Approximately Optimal Mechanisms for Strategyproof Facility Location: Minimizing L_p Norm of Costs

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Abstract

We consider the problem of locating a single facility on the real line. This facility serves a set of agents, each of whom is located on the line, and incurs a cost equal to his distance from the facility. An agent's location is private information that is known only to him. Agents report their location to a central planner who decides where to locate the facility. The planner's objective is to minimize a "social" cost function that depends on the agent-costs. However, agents might not report truthfully; to address this issue, the planner must restrict himself to *strategyproof* mechanisms, in which truthful reporting is a dominant strategy for each agent. A mechanism that simply chooses the optimal solution is generally not strategyproof, and so the planner aspires to to use a mechanism that effectively *approximates* his objective function. This general class of problems was first studied by Procaccia and Tennenholtz and has been the subject of much research since then.

In our paper, we study the problem described above with the social cost function being the L_p norm of the vector of agent-costs. We show that the median mechanism (which is known to be strategyproof) provides a $2^{1-\frac{1}{p}}$ approximation ratio, and that is the optimal approximation ratio among all deterministic strategyproof mechanisms. For randomized mechanisms, we present two results. First, we present a negative result: we show that for integer $\infty > p > 2$, no mechanism—from a rather large class of randomized mechanisms— has an approximation ratio better than that of the median mechanism. This is in contrast to the case of $p = 2$ and $p = \infty$ where a randomized mechanism provably helps improve the worst case approximation ratio. Second, for the case of 2 agents, we show that a mechanism called LRM, first designed by Procaccia and Tennenholtz for the special case of L_∞, provides the optimal approximation ratio among all randomized mechanisms.

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1   Introduction

We consider the problem of locating a single facility on the real line. This facility serves a set of n agents, each of whom is located somewhere on the line as well. Each agent cares about his distance to the facility, and incurs a disutility (equivalently, cost) that is equal to his distance to access the facility. An agent's location is assumed to be private information that is known only to him. Agents report their locations to a central planner who decides where to locate the facility based on the reports of the agents. The planner's objective is to minimize a "social" cost function that depends on the vector of distances that the agents need to travel to access the facility. It is natural for the planner to consider locating the facility at a point that minimizes her objective function, but in that case the agents may not have an incentive to report their locations truthfully. As an example, consider the case of 2 agents located at x_1 and x_2 respectively, and suppose the location that optimizes the planner's objective is the mid-point $(x_1 + x_2)/2$. Then, assuming $x_1 < x_2$, agent 1 has an incentive to report a location $x_1' < x_1$ so that the planner's decision results in the facility being located closer to his true location. The planner can address this issue by restricting herself to a *strategyproof* mechanism: by this we mean that it should be a (weakly) dominant strategy for each agent to report his location truthfully to the central planner. This, of course, is an attractive property, but it comes at a cost: based on the earlier example, it is clear that the planner cannot hope to optimize her objective. One way to avoid this difficulty is to assume an environment in which agents (and the planner) can make or receive payments; in such a case, the planner selects the location of the facility, and also a payment scheme, which specifies the amount of money an agent pays (or receives) as a function of the reported locations of the agents as well as the location of the facility. This option gives the planner the ability to support the "optimal" solution as the outcome of a strategy-proof mechanism by constructing a carefully designed payment scheme in which any potential benefit for a misreporting agent from a change in the location of the facility is offset by an increase in his payment.

There are many settings, however, in which such monetary compensations are either not possible or are undesirable. This motivated Procaccia and Tennenholtz [6] to formulate the notion of *Approximate Mechanism Design without Money*. In this model the planner restricts herself to strategy-proof mechanisms, but is willing to settle for one that does not necessarily optimize her objective. Instead, the planner's goal is to find a mechanism that effectively *approximates* her objective function. This is captured by the standard notion of approximation that is widely used in the CS literature: for a minimization problem, an algorithm is an α-approximation if the solution it finds is guaranteed to have cost at most α times that of the optimal cost ($\alpha \geq 1$).

Procaccia and Tennenholtz [6] apply the notion of approximate mechanism design without money to the facility location problem considered here for two different objectives: (i) *minisum*,

where the goal is to minimize the sum of the costs of the agents; and (ii) *minimax*, where the goal is to minimize the maximum agent cost. They show that for the minimax objective choosing any k-th median—picking the kth largest reported location—is a strategyproof, 2-approximate mechanism. They design a randomized mechanism called LRM (Left-Right-Middle) and show that it is a strategyproof, 3/2-approximate mechanism; futhermore, they show that those mechanisms provide the optimal worst-case approximation ratio possible (among all deterministic and randomized strategyproof mechanisms, respectively). For the *minisum* objective, it is known that choosing the median reported location is optimal and strategyproof [5]. Feldman and Wilf [4] consider the same facility location problem on a line but with the social cost function being the L_2 norm of the agents' costs[1]. They show that the median is a $\sqrt{2}$-approximate strategyproof mechanism for this objective function, and provide a randomized $(1+\sqrt{2})/2$-approximate strategyproof mechanism. In addition, facility location on other topologies such as circles and trees are considered by Alon et al. [1, 2, 3] as well as by Feldman and Wilf [4].

In our paper, we follow the suggestion of Feldman and Wilf [4] and study the problem of locating a single facility on a line, but with the objective function being the L_p norm of the vector of agent-costs (for general $p \geq 1$). We define the problem formally in section 2. In section 3, we show that the median mechanism (which is strategyproof) provides a $2^{1-\frac{1}{p}}$ approximation ratio, and that is the optimal approximation ratio among all deterministic strategyproof mechanisms. We move onto randomized mechanisms in section 4. First, we present a negative result: we show that for integer $\infty > p > 2$, *no* mechanism—from a rather large class of randomized mechanisms— has an approximation ratio better than that of the median mechanism, as the number of agents goes to infinity. It is worth noting that all the mechanisms proposed in literature so far— for minimax, minisum, as well as quadratic mean social cost functions— belong to this class of mechanisms. Next, we consider the case of 2 agents, and show that the LRM mechanism provides the optimal approximation ratio among all randomized mechanisms (that satisfy certain mild assumptions) for this special case, for every $p \geq 1$. Our result for the special case of 2 agents also gives a lower bound on the approximation ratio for all randomized mechanisms. We briefly discuss some directions for further research in section 5.

2   Model

Let $N = \{1, 2, \ldots, n\}$, $n \geq 2$, be the set of agents. Each agent $i \in N$ reports a location $x_i \in \mathbb{R}$. A *deterministic* mechanism is a collection of functions $f = \{f_n | \ n \in \mathbb{N}, n \geq 2\}$ such that each $f_n : \mathbb{R}^n \to \mathbb{R}$ maps each location profile $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to the location of a facility. We will abuse

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[1]Feldman and Wilf actually used the sum of squares of the agents' costs, but most of their results can be easily converted to the L_2 norm. Of course, the approximation ratios they report need to be adjusted as well.

notation and let $f(\mathbf{x})$ denote $f_n(\mathbf{x})$. Under a similar notational abuse, a *randomized* mechanism is a collection of functions f that maps each location profile to a probability distribution over $\mathbb{R}$: if $f(x_1, x_2, \ldots, x_n)$ is the distribution π, then the facility is located by drawing a single sample from π.

Our focus will be on deterministic and randomized mechanisms for the problem of locating a single facility when the location of any agent is *private* information to that agent and cannot be observed or otherwise verified. It is therefore critical that the mechanism be *strategyproof*—it should be optimal for each agent i to report his *true* location x_i rather than something else. To that end we assume that if the facility is located at y, an agent's disutility, equivalently cost, is simply his distance to y. Thus, an agent whose true location is x_i incurs a cost $C(x_i, y) = |x_i - y|$. If the location of the facility is random and according to a distribution π, then the cost of agent i is simply $C(x_i, \pi) = \mathbb{E}_{y \sim \pi}|x_i - y|$, where y is a random variable with distribution π. The formal definition of strategyproofness is now:

Definition 1. *A mechanism f is strategyproof if for each $i \in N$, each $x_i, x_i' \in \mathbb{R}$, and for each* $\mathbf{x_{-i}} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots x_n) \in \mathbb{R}^{n-1}$,

$$C(x_i, f(x_i, \mathbf{x_{-i}})) \leq C(x_i, f(x_i', \mathbf{x_{-i}})),$$

where $(\alpha, \mathbf{x_{-i}})$ denotes a vector with the i-th component being α and the j-th component being x_j for all $j \neq i$.

The class of strategyproof mechanisms is quite large: for example, locating the facility at agent 1's reported location is strategyproof, but is not particularly appealing because it fails almost every reasonable notion of fairness and could also be highly "inefficient". To address these issues, and to winnow the class of acceptable mechanisms, we impose additional requirements that stem from efficiency or fairness considerations. In this paper we assume that locating a facility at y for the location profile is $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ incurs the *social cost*

$$sc(\mathbf{x}, y) \;=\; \left(\sum_{i \in N} |x_i - y|^p \right)^{1/p}, \quad p \geq 1.$$

For a randomized mechanism f that maps x to a distribution π, we define the social cost to be

$$sc(\mathbf{x}, \pi) \;=\; \mathbb{E}_{y \sim \pi} \left(\sum_{i \in N} |x_i - y|^p \right)^{1/p}.$$

For this definition of social cost, our goal now is to find a strategyproof mechanism that does well with respect to minimizing the social cost. A natural mechanism (and this is the approach taken

in the classical literature on facility location) is the "optimal" mechanism: each location profile $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is mapped to $OPT(\mathbf{x})$, defined as[2]

$$OPT(\mathbf{x}) \in \arg\min_{y \in \mathbb{R}} sc(\mathbf{x}, y).$$

This optimal mechanism is not strategyproof as shown in the following example.

Example. Suppose there are two agents located at the points 0 and 1 respectively on the real line. If they report their locations truthfully, the optimal mechanism will locate the facility at $y = 0.5$, for any $p > 1$. Assuming agent 2 reports $x_2 = 1$, if agent 1 reports $x'_1 = -1$ instead, the facility will be located at 0, which is best for agent 1.

Given that strategyproofness and optimality cannot be achieved simultaneously, it is necessary to find a tradeoff. In this paper we shall restrict ourselves to strategyproof mechanisms that approximate the optimal social cost as best as possible. The notion of approxmation that we use is standard in computer science: an α-approximation algorithm is one that is guaranteed to have cost no more than α times the optimal social cost. Formally, the approximation ratio of an algorithm A is $\sup_I \{A(I)/OPT(I)\}$, where the supremum is taken over all possible instances I of the problem; and $A(I)$ and $OPT(I)$ are, respectively, the costs incurred by algorithm A and the optimal algorithm on the instance I.

Our goal then is to design strategyproof (deterministic or randomized) mechanisms whose approximation ratio is as close to 1 as possible.

3   The Median Mechanism

For the location profile $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the median mechanism is a deterministic mechanism that locates the facility at the "median" of the reported locations. The median is unique if n is odd, but not when n is even, so we need to be more specific in describing the mechanism. For odd n, say $n = 2k - 1$ for some $k \geq 1$, the facility is located at $x_{[k]}$, where $x_{[k]}$ is the kth largest component of the location profile. For even n, say $n = 2k$, the "median" can be any point in the interval $[x_{[k]}, x_{[k+1]}]$; to ensure strategyproofness, we need to pick either $x_{[k]}$ or $x_{[k+1]}$, and as a matter of convention we take the median to be $x_{[k]}$. It is well known that the median mechanism is strategyproof [3]. Furthermore, the median mechanism is *anonymous*[4]. Thus we may assume,

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[2] Strictly speaking, the mechanism is not well defined in cases where the social cost at x is minimized by multiple locations y, but we could pick an exogenous tie-braking rule to deal with such cases.

[3] A classical paper of Moulin [5] for a closely related model shows that all deterministic strategyproof mechanisms are essentially generalized median mechanisms.

[4] In an anonymous mechanism, the facility location is the same for two location profiles that are permutations of each other.

without loss of generality, that each agent reports her location truthfully.

Our main result in this section is that, for any $p \geq 1$, the median mechanism uniformly achieves the best possible approximation ratio among all deterministic strategyproof mechanisms. We start with two simple observations, which will be used repeatedly in the proof of this main result.

Lemma 1. *For any real numbers a, b, c with $a \leq b \leq c$, and any $p \geq 1$,*

$$(c - a)^p \leq 2^{p-1}[(c - b)^p + (b - a)^p].$$

Proof. For any $p \geq 1$, $f(x) = x^p$ is a convex function on $[0, \infty)$, and so for any $\lambda \in [0, 1]$ and $x, y \geq 0$,

$$f(\lambda x + (1 - \lambda)y)) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{1}$$

Setting $\lambda = 1/2$, $x = c - b$, and $y = b - a$, and multiplying both sides of the inequality by 2^p gives the result. $\qquad\square$

Lemma 2. *For any non-negative real numbers a and b, and any $p \geq 1$,*

$$(a + b)^p \geq a^p + b^p.$$

Proof. For integer p, the result is a direct consequence of the binomial theorem; the same argument covers the case of rational p as well. Continuity implies the result for all p. $\qquad\square$

Theorem 1. *Suppose there are n agents with the location profile $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Define the social cost of locating a facility at y as $(\sum_{i=1}^{n} |y - x_i|^p)^{\frac{1}{p}}$ for $p \geq 1$. The social cost incurred by the median mechanism is at most $2^{1-\frac{1}{p}}$ times the optimal social cost* [5].

Proof. We may assume that $x_1 \leq \ldots \leq x_n$. Let OPT be a facility location that minimizes the social cost, and let m be the median. The inequality we need to prove is

$$\sum_{i=1}^{n} |m - x_i|^p \leq 2^{p-1} \sum_{i=1}^{n} |OPT - x_i|^p.$$

We do this by pairing each location x_i with its "symmetric" location x_{n+1-i} and arguing that the total cost of these two locations in the median mechanism is within the required bound of their total cost in an optimal solution. For even n, this completes the argument; for odd n the only location without such a pair is the median itself, which incurs *zero* cost in the median mechanism, and so the argument is complete. Formally, the result follows if we can show

$$|m - x_i|^p + |x_{n+1-i} - m|^p \leq 2^{p-1}(|OPT - x_i|^p + |OPT - x_{n+1-i}|^p), \quad \text{forall } i \leq \lfloor n/2 \rfloor.$$

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[5] When $p = \infty$, the median mechanism provides a 2-approximation, as shown in Procaccia and Tennenholtz [6].

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We consider two cases, depending on whether OPT is in the interval $[x_i, x_{n+1-i}]$ or not. In each of these cases, OPT may be above the median or below, but the proof remains identical in each subcase, so we give only one.

1. $x_i \leq m \leq OPT \leq x_{n+1-i}$ or $x_i \leq OPT \leq m \leq x_{n+1-i}$. We will prove the first of these subcases; the proof of the second is identical. Applying Lemma 1 by setting $a = m$, $b = OPT$, and $c = x_{n+1-i}$, we get

$$|x_{n+1-i} - m|^p \leq 2^{p-1}(|x_{n+1-i} - OPT|^p + |OPT - m|^p).$$

Thus,

$$|m - x_i|^p + |x_{n+1-i} - m|^p \leq |m - x_i|^p + 2^{p-1}(|x_{n+1-i} - OPT|^p + |OPT - m|^p)$$
$$\leq 2^{p-1}(|m - x_i|^p + |x_{n+1-i} - OPT|^p + |OPT - m|^p)$$
$$\leq 2^{p-1}(|x_{n+1-i} - OPT|^p + |OPT - x_i|^p),$$

where the last inequality is obtained by applying Lemma 2 to the terms $|m - x_i|^p$ and $|OPT - m|^p$.

2. $OPT \leq x_i \leq m \leq x_{n+1-i}$ or $x_i \leq m \leq x_{n+1-i} \leq OPT$. Again, we prove only the first subcase. Note that

$$|x_{n+1-i} - m|^p + |m - x_i|^p \leq |x_{n+1-i} - x_i|^p$$
$$\leq |OPT - x_{n+1-i}|^p$$
$$\leq 2^{p-1}(|OPT - x_i|^p + |OPT - x_{n+1-i}|^p)$$

where the first inequality follows from Lemma 2. (Note that Lemma 1 is not used in the proof of this case.)

We end this section by showing that *no* deterministic and strategyproof mechanism can give a better approximation to the social cost.

Lemma 3. *Consider the case of two agents and suppose the location profile is (x_1, x_2) with $x_1 < x_2$. For $p \geq 1$, suppose the social cost of locating a facility at y is $(|x_1 - y|^p + |x_2 - y|^p)^{1/p}$. Any determinstic mechanism whose approximation ratio is better than $2^{1 - \frac{1}{p}}$ for $p > 1$ must locate the facility at y for some $y \in (x_1, x_2)$.*

Proof. The function $f(y) = |y - x_1|^p + |y - x_2|^p$ is strictly convex, and its unique minimizer is $y^* = (x_1 + x_2)/2$, with the corresponding value $f(y^*) = |x_2 - x_1|^p/2^{p-1}$. Moreover $f(x_1) = f(x_2) = |x_2 - x_1|^p = 2^{p-1}f(y^*)$. It follows that for the deterministic mechanism to do strictly better than the stated ratio, the facility cannot be located at the reported locations; locating the facility to the left of x_1 or to the right of x_2 only increases the cost of the mechanism, so the only option left for a mechanism to do better is to locate the facility in the interior, i.e., in (x_1, x_2). $\square$

Theorem 2. *Any strategyproof deterministic mechanism has an approximation ratio of at least $2^{1-\frac{1}{p}}$ for the L_p social cost function for any $p \geq 1$[6].*

Proof. The bound holds trivially for $p = 1$. Suppose $p > 1$, and suppose a deterministic strategyproof mechanism yields an approximation ratio strictly better than $2^{1-\frac{1}{p}}$ to the L_p social cost. For the two-agent location profile $x_1 = 0, x_2 = 1$, Lemma 3 implies the facility is located at some $y \in (0, 1)$. Now consider the location profile $x_1 = 0, x_2 = y$. Again, by Lemma 3, the mechanism locate the facility at $y' \in (0, y)$ to guarantee the improved approximation. But if agent 2 is located at $y < 1$, he can misreport his location as 1, forcing the mechanism to locate the facility at y, his true location; this violates strategyproofness. $\square$

4 Randomized Mechanisms

Recall that when the social cost is measured by the L_2 norm or the L_∞ norm, randomization provably improves the approximation ratio. In the former case, Feldman and Wilf [4] describe an algorithm whose approximation ratio is $(\sqrt{2}+1)/2$; for the latter, Procaccia and Tennenholtz [6] design an algorithm with an approximation ratio of $3/2$. The mechanisms in both cases are simple and somewhat similar, placing non-negative probabilities *only* on the optimal and reported locations, where these probabilities are independent of the reported location profile. There are two reasonable ways of choosing the reported locations: one is via dictatorships and the other is via generalized medians. In this section we show that neither is enough; namely, randomizing over dictatorships, generalized medians and the optimal location does not improve the approximation ratio of the median mechanism for *any* integer $p \in (2, \infty)$. For the case of 2 agents we show that the best approximation ratio is given by the LRM mechanism among all strategyproof mechanisms. Extending this analysis even to the case of 3 agents appears to be non-trivial.

4.1 Mixing Dictatorships and Generalized Medians with the Optimal Location

Theorem 3. *Suppose we are given non-negative numbers $p_j^{n'}$, $p_j^{n''}$, and p_{OPT}^n with $p_{OPT}^n + \sum_{j \in N} p_j^{n'} + \sum_{j \in N} p_j^{n''} = 1$ for each n. For the problem with n agents and reported profile $(x_1, x_2, \ldots, x_n)$ let f be the strategyproof randomized mechanism that locates the facility at OPT with probability p_{OPT}^n, at x_j with probability $p_j^{n'}$, and at $x_{[j]}$ with probability $p_j^{n''}$[7], where OPT is the optimal location for the profile $(x_1, x_2, \ldots, x_n)$. Then, for any finite integer $p > 2$, the approximation ratio of f is at least $2^{1-\frac{1}{p}}$.*

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[6]The lower bound of 2 on the approximation ratio also holds when $p = \infty$, see Procaccia and Tennenholtz [6].

[7]When a location appears more than once in $x_1, \ldots, x_n, x_{[1]}, \ldots, x_{[n]}, OPT$, the probabilities add up.

Proof. Fix $n = 2k$, with $k \in \mathbb{N}$. For notational convenience, let $p_j^n = p_j^{n'} + p_j^{n''}$ (we will use this to analyze profiles where $x_j = x_{[j]}$). For $j = 1, \ldots, k$, define the profile $\mathbf{x}^j$ as follows (where a_j is a parameter to be defined shortly): agents 1 through j are located at $-a_j$; agents $j + 1$ through k are located at 0; agents $k + 1$ through $2k - j + 1$ are located at 1; and agents $2k - j + 2$ through $2k$ are located at $1 + a_j$. (Note the *slight* asymmetry in the location of the agents: while k agents are at or below zero, and k agents are at or above 1, there is an additional agent at 1 compared to zero and so one less agent at $1 + a_j$ compared to $-a_j$.) Now, a_j is chosen to be the smallest positive root of the function $g_j(\alpha) = j\alpha^{p-1} - (k - j + 1) - (j - 1)(1 + \alpha)^{p-1}$; such an a_j must exist by the intermediate value theorem, as $g_j(0) < 0$ and $g_j(\alpha)$ is a continuous function of α with $g_j(\alpha) \to \infty$ as $\alpha \to \infty$.

We show that the optimal mechanism locates the facility at zero for the profile $\mathbf{x}^j$, i.e., $OPT = 0$. Note that the social cost for this profile, when locating the facility at $z \in [0, 1]$, is $j(z + a_j)^p + (k - j)z^p + (k - j + 1)(1 - z)^p + (j - 1)(1 + a_j - z)^p$, and when $z \in (-a_j, 0)$ the social cost becomes $j(z + a_j)^p + (k - j)(-z)^p + (k - j + 1)(1 - z)^p + (j - 1)(1 + a_j - z)^p$. Note that the social cost function is differentiable for $z \in (0, 1)$ and for $z \in (-a_j, 0)$. The left and right derivatives at 0 are both $pja_j^{p-1} - p(k - j + 1) - p(j - 1)(1 + a_j)^{p-1}$, and thus the social cost function is differentiable on $(-a_j, 1)$ with its derivative at $z = 0$ equal to zero (by our choice of a_j). The social cost function can also easily be verified to be twice differentiable on $(-a_j, 1)$, with a positive second derivative at $z = 0$, and thus we have a local minimum at $z = 0$. The fact that this is a global minimum now follows from strict convexity of the social cost function $\|\mathbf{x}^j - z(1, \ldots, 1)\|_p$ (for all $z \in \mathbb{R}$). Thus, indeed, $OPT = 0$.

We now attempt to bound p_{OPT}. For each profile $\mathbf{x}^j$, consider the profile $\mathbf{x}^{j'}$ that differs only in the location of agent j: namely, $x_j^{j'} = 0$ instead of $-a_j$. Note that on this profile, $OPT = 0.5$ by symmetry. Strategyproofness implies that a deviation from profile $\mathbf{x}^{j'}$ to profile $\mathbf{x}^j$ should not be beneficial for agent j, namely $a_j p_j^n - \frac{1}{2}p_{OPT}^n \geq 0$ (where a_j is the increase in agent j's cost caused by that deviation when the facility is built in his reported location, and $\frac{1}{2}$ is the decrease in his cost caused by that deviation when the facility is located at OPT), which implies $p_j^n \geq \frac{p_{OPT}^n}{2a_j}$. Defining a_j for $j = k + 1, \ldots, 2k$ in a symmetric fashion, we see that the same inequality holds for j in that range, and that $a_j = a_{2k-j+1}$. Summing those inequalities up, we get:

$$1 - p_{OPT}^n = \sum_{j=1}^{2k} p_j^n \geq \sum_{j=1}^{2k} \frac{p_{OPT}^n}{2a_j} = 2\sum_{j=1}^{k} \frac{p_{OPT}^n}{2a_j} = \sum_{j=1}^{k} \frac{p_{OPT}^n}{a_j}$$

$$p_{OPT}^n \leq \frac{1}{1 + \sum_{j=1}^{k} \frac{1}{a_j}}$$

Now, we claim it is enough to show that as $n \to \infty$ (or equivalently, as $k \to \infty$), $\sum_{j=1}^{k} \frac{1}{a_j} \to \infty$. The inequality then implies that $p_{OPT}^n \to 0$. Consider the profile which locates k agents at 0 and

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k agents at 1. The social cost of locating the facility at OPT on this profile is $\sqrt[p]{n}/2$, while the social cost of locating the facility at an agent's location is $\sqrt[p]{n}2^{-\frac{1}{p}}$; thus, the approximation ratio of f on this profile is $\frac{p_{OPT}^n \sqrt[p]{n}/2 + (1 - p_{OPT}^n) \sqrt[p]{n}2^{-\frac{1}{p}}}{\sqrt[p]{n}/2} = 2^{1-\frac{1}{p}} - (2^{1-\frac{1}{p}} - 1)p_{OPT}^n$. Thus, as $n \to \infty$, the approximation ratio on these profiles approaches $2^{1-\frac{1}{p}}$, completing the proof.

We are left with the task of showing that $\lim_{k \to \infty} \sum_{j=1}^{k} \frac{1}{a_j} = \infty$. To do so, we first show that for $j \geq k^{\frac{1}{p-1}} + 1$, $2^{p-1}(j-1) > a_j$. Recall that a_j was defined as the smallest positive root of $g_j(\alpha)$, and that $g_j(0) < 0$. Thus, it is enough to show that for j in the appropriate range, $g_j(2^{p-1}(j-1)) > 0$. For notational convenience, we denote $Q = 2^{p-1}$.

$$g_j(Q(j-1)) = jQ^{p-1}(j-1)^{p-1} - (k-j+1) - (j-1)(1+Q(j-1))^{p-1}$$

$$= Q^{p-1}(j-1)^{p-1} - k - (j-1)\sum_{i=1}^{p-2}\binom{p-1}{i}(Q(j-1))^{p-1-i}$$

$$\geq Q^{p-1}(j-1)^{p-1} - (j-1)^{p-1} - (j-1)\sum_{i=1}^{p-2}\binom{p-1}{i}(Q(j-1))^{p-1-i}$$

$$> Q^{p-1}(j-1)^{p-1} - (j-1)^{p-1} - (j-1)\sum_{i=1}^{p-2}\binom{p-1}{i}(Q(j-1))^{p-2}$$

$$> Q^{p-1}(j-1)^{p-1} - (j-1)\sum_{i=1}^{p-1}\binom{p-1}{i}(Q(j-1))^{p-2}$$

$$= Q^{p-1}(j-1)^{p-1} - (j-1)(Q(j-1))^{p-2}\sum_{i=1}^{p-1}\binom{p-1}{i}$$

$$> Q^{p-1}(j-1)^{p-1} - (j-1)(Q(j-1))^{p-2}2^{p-1} = 0.$$

Now,

$$\lim_{k\to\infty}\sum_{j=1}^{k}\frac{1}{a_j} > \lim_{k\to\infty}\sum_{j=\lceil k^{\frac{1}{p-1}}+1\rceil}^{k}\frac{1}{2^{p-1}j}$$

$$= \frac{1}{2^{p-1}}\lim_{k\to\infty}\sum_{j=\lceil k^{\frac{1}{p-1}}+1\rceil}^{k}\frac{1}{j}$$

$$\geq \frac{1}{2^{p-1}}\lim_{k\to\infty}\int_{k^{\frac{1}{p-1}}+2}^{k}\frac{1}{t}dt$$

$$= \frac{1}{2^{p-1}}\Big(\lim_{k\to\infty}\int_{k^{\frac{1}{p-1}}}^{k}\frac{1}{t}dt - \lim_{k\to\infty}\int_{k^{\frac{1}{p-1}}}^{k^{\frac{1}{p-1}}+2}\frac{1}{t}dt\Big)$$

$$= \frac{1}{2^{p-1}}\Big(\Big(\lim_{k\to\infty}(1-\frac{1}{p-1})\Big)\ln k\Big) - 0\Big) = \infty$$

which completes our proof. $\qquad\square$

4.2   Optimality of the LRM Mechanism for 2 Agents

Our next result shows that the LRM mechanism provides the best possible approximation ratio among all shift and scale invariant (defined below) startegyproof mechanisms for the case of 2 agents for all L_p social cost functions for $p \geq 1$.

We begin with some definitions: we say that a mechanism f is *shift and scale invariant* if for every location profile $\mathbf{x} = \{x_1, x_2\}$ s.t. $x_1 \leq x_2$ and every $c \in \mathbb{R}$, the following two properties are satisfied:

1. $f(\{x_1 + c, x_2 + c\}) = f(\mathbf{x}) + c$.

2. When $c \geq 0$, we have $f(\{cx_1, cx_2\}) = cf(\mathbf{x})$, and when $c < 0$, we have $f(\{cx_1, cx_2\}) = -cf(\{-x_2, -x_1\})$.

A convenient notation for a given location profile $\mathbf{x}$ is to denote its midpoint as $m_{\mathbf{x}} = \frac{x_1+x_2}{2}$. We say that a mechanism f is *symmetric* if for any location profile $\mathbf{x}$ and for any $y \in \mathbb{R}$, $\mathbb{P}(f(\mathbf{x}) \geq m_{\mathbf{x}} + y) = \mathbb{P}(f(\mathbf{x}) \leq m_{\mathbf{x}} - y)$.

The following lemma allows us to convert any strategyproof mechanism into a symmetric mechanism.

Lemma 4. *Given any strategyproof mechanism, there exists another symmetric strategyproof mechanism with the same approximation ratio.*

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Proof. Given a mechanism f, we define the *mirror mechanism* of f, f_{mirror}, to be such that for any profile $\mathbf{x}$, we have that $\mathbb{P}(f_{mirror}(\mathbf{x}) \geq m_{\mathbf{x}} + b) = \mathbb{P}(f(\mathbf{x}) \leq m_{\mathbf{x}} - b)$ for all $b \in \mathbb{R}$ and location profiles $\mathbf{x}$.

Assume f is a strategyproof mechanism. Symmetry dictates that f_{mirror} must also be strategyproof, since any misreport of the right agent with respect to f induces the same cost as that of an equivalent a misreport of the left agent with respect to f_{mirror} and vice versa. Moreover, since composing two strategproof mechanisms yields a strategyproof mechanism, the mechanism g that picks f with probability $1/2$ and f_{mirror} with probability $1/2$ is a strategyproof mechanism that is also symmetric. Finally, note that g has the same approximation ratio as f for all location profiles, since f_{mirror} has the same approximation ratio as f. $\qquad\square$

From now on, whenever we talk about a shift and scale invariant mechanism, we will also assume that it is symmetric. To simplify our proof of the main result, we will assume in addition that given a reported profile $\mathbf{x} = \{x_1, x_2\}$, the mechanism will only assign a facility location that lies in between x_1 and x_2, i.e. $\mathbb{P}(y \in [x_1, x_2]) = 1$, where y is a random variable representing the facility location assigned by the mechanism. It is worth noting that the main result remains true even without this assumption, although the complete proof is somewhat long and cumbersome, so we will omit it. The next lemma deals with an equivalent condition for strategyproofness with respect to a shift, scale invariant and symmetric mechanism.

Lemma 5. *A shift, scale invariant, and symmetric mechanism f is strategyproof if and only if for any profile $\mathbf{x} = \{x_1, x_2\}$ with $x_1 = 0 < x_2$, the following condition holds:*

$$\int_{(-\infty, x_2)} y dF(y) + \int_{(x_2, \infty)} y dF(y) + x_2 \mathbb{P}(Y = x_2) \geq 0,$$

where $Y = f(\mathbf{x})$ with c.d.f. F.

Proof. By shift invariance, it suffices to check strategyproofness for profiles where $x_1 = 0$ and by symmetry, we can assume without lost of generality that $x_2 \geq 0$. Moreover, any shift, scale invariant mechanism is trivially strategyproof with respect to the profile $\{0, 0\}$ since by definition, the mechanism would place all of the probability mass on 0, which means that no agent has incentive to misreport his location. Thus, we can assume that $x_2 > 0$.

Since the mechanism is symmetric, it suffices to show that agent 2 cannot benefit by deviating from his true location if and only if the aforementioned condition hold. Since $x_2 > 0$, we can denote agent 2's deviation x_2' as cx_2 for some $c \in \mathbb{R}$. Moreover, since $\mathbb{P}(Y \in [x_1, x_2]) = 1$ by assumption, we can further restrict ourselves to the case where $c > 1$ because agent 2 has no incentive to deviate to a location cx_2 where $cx_2 < x_2$ as the mechanism is scale invariant.

When agent 2 reports his location to be cx_2, where $c > 1$, the change in cost incurred by agent 2 is:

$$C_{dev} - C_{orig} = -(c-1)\int_{(-\infty,\frac{x_2}{c})} ydF(y) + \int_{[\frac{x_2}{c},x_2)} ((c+1)y - 2x_2)dF(y) + (c-1)\int_{(x_2,\infty)} ydF(y)$$

$$+ (c-1)x_2\mathbb{P}(Y = x_2)$$

$$= -(c-1)\int_{(-\infty,x_2)} ydF(y) + \int_{[\frac{x_2}{c},x_2)} (2cy - 2x_2)dF(y) + (c-1)\int_{(x_2,\infty)} ydF(y) +$$

$$(c-1)x_2\mathbb{P}(Y = x_2)$$

$$\geq -(c-1)\int_{(-\infty,x_2)} ydF(y) + (c-1)\int_{(x_2,\infty)} ydF(y) + (c-1)x_2\mathbb{P}(Y = x_2)$$

Hence, when condition 1 holds, we have that $-(c-1)\int_{(-\infty,x_2)} ydF(y) + (c-1)\int_{(x_2,\infty)} yF(y) + (c-1)x_2\mathbb{P}(Y = x_2) \geq 0$, which means that $C_{dev} - C_{orig} \geq 0$.

To prove the other direction, suppose the condition does not hold, then there exists $\epsilon > 0$ small enough such that $-\int_{(-\infty,x_2)} ydF(y) + \int_{(x_2,\infty)} ydF(y) + x_2\mathbb{P}(Y = x_2) \leq -\epsilon$ for some $x_2 > 0$. We choose $c > 1$ s.t. $\mathbb{P}(Y \in [\frac{x_2}{c}, x_2)) < \frac{\epsilon}{4x_2}$, then we have that

$$C_{dev} - C_{orig} = -(c-1)\int_{(-\infty,x_2)} ydF(y) + \int_{[\frac{x_2}{c},x_2)} (2cy - 2x_2)dF(y) + (c-1)\int_{(x_2,\infty)} ydF(y)$$

$$+ (c-1)x_2\mathbb{P}(Y = x_2)$$

$$\leq (c-1)(-\int_{(-\infty,x_2)} ydF(y) + \int_{[\frac{x_2}{c},x_2)} (2x_2)dF(y) + \int_{(x_2,\infty)} ydF(y) + x_2\mathbb{P}(Y = x_2))$$

$$< -(c-1)\frac{\epsilon}{2} < 0$$

which contradicts strategyproofness of the mechanism. [8] $\qquad\square$

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[8]Notice that given any shift, scale invariant, and symmetric mechanism f, in order to check whether f is a strategyproof mechanism, it suffices to check whether f is strateyproof for one particular profile. Without lost of generality, we can assume that $x_1 = 0$ and $x_2 = 1$. Here is a short proof of the claim. By the same argument as before, it suffices to check strategyproofness for all profiles $\{x_1, x_2\}$, where $x_2 > x_1 = 0$. Let $Y = f(\{0, 1\})$, then $f(\{0, x_2\}) = x_2Y$. The mechanism is strategyproof with respective to the profile $\{x_1, x_2\}$ if and only if for all $c \in \mathbb{R}$, we have that

$$E[|cx_2Y - x_2|] \geq E[|x_2Y - x_2|].$$

Since $x_2 > 0$, this follows directly from the strategyproofness condition for the profile $\{0, 1\}$:

$$E[|cY - 1|] \geq E[|Y - 1|] \ \forall c \in \mathbb{R}.$$

Given a strategyproof, shift, scale invariant and symmetric mechanism, the next lemma demonstrates how to find another strategyproof, shift, scale invariant and symmetric mechanism that restricts the probability assignment to $x_1, x_2,$ and $m_{\mathbf{x}}$ for all profile $\mathbf{x}$ and simultaneously gives a better approximation than the original mechanism.

Lemma 6. *Let f be a strategyproof shift, scale invariant and symmetric mechanism, where $\mathbb{P}(f(\mathbf{x}) \in [x_1, x_2]) = 1$ for location profile $\mathbf{x} = \{x_1, x_2\}$ with $x_2 > x_1$, then there exists another strategyproof mechanism g such that $\mathbb{P}(g(\mathbf{x}) \in \{x_1, x_2, m_{\mathbf{x}}\}) = 1$ for the (and thus every) location profile $\mathbf{x}$ and that $E[sc(g(\mathbf{x}), \mathbf{x})] \leq E[sc(g(\mathbf{x}), \mathbf{x})]$. Furthermore, g satisfies shift, scale invariance and symmetry.*

Proof. Now, let g be the mechanism that satisfies $\mathbb{P}(g(\mathbf{x}) = x_1) = \mathbb{P}(f(\mathbf{x}) = x_1)$, $\mathbb{P}(g(\mathbf{x}) = x_2) = \mathbb{P}(f(x) = x_2)$, $\mathbb{P}(g(\mathbf{x}) = m_x) = 1 - \mathbb{P}(g(\mathbf{x}) = x_1) - \mathbb{P}(g(\mathbf{x}) = x_2)$. Note that since m_x minimizes the social cost function for the profile $\mathbf{x}$, g certainly provides a weakly better approximation ratio than f. By shift invariance, we can assume wlog that $0 = x_1 \leq x_2$, then an alternative way to show strategyproofness is to check to see that the condition of the lemma 1 is satisfied by g. Since f is a strategyproof mechanism, the condition implies that

$$0 \leq - \int_{(0,x_2)} y d(F(y)) + x_2 \mathbb{P}(f(\mathbf{x}) = x_2)$$

$$= - \int_{(-\frac{x_2-x_1}{2}, \frac{x_2-x_1}{2})} (m_{\mathbf{x}} + u) d(F(m_{\mathbf{x}} + u)) + x_2 \mathbb{P}(f(\mathbf{x}) = x_2)$$

$$= -m_{\mathbf{x}} \mathbb{P}(f(\mathbf{x}) \in (x_1, x_2)) + - \int_{(-\frac{x_2-x_1}{2}, \frac{x_2-x_1}{2})} u d(F(m_{\mathbf{x}} + u)) + x_2 \mathbb{P}(f(\mathbf{x}) = x_2)$$

$$= -m_{\mathbf{x}}(1 - \mathbb{P}(g(\mathbf{x}) = x_1) - \mathbb{P}(g(\mathbf{x}) = x_2)) + x_2 \mathbb{P}(g(\mathbf{x}) = x_2)$$

Note, $- \int_{(-\frac{x_2-x_1}{2}, \frac{x_2-x_1}{2})} u d(F(m_{\mathbf{x}} + u)) = 0$ because the distribution is symmetric around $m_{\mathbf{x}}$. Hence, the condition is satisfied for the mechanism g. $\square$

Thus, g is a symmetric strategyproof mechanism that provides a weakly better approximation ratio than f and which satisfies $\mathbb{P}(g(\mathbf{x}) \in \{x_1, x_2, m_{\mathbf{x}}\}) = 1$ for every location profile $\mathbf{x}$.

Now we are ready to prove the main theorem.

Theorem 4. *The LRM mechanism gives the best approximation ratio among all strategyproof mechanisms that are shift and scale invariant.*

Proof. By the previous lemma, it suffices to search among the class of strategyproof shift, scale invariant and symmetric mechanisms where any element f of the class satisfies the property that $\mathbb{P}(f(\mathbf{x}) \in \{x_1, x_2, m_{\mathbf{x}}\}) = 1$. It is not difficult to see that in order to enforce strategyproofness, we

must have that $\mathbb{P}(f(\mathbf{x}) \in \{x_1, x_2\}) \geq 0.5$, which implies that among all such mechanisms, LRM provides the best approximation ratio of $0.5(2^{1-\frac{1}{p}} + 1)$. $\qquad\square$

An immediate consequence of Theorem 4 is the following corollary.

Corollary 1. *Any strategyproof shift and scale invariant mechanism has an approximation of at least $0.5(2^{1-\frac{1}{p}} + 1)$ in the worst case.*

5   Discussion

The most important open question in our view is whether or not randomization can help improve the worst-case approximation ratio for general L_p norm cost functions. The case of $p = 1$ is uninteresting because there is an optimal deterministic mechanism; for $p = 2$ and $p = \infty$ we already saw that randomization improves the worst-case approximation ratio, but we do not know if this is simply a happy coincidence, or if one can obtain similar results for all $p > 2$. Our negative result in Section 4 implies that any improvement by randomization would require a different approach than the existing mechanisms.

There are many other natural questions as well: for instance, what happens for more general topologies such as trees or cycles? Is it possible to characterize all randomized strategy-proof mechanisms on specific topologies?

Finally, we believe it is of interest to consider more general cost functions for the individual agents. The properties established for the LRM and many other randomized mechanisms depend on the assumption that agents incur costs that are *exactly equal* to the distance to access the facility. Clearly, this is a very restrictive assumption, and working with more general individual agent costs is a promising direction to broaden the applicability of this class of models.[9]

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