi}_{\frac{\sqrt{\alpha}}{\sigma_{f}}}(l,t)}{\sum_{k\in S}\pi_{k}\boldsymbol{\varphi}_{\frac{\sqrt{\alpha}}{\sigma_{f}}}(k,t)}, \ l\in S, \ t\in[0,\infty).$$

We show that

$$\lim_{n\to\infty} \tilde{\pi}^n \stackrel{\mathrm{d}}{=} \pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$$

To this end we define a function $\Lambda : \mathcal{E}_{\infty} \to \mathcal{E}_{\infty}$ by

(8.58)
$$\Lambda(\boldsymbol{\varphi})(l,t) := \frac{\pi_l \boldsymbol{\varphi}(l,t)}{\sum_{k \in S} \pi_k \boldsymbol{\varphi}(k,t)}, \ l \in S, \ t \in [0,\infty).$$

 Λ is continuous with respect to the metric e_{∞} . Therefore,

$$\lim_{n \to \infty} \tilde{\pi}^n = \lim_{n \to \infty} \Lambda(\tilde{\varphi}^n) \stackrel{\mathrm{d}}{=} \Lambda(\varphi_{\frac{\sqrt{\alpha}}{\sigma_f}}) = \pi_{\frac{\sqrt{\alpha}}{\sigma_f}}$$

where the first equality follows from Eqs. (5.1) and (8.58), and the second equality follows from Eq. (5.4). This completes the proof of Theorem 5.4.

8.4. **Proof of Theorem 5.9.** In order to construct a random variable for which the difference $\sigma_v - \frac{1}{\sigma_f}$ is large, we use a random variable that has expectation 1 and has no variance. Let z be a random variable with the density $g(x) := C/(1+x^3)$, x > 0, where $C = 2\pi/3^{1.5}$. Then, E[z] = 1 and $\operatorname{Var}[z] = \infty$. We now show that z satisfies Assumption 5.3. The variance of $\frac{g'(z)}{g(z)}z$ is given by

(8.59)
$$\sigma_f^2 = \int_0^\infty \left(\frac{g'(x)}{g(x)}x\right)^2 g(x)dx = \int_0^\infty \left(\frac{3x^3}{1+x^3}\right)^2 g(x)dx < \infty$$

and there exists a constant D_1 such that for every x > 0

(8.60)
$$\left| \left(\frac{g'(x)}{g(x)} \right)'' x^3 \right|, \left| \frac{xg(x)}{1 - G(x)} \right| \le D_1,$$

where G is the cdf of z. The random variable z fails to satisfy Assumption 4.2.2 since $\operatorname{Var}[z] = \infty$. Let $\mathbb{I}_A(x)$ be a function that equals 1 if $x \in A$ and 0 otherwise and fix y > 1. We now construct a y-dependent random variable that satisfies Assumptions 4.2, 5.1, and 5.3, whose density is 'similar' to the function $g_y(x) := g(x)\mathbb{I}_{\{0 < x < y\}}(x) + e^{-x}\mathbb{I}_{\{y < x\}}(x)$, and for which the difference $\sigma_v - \frac{1}{\sigma_f}$ is large. One may notice that for sufficiently large y, the function g_y is not a density function since $\int_0^\infty g_y(x)dx < 1$. Moreover, for large y's the 'expectation' is not one as $\int_0^\infty xg_y(x)dx < 1$. In order to construct a density 'similar' to g_y we add to g_y a function h_y that is a sum of two functions. Each of these two functions has a significant contribution only to one of the two integrals mentioned above. Let u, C_2, C_3 be positive constants and define the function $h(x) := uC_1\mathbb{I}_{\{1/u < x < 2/u\}}(x) + uC_2\mathbb{I}_{\{u < x < u + 1/u^2\}}(x)$. For a sufficiently large u one has $\int_0^\infty h(x)dx = C_1 + C_2/u \approx C_1$ and $\int_0^\infty xh(x)dx = 3C_1/2u + C_2(1 + 1/2u^3) \approx C_2$. Therefore, for sufficiently large y one can construct a \mathcal{C}^3 function e_y that satisfies the following conditions:

- (C1) $e_y \approx g_y + h_y$, where h_y admits the same form as h with some proper y-dependent parameters u, C_1 , and C_2 , where u > 4 for every y,
- (C2) $\int_0^\infty e_y(x)dx = 1;$ (C3) $\int_0^\infty x e_y(x)dx = 1;$
- (C4) there exists $w := w_y > 0$ such that for every x > w one has $e_y(x) = g_y(x) = e^{-x}$;
- (C5) for every 2/3 < x < 1 one has $e_y(x) = g_y(x)$; and

(C6) there exists a positive parameter D_2 such that $\left|\frac{e'_y(x)}{e_y(x)}x\right|, \left|\left(\frac{e'_y(x)}{e_y(x)}\right)''x^3\right|, \text{ and } \left|\frac{xe_y(x)}{1-E_y(x)}\right|$ are bounded from above by D_2 on the interval (0, w), where E_y is the cdf that is associated with the pdf e_y .

Conditions (C1)–(C3) can hold by the preceding discussion. To see why one can choose e_y that satisfies Conditions (C4) and (C5) notice that h_y is nonzero only over $(1/u, 2/u) \cup$ $(u, u + 1/u^2)$. Therefore, e_y can be chosen to be equal to g_y on any subinterval of the complement of $(1/u, 2/u) \cup (u, u+1/u^2)$. Condition (C4) can hold by taking $w = u+1/u^2$. and Condition (C5) can hold since u > 4 for every y by Condition (C1). Condition (C6) can hold by Eq. (8.60) and by Condition (C1).

Let $v := v_y$ be a random variable that is associated with the pdf e_y . We show that v satisfies Assumptions 4.2 and 5.3. The variance of v is finite since

(8.61)
$$\sigma_{v[y]}^{2} = \int_{0}^{\infty} x^{2} e_{y}(x) dx \approx \int_{0}^{\infty} x^{2} (g_{y}(x) + h_{y}(x)) dx$$
$$= \int_{0}^{y} x^{2} g(x) dx + \int_{y}^{\infty} x^{2} e^{-x} dx + \int_{0}^{\infty} x^{2} h_{y}(x) dx < \infty$$

The variance of $\frac{e'_y(v)}{e_u(v)}v$ is also finite since by Conditions (C5) and (C6) one has

$$\sigma_{f[y]}^{2} = \int_{0}^{\infty} \left(\frac{e_{y}'(x)}{e_{y}(x)}x\right)^{2} e_{y}(x)dx = \int_{0}^{w} \left(\frac{e_{y}'(x)}{e_{y}(x)}x\right)^{2} e_{y}(x)dx + \int_{w}^{\infty} \left(\frac{e_{y}'(x)}{e_{y}(x)}x\right)^{2} e_{y}(x)dx \\ \leq D_{2}w + e^{-w} < \infty.$$

Assumptions 5.3.2 and 5.3.3 also follow by Conditions (C5) and (C6).

We now show that by taking large y's, the difference $\sigma_{v[y]} - \frac{1}{\sigma_{f[y]}}$ becomes large. To this end, we show that $\lim_{y\to\infty} \left(\sigma_{v[y]} - \frac{1}{\sigma_{f[v]}}\right) = \infty$. Let y be such that Conditions (C1)–(C6) hold. As in Eq. (8.61) one has

(8.62)

$$\sigma_{v[y]}^2 \approx \int_0^\infty x^2 (g_y(x) + h_y(x)) dx \ge \int_0^y x^2 g_y(x) dx = \int_0^y x^2 g(x) dx = C \ln(1+y^3)/3.$$

By Condition (C5) one has

$$\sigma_{f[y]}^2 = \int_0^\infty \left(\frac{e_y'(x)}{e_y(x)}x\right)^2 e_y(x)dx \ge \int_{2/3}^1 \left(\frac{e_y'(x)}{e_y(x)}x\right)^2 e_y(x)dx = \int_{2/3}^1 \left(\frac{g'(x)}{g(x)}x\right)^2 g(x)dx$$
$$= \int_{2/3}^1 \left(\frac{3x^4}{1+x^3}x\right)^2 g(x)dx := D_3 < \infty.$$

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Notice that D_3 is independent of y, and therefore

(8.63)
$$\frac{1}{\sigma_{f[y]}^2} \le \frac{1}{D_3}$$

From Eqs. (8.62) and (8.63) one concludes that $\lim_{y\to\infty} \left(\sigma_{v[y]} - \frac{1}{\sigma_{f[v]}}\right) = \infty$ and the result follows.

8.5. **Proof of Theorem 6.4.** Let²² $\Omega' := \Omega_{\theta} \times \Omega_W$. This probability space is the basis for the proof of Theorem 6.4. From Eqs. (8.7) and (8.56) it follows that²³

(8.64)
$$\lim_{n \to \infty} \tilde{\boldsymbol{\varphi}}^n = \hat{\boldsymbol{\varphi}}, \quad \Omega' \text{-u.o.c.}$$

8.5.1. **Proof of Eq. (6.9)**. By using this convergence we show now that on this probability space $\lim_{n\to\infty} \tau_D^n(\pi) = \tau_D(\pi)$, Ω' -a.s.

Lemma 8.6. Fix T > 0. Under Assumptions 4.1, 4.2, 4.3, 5.1, 5.3, 6.1, and 6.3,

$$\lim_{n \to \infty} (\tilde{\tau}_D^n(\pi) \wedge T) = (\tilde{\tau}_D(\pi) \wedge T), \quad \Omega'\text{-a.s.}$$

Proof. If $\pi \notin D$, then $\tilde{\tau}_D^n = \tilde{\tau}_D = 0$. For the case $\pi \in D$ we express the stopping times $\tilde{\tau}_D^n$ and $\tilde{\tau}_D$ in a more convenient way. From Eqs. (3.3) and (5.1) it follows that

(8.65)
$$\frac{\tilde{\boldsymbol{\pi}}^n(t)}{1-\tilde{\boldsymbol{\pi}}^n(t)} = \frac{\pi}{1-\pi} \tilde{\boldsymbol{\varphi}}^n(t)$$

and

(8.66)
$$\frac{\hat{\pi}(t)}{1 - \hat{\pi}(t)} = \frac{\pi}{1 - \pi} \hat{\varphi}(t).$$

Eqs. (8.65) and (8.66) imply that

(8.67)
$$\tilde{\tau}_D(\pi) = \inf \left\{ t \, | \hat{\boldsymbol{\varphi}}(t) \notin \bigcup_j \left(c_j, d_j \right) \right\} = \inf \left\{ t \, | \hat{\boldsymbol{\varphi}}(t) \notin \left(c_i, d_i \right) \right\}$$

and

(8.68)
$$\tilde{\tau}_D^n(\pi) = \inf \left\{ t \, | \tilde{\boldsymbol{\varphi}}^n(t) \notin \cup_j (c_j, d_j) \right\},$$

where for every index $j, c_j := \frac{1-\pi}{\pi} \cdot \frac{a_j}{1-a_j}$ and $d_j := \frac{1-\pi}{\pi} \cdot \frac{b_j}{1-b_j}$. The second equality in Eq. (8.67) follows since $(\hat{\varphi}(t))$ is a continuous process w.r.t. the parameter t (see Eq. (8.2)). In order to prove that $\lim_{n\to\infty} (\tilde{\tau}_D^n(\pi) \wedge T) = (\tilde{\tau}_D(\pi) \wedge T), \Omega'$ -a.s., we distinguish between two possibilities: $\tilde{\tau}_D(\pi)(\omega) > T$ and $\tilde{\tau}_D(\pi)(\omega) \leq T$. Fix²⁴ $\omega \in \Omega'$.

If $\tilde{\tau}_D(\pi)(\omega) > T$ then, since $\hat{\varphi}(t)(\omega)$ is continuous w.r.t. t, it follows that the supremum

$$M(T)(\omega) := \sup_{0 \le t \le T} \hat{\varphi}(t)(\omega)$$

and the infimum

$$m(T)(\omega) := \inf_{0 \le t \le T} \hat{\varphi}(t)(\omega),$$

²³The process $\hat{\varphi}$ was defined in Section 6 as $\lim_{n \to \infty} \tilde{\varphi}^n = \varphi_{\frac{\sqrt{\alpha}}{\sigma_f}}$.

²²See the paragraph preceding Eq. (8.54) for the definition of Ω_W .

²⁴The following properties that we state hold for almost every $\omega \in \Omega'$. We chose $\omega \in \Omega'$ for which these properties hold.

are attained and satisfy $c_i < m(T)(\omega) < M(T)(\omega) < d_i$. Moreover, Eq. (8.64) implies that for every $0 < \delta < \min\{d_i - M(T)(\omega), m(T)(\omega) - c_i\}$ there exists $N_{\delta} > 0$ such that for every $n > N_{\delta}$ and every $t \in [0, T]$ one has

$$|\tilde{\boldsymbol{\varphi}}^n(t)(\omega) - \hat{\boldsymbol{\varphi}}(t)(\omega)| < \delta,$$

and therefore,

$$c_i < \tilde{\boldsymbol{\varphi}}^n(t)(\omega) < d_i.$$

Hence, $\tilde{\tau}_D^n(\omega) > T$, and consequently $(\tilde{\tau}_D^n(\omega) \wedge T) = (\tilde{\tau}_D(\omega) \wedge T)$.

If $\tilde{\tau}_D(\tilde{\pi})(\omega) \leq T$ we assume without loss of generality²⁵ that $\hat{\varphi}(\tilde{\tau}_D(\pi))(\omega) = d_i$. Fix $\epsilon > 0$. Denote

$$\delta_1 = \delta_1(\omega) := d_i - \sup_{0 \le t \le \tilde{\tau}_D(\omega) - \epsilon} \hat{\varphi}(t)(\omega)$$

and

$$\delta_2 = \delta_2(\omega) := \inf_{0 \le t \le \tilde{\tau}_D(\omega) - \epsilon} \hat{\boldsymbol{\varphi}}(t)(\omega) - c_i.$$

By the continuity of $\hat{\varphi}(t)(\omega)$ with respect to t, and by the definition of $\tilde{\tau}_D$ it follows that $\delta_1, \delta_2 > 0$. Denote

$$\delta_3 = \delta_3(\omega) := \sup_{\tilde{\tau}_D(\omega) \le t \le \tilde{\tau}_D(\omega) + \epsilon} \hat{\varphi}(t)(\omega) - d_i.$$

From Eq. (8.2) and the fluctuations of the Brownian motion, it follows that $\delta_3 > 0$. Let

$$\delta_4 := \frac{1}{2} \left(c_{i+1} - d_i \right).$$

Assumption 6.3 implies that $\delta_4 > 0$. Denote

$$\tau_d(\pi) := \inf \left\{ t \left| \hat{\varphi}(t) = d + (\delta_4 \wedge \delta_3) \right\} \right\}.$$

Then clearly one has $\tilde{\tau}_D(\omega) < \tau_d(\omega) < \tilde{\tau}_D(\omega) + \epsilon$. Let $\delta := \min\{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon\}$. From Eq. (8.64) it follows there exists $N_{\delta} > 0$ such that for every $n > N_{\delta}$ and every $t \in [0, T]$,

$$|\tilde{\boldsymbol{\varphi}}^n(t)(\omega) - \hat{\boldsymbol{\varphi}}(t)(\omega)| < \delta$$

Therefore, for every such $n > N_{\delta}$ and every $t \in [0, \tilde{\tau}_D(\omega) - \epsilon]$ one has

(8.69)
$$c_i < \tilde{\varphi}^n(t)(\omega) < d_i,$$

and at time τ_d one has

(8.70)
$$d_i < \tilde{\boldsymbol{\varphi}}^n(\tau_d)(\omega) < d_{i+1}.$$

Since $\tilde{\tau}_D(\omega) < \tau_d(\omega) < \tilde{\tau}_D(\omega) + \epsilon$, Eqs. (8.69)–(8.70) yield

$$\left|\left(\tilde{\tau}_D^n(\omega)\wedge T\right)-\left(\tilde{\tau}_D(\omega)\wedge T\right)\right|<\epsilon.$$

As a corollary, we get that $\lim_{n\to\infty} \tilde{\tau}_D^n(\pi) \stackrel{\mathrm{d}}{=} \tilde{\tau}_D(\pi)$. This completes the proof of Eq. (6.9).

²⁵The proof for $\hat{\varphi}(\tilde{\tau}_D(\pi))(\omega) = c_i$ is similar and is therefore omitted.

8.5.2. **Proof of Eq. (6.10)**. To avoid cumbersome notation we write $\tilde{\tau} := \tilde{\tau}_D$ and $\tilde{\tau}^n := \tilde{\tau}_D^n$. In order to prove that $V_{\tilde{\tau}^n}^n(\pi)$ converges to $V_{\tilde{\tau}}(\pi)$ we will bound the expression $|V_{\tilde{\tau}^n}^n(\pi) - V_{\tilde{\tau}}(\pi)|$ by other terms for which the convergence is easier to prove. By the triangle inequality, for every index $n \in \mathbb{N}$ and every time T > 0,

$$(8.71) \qquad |V_{\tilde{\tau}^{n}}^{n}(\pi) - V_{\tilde{\tau}}(\pi)| \\ \leq \left| E^{\pi} \left[\int_{0}^{\tilde{\tau}^{n}} r e^{-rt} k^{n}(\tilde{\pi}^{n}(t)) dt - \int_{0}^{\tilde{\tau}^{n} \wedge T} r e^{-rt} k^{n}(\tilde{\pi}^{n}(t)) dt \right] \right|$$

(8.72)
$$+ \left| E^{\pi} \left[\int_{0}^{\infty} r e^{-rt} k^{n}(\tilde{\pi}^{n}(t)) dt - \int_{0}^{\infty} r e^{-rt} k^{n}(\tilde{\pi}^{n}(t)) dt \right] \right|$$

(8.73)
$$+ \left| E^{\pi} \left[\int_{0}^{\tau \wedge T} r e^{-rt} k^{n} (\tilde{\boldsymbol{\pi}}^{n}(t)) dt - \int_{0}^{\tau \wedge T} r e^{-rt} k(\hat{\boldsymbol{\pi}}(t)) dt \right] \right|$$

(8.74)
$$+ \left| E^{\pi} \left[\int_{0}^{\tau \wedge t} r e^{-rt} k(\hat{\boldsymbol{\pi}}(t)) dt - \int_{0}^{\tau} r e^{-rt} k(\hat{\boldsymbol{\pi}}(t)) dt \right] \right|$$

(8.75)
$$+ \left| E^{\pi} \left[r e^{-r\tilde{\tau}^n} \frac{1}{n} K^n(\tilde{\boldsymbol{\pi}}^n(\tilde{\tau}^n)) - r e^{-r\tilde{\tau}} K(\hat{\boldsymbol{\pi}}(\tilde{\tau})) \right] \right|$$

We now show that each of the terms converges to zero. That is, for every fixed $\epsilon > 0$, there exists $N_{\epsilon} > 0$ such that for every $n > N_{\epsilon}$, each of the terms is bounded by ϵ . We divide the proof into four parts.

Part I: First and fourth terms. In this part we show that for every $n \in \mathbb{N}$ and for sufficiently large T, if the DM cannot operate the system after time T, then his expected loss by using the stopping time $(\tilde{\tau}^n \wedge T)$ is close up to ϵ to the expected loss from the integral cost part without the limitation of the maximal time of operating the system. From Remark 6.2 it follows that the sequence $\{k^n(\tilde{\pi}^n(t))\}_{n\in\mathbb{N}}$ is bounded by C_k . Therefore, for every T > 0,

(8.76)
$$\left| E^{\pi} \left[\int_{0}^{\tilde{\tau}^{n}} r e^{-rt} k^{n}(\tilde{\pi}^{n}(t)) dt - \int_{0}^{\tilde{\tau}^{n} \wedge T} r e^{-rt} k^{n}(\tilde{\pi}^{n}(t)) dt \right] \right|$$
$$\leq E^{\pi} \left| \int_{\tilde{\tau}^{n} \wedge T}^{\tilde{\tau}^{n}} r e^{-rt} k^{n}(\tilde{\pi}^{n}(t)) dt \right|$$
$$\leq C_{k} E^{\pi} \left| \int_{\tilde{\tau}^{n} \wedge T}^{\tilde{\tau}^{n}} r e^{-rt} dt \right|$$
$$\leq C_{k} E^{\pi} \left| \int_{T}^{\infty} r e^{-rt} dt \right| = C_{k} e^{-rT}.$$

The last term on Eq. (8.76) converges to zero as T goes to infinity, and so there exists a constant $T := T_{\epsilon}$ such that for every $n \in \mathbb{N}$

$$\left| E^{\pi} \left[\int_{0}^{\tilde{\tau}^{n}} r e^{-rt} k^{n}(\tilde{\boldsymbol{\pi}}^{n}(t)) dt - \int_{0}^{\tilde{\tau}^{n} \wedge T} r e^{-rt} k^{n}(\tilde{\boldsymbol{\pi}}^{n}(t)) dt \right] \right| < \epsilon.$$

Similarly, one can choose T_{ϵ} to be such that, in addition,

$$\left| E^{\pi} \left[\int_{0}^{\tilde{\tau}} r e^{-rt} k(\hat{\boldsymbol{\pi}}(t)) dt - \int_{0}^{\tilde{\tau} \wedge T} r e^{-rt} k(\hat{\boldsymbol{\pi}}(t)) dt \right] \right| < \epsilon.$$

Part II: Second term. We now show that for sufficiently large $n \in \mathbb{N}$, by changing

the $\mathcal{F}_{nt}^{L^n}$ -adapted stopping time $\tilde{\tau}^n$ to the $\mathcal{F}_t^{\tilde{\pi}}$ -adapted stopping time $\tilde{\tau}$, the expected integral cost does not change by much:

(8.77)
$$\left| E^{\pi} \left[\int_{0}^{\tilde{\tau}^{n} \wedge T} r e^{-rt} k^{n} (\tilde{\boldsymbol{\pi}}^{n}(t)) dt - \int_{0}^{\tilde{\tau} \wedge T} r e^{-rt} k^{n} (\tilde{\boldsymbol{\pi}}^{n}(t)) dt \right] \right|$$
$$\leq E^{\pi} \left| \int_{\tilde{\tau} \wedge T}^{\tilde{\tau}^{n} \wedge T} r e^{-rt} k^{n} (\tilde{\boldsymbol{\pi}}^{n}(t)) dt \right|$$
$$\leq r C_{k} E^{\pi} \left| (\tilde{\tau}^{n} \wedge T) - (\tilde{\tau} \wedge T) \right|.$$

From Lemma 8.6 one has $\lim_{n\to\infty} (\tilde{\tau}^n \wedge T) = (\tilde{\tau} \wedge T)$, Ω' -a.s. Therefore, by the bounded convergence theorem, there exists $N_{\epsilon} > 0$ such that for every $n > N_{\epsilon}$ the last term in Eq. (8.77) is smaller than ϵ .

Part III: Third term. In this part we show that if the DM cannot operate the system after time T, then his expected integral cost from the *n*-th system and by using the $\mathcal{F}_t^{\tilde{\pi}}$ -adapted stopping time $\tilde{\tau}$ is close to the expected integral cost of the limit problem using the same stopping time $\tilde{\tau}$:

(8.78)
$$\left| E^{\pi} \left[\int_{0}^{\tilde{\tau} \wedge T} r e^{-rt} k^{n}(\tilde{\pi}^{n}(t)) dt - \int_{0}^{\tilde{\tau} \wedge T} r e^{-rt} k(\tilde{\pi}(t)) dt \right] \right|$$
$$\leq E^{\pi} \left[\int_{0}^{\tilde{\tau} \wedge T} r e^{-rt} \left| k^{n}(\tilde{\pi}^{n}(t)) - k(\hat{\pi}(t)) \right| dt \right]$$
$$\leq E^{\pi} \left[T \sup_{0 \leq t \leq T} \left| k^{n}(\tilde{\pi}^{n}(t)) - k(\hat{\pi}(t)) \right| \right].$$

From Eqs. (8.64), (8.65), and (8.66) it follows that $\lim_{n\to\infty} \tilde{\pi}^n(t) = \hat{\pi}(t)$, Ω' -u.o.c. Moreover, by Assumption 6.1.1, the functions k^n converge uniformly on [0, 1] to k, and therefore, $\lim_{n\to\infty} \sup_{0\le t\le T} |k^n(\tilde{\pi}^n(t)) - k(\hat{\pi}(t))| = 0$, Ω' -a.s. The bounded convergence theorem implies that for sufficiently large $n \in \mathbb{N}$, the last term in Eq. (8.78) is smaller than ϵ .

Part IV: Fifth term. In this part we show that for sufficiently large $n \in \mathbb{N}$, the expected terminal cost from the *n*-th system using the stopping time $\tilde{\tau}^n$ is relatively close to the expected terminal cost from the limit system using the stopping time $\tilde{\tau}$. To this end, we show that $\lim_{n\to\infty} re^{-r\tilde{\tau}^n} \frac{1}{n} K^n(\tilde{\pi}^n(\tilde{\tau}^n)) = re^{-r\tilde{\tau}} K(\hat{\pi}(\tilde{\tau})), \Omega'$ -a.s. From Remark 6.2 and the bounded convergence theorem it will follow that there exists $N_{\epsilon} > 0$ such that, for every $n > N_{\epsilon}$,

$$\left| E^{\pi} \left[r e^{-r\tilde{\tau}^n} \frac{1}{n} K^n(\tilde{\boldsymbol{\pi}}^n(\tilde{\tau}^n)) - r e^{-r\tilde{\tau}} K(\hat{\boldsymbol{\pi}}(\tilde{\tau})) \right] \right| < \epsilon.$$

From Lemma 8.6 it follows that

(8.79)
$$P(\omega \in \Omega' \mid \forall T \in \mathbb{N} \ \lim_{n \to \infty} (\tilde{\tau}^n(\pi)(\omega) \wedge T) = (\tilde{\tau}(\pi)(\omega) \wedge T)) = 1.$$

Fix $\omega \in \Omega'$ such that for every $T \in \mathbb{N}$ one has $\lim_{n \to \infty} (\tilde{\tau}^n(\omega) \wedge T) = (\tilde{\tau}(\omega) \wedge T)$ and $\lim_{n \to \infty} \tilde{\pi}^n(\omega) = \hat{\pi}(\omega)$ u.o.c. We divide the proof into two cases: $\tilde{\tau}(\omega) = \infty$ and $\tilde{\tau}(\omega) < \infty$. If $\tilde{\tau}(\omega) = \infty$ then, since by Lemma 8.6 one has $\lim_{n \to \infty} (\tilde{\tau}^n(\omega) \wedge T) = (\tilde{\tau}(\omega) \wedge T) = T$, it follows that there exists $N_{\epsilon} > 0$ such that, for every $n > N_{\epsilon}$, $|(\tilde{\tau}^n(\omega) \wedge T) - T| < 1$. Let T be such that $re^{-r(T-1)} < \frac{\epsilon}{C_K}$. Then for every $n > N_{\epsilon}$,

$$|re^{-r\tilde{\tau}^{n}}\frac{1}{n}K^{n}(\tilde{\pi}^{n}(\tilde{\tau}^{n}))(\omega) - re^{-r\tilde{\tau}}K(\hat{\pi}(\tilde{\tau}))(\omega)|$$

$$= |re^{-r\tilde{\tau}^{n}}\frac{1}{n}K^{n}(\tilde{\pi}^{n}(\tilde{\tau}^{n}))(\omega)|$$

$$\leq C_{K}re^{-r(T-1)} \leq \epsilon.$$

If $\tilde{\tau}(\omega) < \infty$ then, since by Lemma 8.6 $\lim_{n \to \infty} \tilde{\tau}^n(\omega) = \tilde{\tau}(\omega)$, Ω' -a.s., it follows that for sufficiently large $n \in \mathbb{N}$ the following two conditions hold:

(8.80)
$$\tilde{\tau}^n(\omega) < \tilde{\tau}(\omega) + 1,$$

(8.81)
$$|e^{-r\tilde{\tau}^n(\omega)} - e^{-r\tilde{\tau}(\omega)}| < \frac{\epsilon}{2rC_K}.$$

By Assumptions 6.1.3 and 6.1.4, the functions K^n/n converge uniformly on [0, 1] to the continuous function K. Since $\lim_{n\to\infty} \tilde{\pi}^n(t)(\omega) = \hat{\pi}(t)(\omega)$ uniformly on $[0, \tilde{\tau}(\omega) + 1]$, it follows from Eq. (8.80) that for sufficiently large $n \in \mathbb{N}$

(8.82)
$$\left|\frac{1}{n}K^{n}(\tilde{\boldsymbol{\pi}}^{n}(\tilde{\boldsymbol{\tau}}^{n}))(\omega) - K(\hat{\boldsymbol{\pi}}(\tilde{\boldsymbol{\tau}}))(\omega)\right| < \frac{\epsilon}{2rC_{K}}$$

By combining Eqs. (8.81)–(8.82) one concludes that there exists $N_{\epsilon} > 0$ such that for every $n > N_{\epsilon}$,

$$\begin{aligned} |re^{-r\tilde{\tau}^{n}}\frac{1}{n}K^{n}(\tilde{\boldsymbol{\pi}}^{n}(\tilde{\tau}^{n})) - re^{-r\tilde{\tau}}K(\hat{\boldsymbol{\pi}}(\tilde{\tau}))|(\omega) \\ &\leq re^{-r\tilde{\tau}^{n}(\omega)}|\frac{1}{n}K^{n}(\tilde{\boldsymbol{\pi}}^{n}(\tilde{\tau}^{n}))(\omega) - K(\hat{\boldsymbol{\pi}}(\tilde{\tau}))(\omega)| + |K(\hat{\boldsymbol{\pi}}(\tilde{\tau}))(\omega)||re^{-r\tilde{\tau}^{n}(\omega)} - re^{-r\tilde{\tau}(\omega)}| \leq \epsilon \end{aligned}$$

This completes the proof of Eq. (6.10).

References

- E. Bayraktar, S. Dayanik, and I. Karatzas. Adaptive poisson disorder problem. Ann. Appl. Probab., 16:1190–1261, 2006.
- [2] D. Bergemann and J. Välimäki. Market diffusion with two-sided learning. RAND Journal of Economics, 28:773–795, 1997.
- [3] D. A. Berry and B. Fristedt. Bandit Problems: Sequential Allocation of Experiments. Chapman and Hall: New York, 1985.
- [4] P. Billingsley. Convergence of Probability Measures. J. Wiley & Sons:New York, 2nd edition, 1999.
- [5] P. Bolton and C. Harris. Strategic experimentation. *Econometrica*, 67:349–374, 1999.
- [6] B. Buonaguidi and P. Muliere. Sequential testing problems for lévy processes. Sequential Analysis, 32:47–70, 2013.
- [7] H. Chen and D. D. Yao. Fundamentals of Queuing Networks: Performance, Asymptotics, and Optimization. Springer:Berlin, 2001.
- [8] A. Cohen and E. Solan. Bandit problems with lévy processes. Mathematics of Operations Research, 38:92–107, 2013.
- [9] S. Dayanik. Wiener disorder problem with observations at fixed discrete time epochs. Math. Oper. Res., 35(4):756-785, 2010.
- [10] L. Felli and C. Harris. Learning, wage dynamics and firm-specific human capital. Journal of Political Economy, 104:838–868, 1996.
- [11] P. V. Gapeev. The disorder problem for compound Poisson processes with exponential jumps. Annals of Applied Probability, 15:487–499, 2005.
- [12] P. V. Gapeev and G. Peskir. The Wiener sequential testing problem with finite horizon. Stochastics and Stochastic Reports, 76:59–75, 2004.
- [13] P. V. Gapeev and G. Peskir. The wiener disorder problem with finite horizon. Stochastic Processes and their Applications, 116(12):1770–1791, 2006.

- [14] P. V. Gapeev and A. N. Shiryaev. On the sequential testing problem for some diffusion processes. Stochastics: an international journal of probability and stochastic processes, 83:519–535, 2011.
- [15] J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer: Berlin, 1987.
- [16] B. Jovanovic. Job matching and the theory of turnover. *Journal of Political Economy*, 87:972–990, 1979.
- [17] R. E. Kalman and R. S. Bucy. New results in linear filtering and prediction theory. Journal of Basic Engineering, 83:95–108, 1961.
- [18] I. A. Karatzas and S. E. Shreve. Brownian Motion and Stochastic Calculus. Springer:New York, 2nd edition, 1991.
- [19] G. Keller and S. Rady. Optimal experimentation in a changing environment. Review of Economic Studies, 66:475–507, 1999.
- [20] G. Moscarini. Job matching and the wage distribution. *Econometrica*, 73:481–516, 2005.
- [21] G. Peskir and A. N. Shiryaev. Solving the Poisson disorder problem. In K. Sandmann and P. Schönbucher, editors, Advances in Finance and Stochastics. Essays in Honour of Dieter Sondermann, pages 295–312. Springer: Berlin, 2002.
- [22] N. G. Polson and G. O. Roberts. Bayes factors for discrete observations from diffusion processes. *Biometrica*, 11:11–26, 1994.
- [23] S. O. Sezer. On the Wiener disorder problem. Ann. Appl. Probab., 20(4):1537–1566, 2010.
- [24] A. N. Shiryaev. Optimal Stopping Rules. Springer: Berlin, 1978.
- [25] M. H. Vellekoop and J. M. C. Clark. Optimal speed of detection in generalized wiener disorder problems. Stochastic Processes and Their Applications, 95(1):25–54, 2001.
- [26] W. Whitt. Staffing a call center with uncertain arrival rate and absenteeism. Production and Operations Management, 15(1):88–102, 2006.
- [27] M. Zakai. On the optimal filtering of diffusion processes. Z. Wahrsch. Verw. Gebiete, 11:230–243, 1969.
- [28] M. V. Zhitlukhin and A. N. Shiryaev. A Bayesian sequential testing problem of three hypotheses for Brownian motion. *Statistics and Risk Modeling*, 28:227–249, 2011.