## Nonzero-sum Risk-sensitive Stochastic Games on a Countable State Space

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The infinite horizon risk-sensitive discounted-cost and ergodic-cost nonzero-sum stochastic games for controlled Markov chains with countably many states are analyzed. For the discounted-cost game, we prove the existence of Nash equilibrium strategies in the class of Markov strategies under fairly general conditions. Under an additional geometric ergodicity condition and a small cost criterion, the existence of Nash equilibrium strategies in the class of stationary Markov strategies is proved for the ergodic-cost game.

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1. Introduction We study risk-sensitive nonzero-sum stochastic games on the infinite time horizon on a countable state space. Risk-sensitive cost criterion plays an important role in many applications including mathematical finance (see, e.g., Bilelecki and Pliska [8], Nagai [32]). In this criterion one investigates 'exponential of integral' cost which takes into account the attitude of the controller with respect to risk. The study of this kind of cost criteria was first initiated by Bellman [6], p. 329 for the finite-state space case. Howard and Matheson [26] did an in-depth analysis for the first time in the finite-state space case where each controlled chain is irreducible and aperiodic. Rothblum [33] extended it to the general non-irreducible finite-state space case. In the past two decades, there has been a renewed interest in this type of cost criteria as, when the 'risk factor' is strictly positive, i.e., in the risk-averse case, the use of the exponential reduces the possibility of rare but devastating large excursions of the state process. Though this criterion has been studied extensively in the literature of Markov decision processes (see, e.g., Borkar and Meyn [13], Cavazos-Cadena and Fernandez-Gaucherand [14], Di Masi and Stettner [15, 16, 17], Fleming and Hernández-Hernández [21], Fleming and McEneaney [22], Hernández-Hernández and

Marcus [24], Whittle [35, 36]), the corresponding results on stochastic games seem to be limited (see e.g., Basar [3], El-Karoui and Hamadene [18], Jacobson [27], James et. al. [28], and, Klompstra [30]). The general Linear-Exponential-Gaussian (LEG) control problem for discrete time with perfect state observation is treated in Jacobson [27] where an equivalence of this with deterministic zero-sum quadratic-cost games was shown. Whereas this paper addresses the undiscounted case, the corresponding discounted case was addressed by Hansen and Sargent [23]. This was further extended to studying the Nash equilibrium for a two-person discrete-time nonzero-sum game with quadratic-exponential cost criteria in Klompstra [30], the analogue of the Linear-Exponential-Quadratic-Gaussian (LEQG) control problem studied in Whittle [34, 35]. The papers of Basar [3] and El-Karoui et. al. [18] deal with stochastic differential games on the finite time horizon. In James et. al. [28], the finite-horizon risk-sensitive stochastic optimal control problem for discrete-time nonlinear systems was studied and its relation to a deterministic partially observed dynamic game was established. To the best of our knowledge, the general case of infinite-horizon risk-sensitive stochastic zero-sum games for both discounted as well as ergodic cost criteria was first addressed by the current authors for the differential games setup in Basu and Ghosh [4] and for the discrete-time countable state space case in Basu and Ghosh [5]. The case of nonzero-sum games in such setups still remains open. In this work, we address this novel problem and analyze the generic nonzerosum case for the discrete-time countable state space setup using totally different mathematical techniques for its solution as compared to the corresponding zero-sum case. We study this problem with two players as the analysis can be routinely extended to three or more player case without introducing any technical novelty but at the cost of further notational complication.

We use the results of Balaji and Meyn [2] to extend the work of Borkar and Meyn [13] from one-controller case to two-controller case in a fully competitive setup. In other words we study nonzero-sum risk-sensitive stochastic games on the infinite planning horizon with both discounted and ergodic cost criteria. We would like to elucidate that both Balaji and Meyn [2] as well as Borkar and Meyn [13] have obtained the desired results under a "norm-like" or "near-monotone" condition on the running cost and a Lyapunov-type stability condition. This "norm-like" condition has been crucially used in the proofs therein as to ensure that the 'relative value functions' are bounded from below. However, for our case, we have performed the analysis without this "norm-like" condition on the running cost since it would not be suitable for our purpose as it would invariably favour one of the competing players. This change makes our analysis totally novel and substantially different from those in the existing literature. Also, in most of the existing literature in this domain see, e.g., Balaji and Meyn [2], Borkar and Meyn [13], Cavazos-Cadena and Fernandez-Gaucherand [14], Di Masi and Stettner [16], and, Hernández-Hernández and Marcus [24], the 'risk factor' is assumed to be sufficiently small. We make an assumption, for the ergodic game only, on the smallness of the cost function as in Basu and Ghosh [4] which essentially implies that the 'risk factor' cannot be too large.

Under certain assumptions, we have established the existence of Nash equilibria for both criteria. We obtain our results by studying the corresponding Bellman equations. Note that if player I announces that he is using a stationary/Markov strategy then for player II the game problem reduces to a Markov decision problem (MDP). Then by the results of Borkar and Meyn [13], Di Masi and Stettner [15] and Hernández-Hernández and Marcus [24], player II has optimal stationary/Markov strategies. Such a strategy of player II is called an 'optimal response' of player II corresponding to the announced strategy of player I. Optimal responses of player I to the announced strategies of player II are obtained analogously. Thus, for a given pair of strategies of the two players, there exists a set of optimal responses. This defines a point-to-set map. Any fixed point of this map is clearly a Nash equilibrium. In this paper, we establish the existence of such a fixed point thereby establishing the existence of Nash equilibria for relevant cases. For the sake of notational simplicity, we consider two-player games only. All our results extends to multi-player games in a routine manner.

The rest of our paper is structured as follows. Section 2 deals with the description of the problem. The discounted cost criterion is studied in Section 3. Section 4 deals with the ergodic cost criterion. We conclude our paper in Section 5 with a summary and possible future directions of work.

2. Problem Description A two-person stochastic game is determined by six objects  $(X, U, V, r_1, r_2, q)$  where  $X \stackrel{def}{=} \{0, 1, 2, ...\}$  is a countable state space; U and V are action spaces of players I and II, resp., assumed to be compact metric spaces;  $r_1(\text{resp. } r_2) : X \times U \times V \mapsto \mathbb{R}$  is the one-stage cost function for player I (resp. II) assumed to be bounded and jointly continuous in  $(u, v) \in U \times V$  for each  $k \in X$ . Let  $\mathcal{P}(X)$  be the space of probability measures on X endowed with the Prohorov topology (see, e.g., Borkar [12]). Let  $q : X \times U \times V \mapsto \mathcal{P}(X)$  be the transition stochastic kernel which is assumed to be jointly continuous in  $(u, v) \in U \times V$  in the topology of weak convergence for each  $k \in X$ . The game is played is as follows: At each stage (time instant) players observe the current state  $k \in X$  of the system and then players I and II independently choose actions  $u \in U$ ,  $v \in V$ , resp. As a result of this, two things happen:

(i) the player I (resp. II) pays an immediate cost  $r_1(k, u, v)$  (resp.  $r_2(k, u, v)$ ),

(ii) the system moves to a new state  $k' \in X$  with distribution  $q(\cdot|k, u, v)$ .

The whole process then repeats from the new state k'. Cost accumulates throughout the course of the game. The planning horizon or total number of stages is infinite, and each player wants to minimize his infinite-horizon multiplicative expected cost to be described shortly. At each stage the players choose their actions independently on the basis of past information. The available information for decision making at time  $t \in \mathbb{N}_0 \stackrel{def}{=} \{0, 1, 2...\}$  is given by the history of the process up to that time

$$h_t \stackrel{def}{=} (k_0, (u_0, v_0), k_1, (u_1, v_1), \dots, (u_{t-1}, v_{t-1}), k_t) \in H_t,$$

where  $H_0 = X$ ,  $H_t = H_{t-1} \times (U \times V \times X), \dots, H_{\infty} = (U \times V \times X)^{\infty}$  are the history spaces. A strategy for player I is a sequence  $\mu \stackrel{def}{=} {\{\mu_t : H_t \mapsto \mathcal{P}(U)\}_{t \in \mathbb{N}}}$  of stochastic kernels. The set of all such strategies for player I is denoted by  $\Pi_1$ . Given any  $t, T \in \mathbb{N}_0$  with  $T \ge t+1$ , the set of strategies  ${\{\mu_s : H_s \mapsto \mathcal{P}(U)\}_{t \le s \le T}}$  played from time t to time T is denoted by  $\Pi_1^{t,T}$ . Hence,  $\Pi_1^{0,\infty} \equiv \Pi_1$ .

DEFINITION 1. A strategy  $\mu$  for player I is called a Markov strategy if

$$\mu_t(h_{t-1}, u, v, k)(\cdot) = \mu_t(h'_{t-1}, u', v', k)(\cdot)$$

for all  $h_{t-1}, h'_{t-1} \in H_{t-1}, u, u' \in U, v, v' \in V, k \in X, t \in \mathbb{N}_0$ . Thus a Markov strategy for player I can be identified with a sequence of measurable maps, denoted by  $\mu \equiv \{\mu_t : X \mapsto \mathcal{P}(U)\}_{t \in \mathbb{N}}$ . A Markov strategy  $\{\mu_t\}$  is called stationary Markov if  $\mu_t = \mu : X \mapsto \mathcal{P}(U)$  for all t. A stationary Markov strategy is called pure or deterministic if  $\mu : X \mapsto U$ .

Let  $\mathcal{M}_1, \mathcal{S}_1, D_1$  denote the set of Markov, stationary Markov and deterministic strategies strategies for player I. The strategy sets  $\Pi_2, \Pi_2^{t,T}, \mathcal{M}_2, \mathcal{S}_2, D_2$  for player II are defined analogously. The spaces  $\mathcal{M}_i, \mathcal{S}_i, i = 1, 2$  are endowed with the product topologies derived from the Prohorov topology on the underlying spaces  $\mathcal{P}(U)$  (resp.  $\mathcal{P}(V)$ ) respectively. Since U, V are compact metric spaces, it follows that  $\mathcal{M}_i, \mathcal{S}_i, i = 1, 2$  are also compact metric spaces. Note that going forward we sometimes use barbarism of notation and denote a stationary strategy  $\{\mu, \mu, \ldots\}$  as  $\{\mu\}$  or only  $\mu$  when the context is clear.

Given an initial distribution  $\pi_0 \in \mathcal{P}(X)$  and a pair of strategies  $(\mu, \nu) \in \Pi_1 \times \Pi_2$ , the corresponding state and action processes  $\{X_t\}, \{U_t\}, \{V_t\}$  are stochastic processes defined on the canonical space  $(H_{\infty}, \mathcal{B}(H_{\infty}), P_{\pi_0}^{\mu,\nu})$   $(\mathcal{B}(H_{\infty}) = \text{Borel } \sigma\text{-field on } H_{\infty})$  via the projections  $X_t(h_{\infty}) = k_t, U_t(h_{\infty}) =$  $u_t, V_t(h_{\infty}) = v_t$ , where  $P_{\pi_0}^{\mu,\nu}$  is uniquely determined by  $\mu, \nu$  and  $\pi_0$  by Ionescu Tulcea's Theorem (see, e.g., Proposition 7.28 of Bertsekas and Shreve [10]). When  $\pi_0 = \delta_k, \ k \in X$ , we simply write  $P_k^{\mu,\nu}$ .

Let  $(\mu, \nu) \in \Pi_1 \times \Pi_2$  and let  $(X_s^{(t)}, U_s^{(t)}, V_s^{(t)} : s \ge t)$  be the corresponding process starting from  $X_t^{(t)} = k \in X$  for given  $t \ge 0$ . We omit the superscript when t = 0 for notational convenience.

DEFINITION 2. The risk-sensitive discounted cost for player I (i = 1) and II (i = 2) is defined by

$$\rho_i^{\mu,\nu}(\theta,(k,t)) = \frac{1}{\theta} \ln E_{k,t}^{\mu,\nu} \left[ e^{\theta \sum_{s=t}^{\infty} \alpha^{s-t} r_i(X_s^{(t)}, U_s^{(t)}, V_s^{(t)})} \right],\tag{1}$$

where  $\theta \in (0, \Theta]$ ,  $\Theta > 0$  is the 'risk-sensitive' parameter,  $\alpha \in [0, 1]$  is the 'discount factor' and  $E_{k,t}^{\mu,\nu}$  denotes the expectation with respect to  $P_{k,t}^{\mu,\nu}$ . When t = 0, we omit this subscript 't' for notational convenience. The corresponding risk-sensitive ergodic cost for player i, i = 1, 2 is defined by

$$\beta_i^{\mu,\nu}(\theta,k) = \limsup_{T \to \infty} \frac{1}{\theta T} \ln E_k^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{T-1} r_i(X_t, U_t, V_s)} \right].$$
(2)

DEFINITION 3. Given a  $(\theta, k) \in (0, \Theta] \times X$ , a pair of strategies  $(\mu^*, \nu^*) \in \Pi_1 \times \Pi_2$  is called a Nash equilibrium (for the cost criteria above) if

$$L_{1}^{\mu^{*},\nu^{*}}(\theta,k) \leq L_{1}^{\mu,\nu^{*}}(\theta,k), \ \mu \in \Pi_{1},$$
  
and  
$$L_{2}^{\mu^{*},\nu^{*}}(\theta,k) \leq L_{2}^{\mu^{*},\nu}(\theta,k), \ \nu \in \Pi_{2},$$
(3)

where for  $i = 1, 2, L_i = \rho_i$  or  $L_i = \beta_i$  as the case may be.

For a given  $\theta \in (0, \Theta]$ , a pair  $(\mu^*(\theta, \cdot), \nu^*(\theta, \cdot))$  of stationary/Markov strategies (depending on  $\theta$ ) is said to be Nash equilibrium strategies if these measurable maps constitute a Nash equilibrium for any initial  $k \in X$ .

We shall first establish the existence of Nash equilibria in the class of Markov strategies for cost criterion (1). Under additional ergodicity and smallness of cost conditions, we establish the existence of Nash equilibria in the class of stationary Markov strategies for cost criterion (2).

Given a topological space Y, we denote by B(Y) and  $C_b(Y)$  the Banach spaces of bounded measurable and bounded continuous functions on Y respectively, each equipped with the sup-norm metric.

The following result established in Section II.3, Chapter II (pp. 25-30) of Borkar [11] plays a crucial role in Sections 3 and 4. Given any set of stochastic process  $\{Y_t^{(1)}\}, \{Y_t^{(2)}\}, \ldots$  on  $(H_{\infty}, \mathcal{B}(H_{\infty}))$  driven by  $(\mu, \nu) \in \Pi_1 \times \Pi_2$  we denote their joint law by  $\mathcal{L}^{\mu,\nu}\{(Y_t^{(1)}, Y_t^{(2)}, \ldots), t \ge 0\}$ .

PROPOSITION 1. For a fixed initial distribution  $\pi_0 \in \mathcal{P}(X)$ , and given  $(\mu, \nu) \in \mathcal{M}_1 \times \mathcal{M}_2$ , the map

$$\mathcal{M}_1 \times \mathcal{M}_2 \ni (\mu, \nu) \mapsto \mathcal{L}^{\mu, \nu} \{ (X_t, U_t, V_t), t \ge 0 \}$$

is jointly continuous in  $(\mu, \nu)$ , where  $\{X_t, t \ge 0\}$  denotes the state process with initial law  $\pi_0$  and  $\{U_t, t \ge 0\}$ ,  $\{V_t, t \ge 0\}$  are the corresponding control processes.

**3.** Discounted Game In this section, we study the cost criterion (1). To this end, we first consider the risk-sensitive exponential cost criterion for player I (i = 1) and II (i = 2) given by:

$$\zeta_{i}^{\mu,\nu}(\theta,(k,t)) \stackrel{def}{=} e^{\theta \rho_{i}^{\mu,\nu}(\theta,(k,t))} = E_{k,t}^{\mu,\nu} \left[ e^{\theta \sum_{s=t}^{\infty} \alpha^{s-t} r_{i}(X_{s}^{(t)},U_{s}^{(t)},V_{s}^{(t)})} \right],\tag{4}$$

where  $\{X_s^{(t)}: s \ge t\}$  is the state process and  $\{(U_s^{(t)}, V_s^{(t)}): s \ge t\}$  are the control processes under  $(\mu, \nu) \in \Pi_1^{t,\infty} \times \Pi_2^{t,\infty}$  starting from  $k \in X$ .

Given strategies  $(\mu, \nu) \in \Pi_1^{t,\infty} \times \Pi_2^{t,\infty}$ , consider the following Bellman equations for the exponential cost (4) for players I and II (resp.):

$$\phi_1(\theta, (k, t)) = \inf_{\xi \in \mathcal{P}(U)} \left[ \int_U \int_V e^{\theta r_1(k, u, v)} \left( \sum_{j \in X} \phi_1(\theta \alpha, (j, t+1)) q(j|k, u, v) \right) \xi(du) \nu_t[h_t](dv) \right], \quad (5)$$

with

$$\lim_{\theta \to 0} \phi_1(\theta, (k, t)) = 1, \ \forall k \in X, \ \forall t,$$
(6)

and

$$\phi_2(\theta, (k,t)) = \inf_{\chi \in \mathcal{P}(V)} \left[ \int_U \int_V e^{\theta r_2(k,u,v)} \left( \sum_{j \in X} \phi_2(\theta\alpha, (j,t+1))q(j|k,u,v) \right) \mu_t[h_t](du)\chi(dv)) \right], \quad (7)$$

with a boundary condition analogous to (6).

Note that under  $(\mu, \nu) \in \Pi_1^{t,\infty} \times \Pi_2^{t,\infty}$ , the corresponding chain  $\{X_t\}$  is inhomogeneous. Hence, we consider the transformed chain  $\{\tilde{X}_t\}$  on  $X \times \mathbb{N}_0$  with the transition kernel  $\tilde{q}$  defined by

$$\tilde{q}((k,t')|(j,t),u,v) = \hat{q}(k|j,u,v)\delta_{t',t+1}, \ u \in U, \ v \in V,$$
(8)

where  $\delta_{s',s} = 1$  if s' = s and 0 otherwise. Now, replacing q by  $\tilde{q}$  and using the arguments used in the proof of Proposition 3.1 of Di Masi and Stettner [15], we get the following result. We omit the routine details.

PROPOSITION 2. Given  $(\mu, \nu) \in \Pi_1^{t,\infty} \times \Pi_2^{t,\infty}$ , there exist unique bounded solutions  $\hat{\phi}_1[\nu], \hat{\phi}_2[\mu]$ to (5) and (7) (resp.) satisfying the boundary condition (6) such that

$$\hat{\phi}_1[\nu](\theta,(k,t)) = \inf_{\tilde{\mu}\in\Pi_1^{t,\infty}} \zeta_1^{\tilde{\mu},\nu}(\theta,(k,t)),\tag{9}$$

and

$$\hat{\phi}_2[\mu](\theta,(k,t)) = \inf_{\tilde{\nu}\in\Pi_2^{t,\infty}} \zeta_2^{\mu,\tilde{\nu}}(\theta,(k,t)).$$
(10)

Moreover, by Remark 3.2 of Di Masi and Stettner [15], given  $(\mu, \nu) \in \Pi_1^{t,\infty} \times \Pi_2^{t,\infty}$  and  $\theta \in (0, \Theta]$ , the minimizing strategies  $\{\mu_t^*[\nu]\}, \{\nu_t^*[\mu]\}$  in (9) and (10) (resp.) are given by

$$\mu_t^*[\nu] = \hat{\mu}[\nu](\theta \alpha^t, (X_t, t)) \in \mathcal{M}_1$$
(11)

and

$$\nu_t^*[\mu] = \hat{\nu}[\mu](\theta \alpha^t, (X_t, t)) \in \mathcal{M}_2$$
(12)

where  $(\hat{\mu}[\nu], \hat{\nu}[\mu]) : (0, \Theta] \times (X \times \mathbb{N}_0) \mapsto \mathcal{P}(U) \times \mathcal{P}(V)$  are measurable (minimizing) selectors (see Beneš [9]) such that

$$\inf_{\xi \in \mathcal{P}(U)} \left[ \int_{U} \int_{V} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} \hat{\phi}_{1}[\nu](\theta \alpha, (j,t+1))q(j|k,u,v) \right) \xi(du) \nu_{t}[h_{t}](dv) \right] \\
= \int_{U} \int_{V} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} \hat{\phi}_{1}[\nu](\theta \alpha, (j,t+1))q(j|k,u,v) \right) \hat{\mu}[\nu](\theta, (k,t))(du) \nu_{t}[h_{t}](dv), \quad (13)$$

and

$$\inf_{\chi \in \mathcal{P}(V)} \left[ \int_{U} \int_{V} e^{\theta r_{2}(k,u,v)} \left( \sum_{j \in X} \hat{\phi}_{2}[\mu](\theta \alpha, (j,t+1))q(j|k,u,v) \right) \mu_{t}[h_{t}](du)\chi(dv) \right] \\
= \int_{U} \int_{V} e^{\theta r_{2}(k,u,v)} \left( \sum_{j \in X} \hat{\phi}_{2}[\mu](\theta \alpha, (j,t+1))q(j|k,u,v) \right) \mu_{t}[h_{t}](du)\hat{\nu}[\mu](\theta, (k,t))(dv). \tag{14}$$

Thus,  $\mu_t^*[\nu] \in \mathcal{M}_1$  (resp.  $\nu_t^*[\mu] \in \mathcal{M}_2$ ) is an optimal response of player I (resp. player II) corresponding to  $\nu \in \Pi_2^{t,\infty}$  (resp.  $\mu \in \Pi_1^{t,\infty}$ ). Hence, without loss of generality, we can effectively consider the strategy pair  $(\mu, \nu) \in \mathcal{M}_1 \times \mathcal{M}_2$  for further analysis of this game.

Considering the minimizing selectors  $(\hat{\mu}[\nu], \hat{\nu}[\mu])$  in (13) and (14) (resp.), we now define the following point-to-set maps  $\mathcal{H}_i : \mathcal{M}_j \mapsto 2^{\mathcal{M}_i}, \ i, j = 1, 2, \ i \neq j$  as follows:

$$\mathcal{H}_1(\nu) \stackrel{\text{def}}{=} \{\{\mu_t^*[\nu]\} : \mu_t^*[\nu] \text{ satisfies (11)}\},\tag{15}$$

and

$$\mathcal{H}_{2}(\mu) \stackrel{def}{=} \{ \{ \nu_{t}^{*}[\mu] \} : \nu_{t}^{*}[\mu] \text{ satisfies } (12) \}.$$
(16)

Define the map  $\mathcal{H} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 : \mathcal{M}_1 \times \mathcal{M}_2 \mapsto 2^{\mathcal{M}_1 \times \mathcal{M}_2}$  as

$$\mathcal{H}(\mu,\nu) \stackrel{def}{=} \{(\{\mu_t^*[\nu]\},\{\nu_t^*[\mu]\}): \mu_t^*[\nu] \text{ satisfies (11) and } \nu_t^*[\mu] \text{ satisfies (12)}\}.$$
(17)

We now prove the existence of Nash equilibria in the class of Markov strategies for the exponential cost criterion (4).

THEOREM 1. Given  $\theta \in (0, \Theta]$ , there exists a pair of Nash equilibrium strategy in  $\mathcal{M}_1 \times \mathcal{M}_2$  for the game corresponding to the cost criterion (4).

**Proof:** Given  $\theta \in (0, \Theta]$ ,  $(\mu, \nu) \in \mathcal{M}_1 \times \mathcal{M}_2$ , both  $\mathcal{H}_1(\nu)$  and  $\mathcal{H}_2(\mu)$  are non-empty and convex implying  $\mathcal{H}(\mu,\nu)$  is also non-empty and convex. Let  $\{(\{\mu_{t,k}^*[\nu]\}_t,\{\nu_{t,k}^*[\mu]\}_t)\}_{k\in\mathbb{N}}$  be a sequence of distinct points of  $\mathcal{H}(\mu,\nu)$  converging to  $(\{\mu_{t,\infty}[\nu]\}_t, \{\nu_{t,\infty}[\mu]\}_t)$ , i.e.,  $(\{\mu_{t,\infty}[\nu]\}_t, \{\nu_{t,\infty}[\mu]\}_t)$  is a limit point of  $\mathcal{H}(\mu,\nu)$ . Since, for each  $k \in \mathbb{N}$ ,  $\mu_{t,k}^*[\nu] = \hat{\mu}_k[\nu](\theta \alpha^t, (X_t, t))$  and  $\nu_{t,k}^*[\mu] = \hat{\nu}_k[\mu](\theta \alpha^t, (X_t, t))$ satisfy (13) and (14) resp., it follows (by linearity) that so do  $\mu_{t,\infty}[\nu]$  and  $\nu_{t,\infty}[\mu]$  resp. implying that they satisfy (11) and (12) resp. The closure property of  $\mathcal{H}(\mu,\nu)$  thus follows. Hence,  $\mathcal{H}$  is a map with non-empty, closed and convex values. Now, by Proposition 1,  $\mathcal{L}^{\mu,\nu}\{X_t, t \geq 0\}$  is jointly continuous in  $(\mu, \nu) \in \mathcal{M}_1 \times \mathcal{M}_2$ . Note that  $\hat{\phi}_1[\nu]$  as obtained in (9) has a minimizing Markov strategy (11) and  $\phi_2[\mu]$  as obtained in (10) has a minimizing Markov strategy (12), i.e., the minimizations in (9) and (10) can be effectively considered over  $\mathcal{M}_1$  and  $\mathcal{M}_2$  (resp.) which are compact sets. Also,  $r_i$ , i = 1, 2 are bounded functions. Hence, by Berge Maximum Theorem (see Theorem 17.31 of Aliprantis and Border [7]),  $\hat{\phi}_1[\nu]$  and  $\hat{\phi}_2[\mu]$  are continuous in  $\nu$  and  $\mu$  respectively. Now consider a sequence  $\{(\mu_k, \nu_k)\}_{k \in \mathbb{N}}$  in  $\mathcal{M}_1 \times \mathcal{M}_2$  converging to  $(\mu_\infty, \nu_\infty) \in \mathcal{M}_1 \times \mathcal{M}_2$ . Let  $\mathcal{H}(\mu_k, \nu_k) \ni (\tilde{\mu}_k, \tilde{\nu}_k) \rightarrow \mathcal{M}_2$  $(\tilde{\mu}_{\infty}, \tilde{\nu}_{\infty}) \in \mathcal{M}_1 \times \mathcal{M}_2$ . From the continuity of  $q(\cdot|\cdot, u, v), \hat{\phi}_1[\nu]$  and  $\hat{\phi}_2[\mu]$  in  $\mu, \nu$ , it follows that  $(\tilde{\mu}_{\infty}, \tilde{\nu}_{\infty}) \in \mathcal{H}(\mu_{\infty}, \nu_{\infty})$ . Hence the map  $\mathcal{H}$  is upper semi-continuous. Then, by Theorem 1 of Fan [19], there exists a fixed point  $(\mu^*, \nu^*) : \mathbb{N}_0 \times X \mapsto \mathcal{P}(U) \times \mathcal{P}(V)$  of the map  $\mathcal{H}$  defined as

$$(\mu^{*}(t,k),\nu^{*}(t,k)) \stackrel{def}{=} (\hat{\mu}[\nu^{*}](\theta\alpha^{t},(k,t)),\hat{\nu}[\mu^{*}](\theta\alpha^{t},(k,t)))$$
(18)

where  $\hat{\mu}[\nu^*]$  (resp.  $\hat{\nu}[\mu^*]$ ) is any minimizing selector in (13) (resp. (14)). Then  $(\mu^*, \nu^*) \in \mathcal{M}_1 \times \mathcal{M}_2$ is a Nash equilibrium strategy for the cost criterion (4), i.e.,

$$\hat{\phi}_1[\nu^*](\theta,(k,t)) = \zeta_1^{\mu^*,\nu^*}(\theta,(k,t)) \le \zeta_1^{\mu,\nu^*}(\theta,k), \; \forall \mu \in \Pi_1^{t,\infty},$$

and

$$\hat{\phi}_{2}[\mu^{*}](\theta,(k,t)) = \zeta_{2}^{\mu^{*},\nu^{*}}(\theta,(k,t)) \leq \zeta_{2}^{\mu^{*},\nu}(\theta,k), \ \forall \nu \in \Pi_{2}^{t,\infty}.$$

Now we prove the existence of Nash equilibria in the class of Markov strategies for the cost criterion (1).

Given definition (4), the Bellman equations for the discounted cost (1) are:

$$e^{\theta\psi_1(\theta,(k,t))} = \inf_{\xi\in\mathcal{P}(U)} \left[ \int_U \int_V e^{\theta r_1(k,u,v)} \left( \sum_{j\in X} e^{\theta\alpha\psi_1(\theta\alpha,(j,t+1))} q(j|k,u,v) \right) \xi(du) \nu_t[h_t](dv) \right],$$
(19)

for player 1 with the boundary condition

$$\lim_{\theta \to 0} \psi_1(\theta, (k, t)) = \inf_{\mu \in \Pi_1^{t,\infty}} E_{k,t}^{\mu,\nu} \left[ \sum_{s=t}^{\infty} \alpha^{s-t} r_1(X_s^{(t)}, U_s^{(t)}, V_s^{(t)}) \right], \ \forall k \in X, \ \forall t,$$
(20)

and

$$e^{\theta\psi_2(\theta,(k,t))} = \inf_{\chi\in\mathcal{P}(V)} \left[ \int_U \int_V e^{\theta r_2(k,u,v)} \left( \sum_{j\in X} e^{\theta\alpha\psi_2(\theta\alpha,(j,t+1))} q(j|k,u,v) \right) \mu_t[h_t](du)\chi(dv) \right],$$
(21)

for player 2 with the boundary condition

$$\lim_{\theta \to 0} \psi_2(\theta, (k, t)) = \inf_{\nu \in \Pi_2^{t,\infty}} E_{k,t}^{\mu,\nu} \left[ \sum_{s=t}^{\infty} \alpha^{s-t} r_2(X_s^{(t)}, U_s^{(t)}, V_s^{(t)}) \right], \ \forall k \in X, \ \forall t.$$
(22)

THEOREM 2. Given  $\theta \in (0, \Theta]$ , consider the fixed point Markov strategies  $(\mu^*, \nu^*)$  of  $\mathcal{H}$  as in (18) in Theorem 1. Then  $(\mu^*, \nu^*)$  is also a pair of Nash equilibrium strategy for the cost criterion (1), i.e.,

$$\hat{\psi}_1[\nu^*](\theta, (k, t)) = \rho_1^{\mu^*, \nu^*}(\theta, (k, t)) \le \rho_1^{\mu, \nu^*}(\theta, (k, t)), \ \forall \mu \in \Pi_1^{t, \infty},$$

and

$$\hat{\psi}_{2}[\mu^{*}](\theta,(k,t)) = \rho_{2}^{\mu^{*},\nu^{*}}(\theta,(k,t)) \le \rho_{2}^{\mu^{*},\nu}(\theta,(k,t)), \ \forall \nu \in \Pi_{2}^{t,\infty},$$

where  $\hat{\psi}_1[\nu^*]$  and  $\hat{\psi}_2[\mu^*]$  are unique bounded solutions to (19) and (21) (resp.) satisfying the boundary conditions (20) and (22) respectively.

**Proof:** The result follows directly from (4), Proposition 2, Theorem 1 and the fact that log is an increasing function.  $\Box$ 

**4. Ergodic Game** In this section, we study the cost criterion (2). To this end, we make the following assumption:

(A1) The process  $\{X_t\}_{t\in\mathbb{N}_0}$  is an irreducible, aperiodic Markov chain under any pair of stationary Markov strategies  $(\mu, \nu) \in S_1 \times S_2$ .

We also assume the following condition which guarantees uniform ergodicity of controlled Markov processes and is used to study additive average cost problems (see, e.g., Hernández-Lerma [25], Section 3.3 or an equivalent assumption (A1) in Di Masi and Stettner [15] or (3.2) in Hernández-Hernández and Marcus [24]):

(A2) There exists a number  $0 < \delta < 1$  such that for all  $i, j \in X, u, u' \in U$  and  $v, v' \in V$ ,

$$||q(\cdot|i, u, v) - q(\cdot|j, u', v')||_{\mathrm{TV}} \le 2\delta,$$

where  $|| \cdot ||_{TV}$  denotes the total variation norm (see, e.g., Hernández-Lerma [25], Appendix B).

In our further analysis of this problem, the above assumptions always hold.

Under (A1)-(A2), it follows from the Lemma 3.3 of Hernández-Lerma [25] that for any  $(\mu, \nu) \in S_1 \times S_2$ , the corresponding Markov chain  $\{X_t\}$  is uniformly ergodic with a unique invariant probability measure  $\eta[\mu, \nu] \in \mathcal{P}(X)$ , i.e.,

$$||\hat{q}^t(\cdot|k,\mu,\nu) - \eta[\mu,\nu](\cdot)||_{\text{TV}} \le 2\delta^t, \ \forall k \in X, \ \forall t \in \mathbb{N},$$

$$(23)$$

where  $\hat{q}^t(\cdot|k,\mu,\nu)$  denotes the *t*-step transition kernel under  $(\mu,\nu)$  starting from  $k \in X$ .

REMARK 1. It is important to note here that the quantity  $\delta$  on the r.h.s. of the above inequality (23) is independent of the strategy pair  $(\mu, \nu)$  and (23) holds uniformly across all  $(\mu, \nu) \in S_1 \times S_2$  (see also condition C4 of Theorem 4(ii) in Federgruen et. al. [20]). Note that an assumption equivalent to (A2) was also necessary for the corresponding zero-sum case as analyzed in Basu and Ghosh [5]. Equivalent assumptions are quite common in the corresponding literature for the one-controller case (see, e.g., Di Masi and Stettner [15],[16] and [17], Hernández-Hernández and Marcus [24]).

DEFINITION 4. For any  $B \subseteq X$ , define the hitting time of the set B by  $\{X_t\}$  as

$$\tau_B \stackrel{def}{=} \inf\{t \ge 0 : X_t \in B\}.$$
(24)

Denote  $\tau_B = \tau_j$  if  $B = \{j\}$ . Similarly, define the first return time to B by  $\{X_t\}$  as

$$\sigma_B \stackrel{def}{=} \inf\{t \ge 1 : X_t \in B\}.$$
(25)

Denote  $\sigma_B = \sigma_j$  if  $B = \{j\}$ .

Now, Theorem 2.1 of Balaji and Meyn [2], Theorem 16.0.2 of Meyn and Tweedie [31] and Remark 1 together imply the following result:

PROPOSITION 3. Assume (A1)-(A2). Then the following three equivalent statements hold.

(a) For every  $A \in 2^X$ , the chain  $\{X_t\}$  is geometrically recurrent uniformly over all  $(\mu, \nu) \in S_1 \times S_2$ , i.e., there exists some  $R_A > 1$  such that  $\sup_{(\mu,\nu)\in S_1\times S_2} \sup_{k\in X} E_k^{\mu,\nu}[R_A^{\sigma_A}] \leq B_A < \infty$ . In particular, if  $A = \{j\}$ ,  $j \in X$  we write the constants above as  $R_j$  and  $B_j$  respectively. Note that  $B_A, B_j > 1$  by definition.

(b) For every  $A \in 2^X$ , there exists some  $L_A > 0$  such that  $\sup_{(\mu,\nu)\in S_1\times S_2} \sup_{k\in X} E_k^{\mu,\nu}[\sigma_A] \leq L_A < \infty$ . In particular, if  $A = \{j\}, j \in X$  we write the constant above as  $L_j$ .

(c) There exist  $\eta < 1, b < \infty$ , a finite subset  $C \subset X$  and a bounded function  $V : X \to [1, \infty)$  such that

$$\sum_{j} V(j)q(j|i,\mu(i),\nu(i)) \leq \eta V(i) + b\mathbb{I}_{C}(i)$$

where  $\eta, b, C$  and V are all independent of  $(\mu, \nu) \in S_1 \times S_2$ .

REMARK 2. Note that stochastic Lyapunov-type stability assumptions as in Proposition 3(c) was also necessary for the corresponding zero-sum stochastic differential game case as analyzed in Basu and Ghosh [4]. See also Remark 1 for reference to the corresponding literature where equivalent assumptions are used. We also refer the reader to Altman et. al. [1] for examples of such games on countable spaces with additive cost criteria where equivalent stability assumptions were made.

Henceforth we fix  $\theta \in (0, \Theta]$ . Given strategies  $(\mu, \nu) \in \Pi_1 \times \Pi_2$ , consider the following dynamic programming (HJB) equations for the ergodic cost (2) for players I and II (resp.):

$$e^{\theta\lambda_1+V_1(\theta,k)} = \inf_{\xi\in\mathcal{P}(U)} \left[ \int_U \int_V e^{\theta r_1(k,u,v)} \left( \sum_{j\in X} e^{V_1(\theta,j)} q(j|k,u,v) \right) \xi(du) \nu_t[h_t](dv) \right],$$
(26)

and

$$e^{\theta\lambda_2+V_2(\theta,k)} = \inf_{\chi\in\mathcal{P}(V)} \left[ \int_U \int_V e^{\theta r_2(k,u,v)} \left( \sum_{j\in X} e^{V_2(\theta,j)} q(j|k,u,v) \right) \mu_t[h_t](du)\chi(dv) \right],$$
(27)

where  $\lambda_1, \lambda_2$  are scalars.

Under (A1)-(A2), by Corollary 2.3 of Di Masi and Stettner [15], there exists at most one (up to an additive constant) bounded function  $\hat{V}_1[\nu]$  and a unique constant  $\hat{\lambda}_1[\nu]$  for which (26) is satisfied. If it does then, by Proposition 1.1 of Di Masi and Stettner [15] (see also Theorem 2.1 of Hernández-Hernández and Marcus [24]), we have

$$\hat{\lambda}_1[\nu] = \inf_{\mu \in \Pi_1} \beta_1^{\mu,\nu}(\theta,k) = \inf_{\mu \in \mathcal{S}_1} \limsup_{T \to \infty} \frac{1}{\theta T} \ln E_k^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{T-1} r_1(X_t, U_t, V_t)} \right]$$
(28)

and similarly, there exists at most one bounded  $\hat{V}_2[\mu]$  (modulo an additive constant) and a unique  $\hat{\lambda}_2[\mu]$  satisfying (27) such that

$$\hat{\lambda}_{2}[\mu] = \inf_{\nu \in \Pi_{2}} \beta_{2}^{\mu,\nu}(\theta,k) = \inf_{\nu \in \mathcal{S}_{2}} \limsup_{T \to \infty} \frac{1}{\theta T} \ln E_{k}^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{T-1} r_{2}(X_{t},U_{t},V_{t})} \right].$$
(29)

Hence there exist measurable (minimizing) selectors (see Beneš [9])

$$(\hat{\mu}[\nu], \hat{\nu}[\mu]) : X \mapsto \mathcal{P}(U) \times \mathcal{P}(V)$$

satisfying

$$\inf_{\xi \in \mathcal{P}(U)} \left[ \int_{U} \int_{V} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} e^{\hat{V}_{1}[\nu](\theta,j)} q(j|k,u,v) \right) \xi(du) \nu_{t}[h_{t}](dv) \right] \\
= \int_{U} \int_{V} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} e^{\hat{V}_{1}[\nu](\theta,j)} q(j|k,u,v) \right) \hat{\mu}[\nu] [du|k] \nu_{t}[h_{t}](dv),$$
(30)

and

$$\inf_{\chi \in \mathcal{P}(V)} \left[ \int_{U} \int_{V} e^{\theta r_{2}(k,u,v)} \left( \sum_{j \in X} e^{\hat{V}_{2}[\mu](\theta,j)} q(j|k,u,v) \right) \mu_{t}[h_{t}](du) \chi(dv) \right] \\
= \int_{U} \int_{V} e^{\theta r_{2}(k,u,v)} \left( \sum_{j \in X} e^{\hat{V}_{2}[\mu](\theta,j)} q(j|k,u,v) \right) \mu_{t}[h_{t}](du) \hat{\nu}[\mu] [dv|k].$$
(31)

Hence, by Proposition 1.1 of Di Masi and Stettner [15] (see also Theorem 2.1 of Hernández-Hernández and Marcus [24]), given  $(\mu, \nu) \in \Pi_1 \times \Pi_2$ , if there exist bounded solutions to (26) and (27) then the minimizing selector  $\hat{\mu}[\nu]$  and  $\hat{\nu}[\mu]$  in (30) and (31) resp. generates stationary (see (28) and (29)) optimal response strategy of player I (resp. player II) when the other's strategy fixed. Hence, without loss of generality, we can effectively consider the strategy pairs  $(\mu, \nu) \in S_1 \times S_2$  for further analysis of this game. Thus if we consider the minimizing selectors  $\hat{\mu}[\nu]$  and  $\hat{\nu}[\mu]$  in (30) and (31) respectively, we can define the following point-to-set maps  $\mathcal{H}_i : S_j \mapsto 2^{S_i}$ ,  $i, j = 1, 2, i \neq j$ as follows:

$$\mathcal{H}_1(\nu) \stackrel{\text{def}}{=} \{\{\mu^*[\nu]\} : \mu^*[\nu] \text{ satisfies } (30)\},\tag{32}$$

and

$$\mathcal{H}_{2}(\mu) \stackrel{def}{=} \{ \{ \nu^{*}[\mu] \} : \nu^{*}[\mu] \text{ satisfies } (31) \}.$$
(33)

Define the map  $\mathcal{H} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 : \mathcal{S}_1 \times \mathcal{S}_2 \mapsto 2^{\mathcal{S}_1 \times \mathcal{S}_2}$  as

$$\mathcal{H}(\mu,\nu) \stackrel{def}{=} \{ (\{\mu^*[\nu]\}, \{\nu^*[\mu]\}) : \mu^*[\nu] \text{ satisfies (30) and } \nu^*[\mu] \text{ satisfies (31)} \}.$$
(34)

A priori, it is not clear that the sets  $\mathcal{H}_1, \mathcal{H}_2$  and hence  $\mathcal{H}$  are nonempty. We now proceed to prove the existence of unique bounded solutions to (26) and (27). Once this result is established, the nonemptyness of the above sets follow automatically. Then, using arguments similar to Theorem 1, we prove the existence of Nash equilibria by showing that the map  $\mathcal{H}$  has a fixed point  $(\tilde{\mu}, \tilde{\nu})$ . To this end, we make the following definitions (see Balaji and Meyn [2] and Borkar and Meyn [13]).

DEFINITION 5. Given an arbitrarily fixed state  $0 \in X$  and any strategy  $(\mu, \nu) \in S_1 \times S_2$ , let

$$\Lambda_1[\nu](\mu) \stackrel{def}{=} \inf \left\{ \Lambda \in \mathbb{R} : E_0^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\sigma_0 - 1} (r_1(X_t, U_t, V_t) - \Lambda)} \right] \le 1 \right\},\tag{35}$$

and

$$\Lambda_2[\mu](\nu) \stackrel{def}{=} \inf \left\{ \Lambda \in \mathbb{R} : E_0^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\sigma_0 - 1} (r_2(X_t, U_t, V_t) - \Lambda)} \right] \le 1 \right\}.$$
(36)

Also define

$$\Lambda_1^*[\nu] \stackrel{def}{=} \inf_{\mu \in \mathcal{S}_1} \Lambda_1[\nu](\mu), \tag{37}$$

$$h_1^*[\nu](k) \stackrel{def}{=} \inf_{\mu \in \mathcal{S}_1} E_k^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\tau_0} (r_1(X_t, U_t, V_t) - \Lambda_1^*[\nu])} \right],\tag{38}$$

$$w_1^*[\nu](k) \stackrel{def}{=} \arg\min_{\xi \in \mathcal{P}(U)} \left[ \int_U \int_V e^{\theta r_1(k,u,v)} \left( \sum_{j \in X} h_1^*[\nu](j)q(j|k,u,v) \right) \xi(du)\nu\left[dv|k\right] \right], \quad (39)$$

where  $w_1^*[\nu]$  is any minimizer in (39). Similarly, let

$$\Lambda_2^*[\mu] \stackrel{def}{=} \inf_{\nu \in S_2} \Lambda_2[\mu](\nu), \tag{40}$$

$$h_{2}^{*}[\mu](k) \stackrel{def}{=} \inf_{\nu \in S_{2}} E_{k}^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\tau_{0}} (r_{2}(X_{t}, U_{t}, V_{t}) - \Lambda_{2}^{*}[\mu])} \right], \tag{41}$$

$$w_2^*[\mu](k) \stackrel{def}{=} \arg\min_{\chi \in \mathcal{P}(V)} \left[ \int_U \int_V e^{\theta r_2(k,u,v)} \left( \sum_{j \in X} h_2^*[\mu](j)q(j|k,u,v) \right) \mu\left[du|k\right] \chi(dv) \right]$$
(42)

where  $w_2^*[\mu]$  is any minimizer in (42).

Note that since  $\theta$  is fixed we have suppressed the explicit dependence on  $\theta$  in these definitions for notational convenience. A priori, it is not obvious that the quantities defined above are finite. The following Proposition 4 and Lemmata 1 and 2 settle this issue.

First, we make a "small cost" assumption which shall hold for the rest of this paper.

(A3) 
$$||r_i||_{\infty} \leq \frac{\ln R_0}{3\Theta}, \ i = 1, 2$$

where  $R_0 > 1$  is as in part (a) of Proposition 3.

REMARK 3. Note that a similar assumption was also necessary for the corresponding zero-sum stochastic differential game case as analyzed in Basu and Ghosh [4].

Now we state and prove the following important proposition.

PROPOSITION 4. Under (A1)-(A3) and for any  $(\mu, \nu) \in S_1 \times S_2$ ,

$$||r_1||_{\infty} \ge |\Lambda_1^*[\nu]|,\tag{43}$$

and

$$||r_2||_{\infty} \ge |\Lambda_2^*[\mu]|. \tag{44}$$

Moreover,  $\Lambda_1^*[\nu]$  (resp.  $\Lambda_2^*[\mu]$ ) is continuous in  $\nu$  (resp.  $\mu$ ).

**Proof:** We prove it for player I. The result for player II follows analogously. From (35) we obtain, by Jensen's inequality

$$1 \geq E_{0}^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\sigma_{0}-1} (r_{1}(X_{t}, U_{t}, V_{t}) - \Lambda_{1}[\nu](\mu))} \right] \geq e^{-\theta \Lambda_{1}[\nu](\mu) E_{0}^{\mu,\nu}[\sigma_{0}]} e^{\theta E_{0}^{\mu,\nu} \left[ \sum_{t=0}^{\sigma_{0}-1} (r_{1}(X_{t}, U_{t}, V_{t}) \right]} \\ \geq e^{-\theta \Lambda_{1}[\nu](\mu) E_{0}^{\mu,\nu}[\sigma_{0}]} e^{-\theta ||r_{1}||_{\infty} E_{0}^{\mu,\nu}[\sigma_{0}]}$$

which implies

$$e^{\theta||r_1||_{\infty}E_0^{\mu,\nu}[\sigma_0]} > e^{-\theta\Lambda_1[\nu](\mu)E_0^{\mu,\nu}[\sigma_0]}$$

Under (A1)-(A2),  $0 < E_0^{\mu,\nu}[\sigma_0] \le L_0 < \infty$  (see part (b) of Proposition 3) and hence, by taking log on both sides of the last inequality, we get

$$-||r_1||_{\infty} \leq \Lambda_1[\nu](\mu).$$

Now, by (35), for any  $\epsilon > 0$ , we have

$$E_0^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\sigma_0 - 1} (r_1(X_t, U_t, V_t) - \Lambda_1[\nu](\mu) + \epsilon)} \right] > 1$$

implying

$$E_0^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\sigma_0 - 1} (||r_1||_{\infty} - \Lambda_1[\nu](\mu) + \epsilon)} \right] > 1$$

which, in turn, implies

$$E_0^{\mu,\nu}\left[e^{\theta(||r_1||_\infty-\Lambda_1[\nu](\mu)+\epsilon)\sigma_0}\right]>1.$$

Since this is true for any  $\epsilon > 0$ , we have in the limit  $\epsilon \downarrow 0$ ,

$$E_0^{\mu,\nu} \left[ e^{\theta(||r_1||_{\infty} - \Lambda_1[\nu](\mu))\sigma_0} \right] \ge 1$$

implying

$$||r_1||_{\infty} - \Lambda_1[\nu](\mu) \ge 0.$$

Hence we have

$$-||r_1||_{\infty} \le \Lambda_1[\nu](\mu) \le ||r_1||_{\infty}.$$
(45)

The inequality (43) follows by taking the inf over  $\mu \in S_1$ . We now prove the joint continuity in  $(\mu, \nu)$  of  $\Lambda_1[\nu](\mu)$ . Consider a sequence  $\{(\mu^{(n)}, \nu^{(n)})\} \xrightarrow{n \to \infty} (\mu^{(\infty)}, \nu^{(\infty)})$  in  $S_1 \times S_2$ . Let  $\{X_t, t \ge 0\}$  denote the chain starting at 0. By Proposition 1,

$$\mathcal{L}^{\mu^{(n)},\nu^{(n)}}\{X_t,t\geq 0\} \stackrel{n\to\infty}{\longrightarrow} \mathcal{L}^{\mu^{(\infty)},\nu^{(\infty)}}\{X_t,t\geq 0\}.$$

Then, by Skorohod's Theorem (see Theorem 2.2.2 of Borkar [12]), there exists some augmentation  $(\tilde{H}_{\infty}, \mathcal{B}(\tilde{H}_{\infty}), \tilde{\mathcal{P}})$  of the canonical space  $(H_{\infty}, \mathcal{B}(H_{\infty}))$  on which

$$\{X_t^{\mu^{(n)},\nu^{(n)}}, t \ge 0\} \xrightarrow{\tilde{P}\text{-}a.s} \{X_t^{\mu^{(\infty)},\nu^{(\infty)}}, t \ge 0\}$$

where  $\{X_t^{\mu^{(n)},\nu^{(n)}}, t \ge 0\}, n = 1, 2, ..., \infty$  denotes the Markov chain under  $(\mu^{(n)},\nu^{(n)})$  on this augmented space starting at 0. Following (25), let  $\sigma_0^{(n)} \stackrel{def}{=} \inf\{t \ge 1 : X_t^{\mu^{(n)},\nu^{(n)}} = 0\}, n = 1, 2, ..., \infty$ . Then, obviously  $\tilde{P}$ -a.s.

$$\sigma_0^{(n)} \stackrel{n \to \infty}{\longrightarrow} \sigma_0^{(\infty)}.$$

Hence we get the joint continuity under  $(\mu, \nu)$  of  $\sigma_0$ . Hence, for any  $0 < \epsilon \leq ||r_1||_{\infty}$  and along a subsequence  $\{n_k\} \subset \{n\}$  with  $\{(\mu^{(n_k)}, \nu^{(n_k)})\} \xrightarrow{n_k \to \infty} (\mu^{(\infty)}, \nu^{(\infty)})$ , we have by (45) and the joint continuity of  $\sigma_0$  under  $(\mu, \nu)$ 

$$f_{n_{k}} \equiv e^{\theta \sum_{t=0}^{\sigma_{0}^{(n_{k})}-1} \left( r_{1}(X_{t}^{\mu^{(n_{k})},\nu^{(n_{k})}},U_{t}^{\mu^{(n_{k})},\nu^{(n_{k})}},V_{t}^{\mu^{(n_{k})},\nu^{(n_{k})}}) - \Lambda_{1}[\nu^{(n_{k})}](\mu^{(n_{k})}) + \epsilon \right)} \\ \xrightarrow{\tilde{P}-a.s.} e^{\theta \sum_{t=0}^{\sigma_{0}^{(\infty)}-1} \left( r_{1}(X_{t}^{\mu^{(\infty)},\nu^{(\infty)}},U_{t}^{\mu^{(\infty)},\nu^{(\infty)}},V_{t}^{\mu^{(\infty)},\nu^{(\infty)}}) - \Lambda + \epsilon \right)} \equiv f_{\infty},$$
(46)

where  $|\Lambda| \leq ||r_1||_{\infty}$  and  $U_t^{\mu^{(n)},\nu^{(n)}}, V_t^{\mu^{(n)},\nu^{(n)}}$  are the corresponding control sequences under  $(\mu^{(n)},\nu^{(n)}), n = 1, 2, ..., \infty$  on this augmented space. Then

$$0 < f_{n_k} = |f_{n_k}| \le g_{n_k} \stackrel{\text{def}}{=} e^{\theta(2||r_1||_{\infty} + \epsilon)\sigma_0^{(n_k)}} \stackrel{\tilde{P}\text{-}a.s.}{\longrightarrow} e^{\theta(2||r_1||_{\infty} + \epsilon)\sigma_0^{(\infty)}} \stackrel{\text{def}}{=} g_{\infty} \ge |f_{\infty}| = f_{\infty}.$$
(47)

Moreover, by (A3) and part (a) of Proposition 3, we have

$$\sup_{n_k} \tilde{E}\left[g_{n_k}\right] \le \sup_{n_k} \tilde{E}\left[e^{3\Theta||r_1||_{\infty}\sigma_0^{(n_k)}}\right] \le \sup_{n_k} \tilde{E}\left[R_0^{\sigma_0^{(n_k)}}\right] \le B_0 < \infty,\tag{48}$$

which, in particular, implies that  $\{\sigma_0^{(n_k)}\}\$  are uniformly integrable. Given any  $\varepsilon > 0$  it follows from (48) that there exists some large enough constant  $K \equiv K(\varepsilon, B_0) > 0$  such that

$$\sup_{n_k} \tilde{E}\left[g_{n_k} \mathbb{I}_{\{g_{n_k} \ge K\}}\right] \le \varepsilon \tag{49}$$

implying  $\{g_{n_k}\}$  are uniformly integrable.

It follows by Theorem 6.18(iii) of Klenke [29] and (47) that  $\{f_{n_k}\}$  is uniformly integrable.

Also, by (A3) and part (a) of Proposition 3, we have

$$\tilde{E}\left[g_{\infty}\right] \leq \tilde{E}\left[e^{3\Theta||r_{1}||_{\infty}\sigma_{0}^{(\infty)}}\right] \leq \tilde{E}\left[R_{0}^{\sigma_{0}^{(\infty)}}\right] \leq B_{0} < \infty.$$

$$\tag{50}$$

Hence, by Corollary 1.3.1 of Borkar [12], we get

$$\lim_{n_{k}\to\infty} \tilde{E} \left[ e^{\theta \sum_{t=0}^{\sigma_{0}^{(n_{k})}-1} \left( r_{1}(X_{t}^{\mu^{(n_{k})},\nu^{(n_{k})}},U_{t}^{\mu^{(n_{k})},\nu^{(n_{k})}},V_{t}^{\mu^{(n_{k})},\nu^{(n_{k})}}) - \Lambda_{1}[\nu^{(n_{k})}](\mu^{(n_{k})}) + \epsilon \right) \right] \\
= \tilde{E} \left[ e^{\theta \sum_{t=0}^{\sigma_{0}^{(\infty)}-1} \left( r_{1}(X_{t}^{\mu^{(\infty)},\nu^{(\infty)}},U_{t}^{\mu^{(\infty)},\nu^{(\infty)}},V_{t}^{\mu^{(\infty)},\nu^{(\infty)}}) - \Lambda + \epsilon \right)} \right].$$
(51)

Similarly we can show that

$$\lim_{n_{k}\to\infty} \tilde{E} \left[ e^{\theta \sum_{t=0}^{\sigma_{0}^{(n_{k})}-1} \left( r_{1}(X_{t}^{\mu^{(n_{k})},\nu^{(n_{k})}},U_{t}^{\mu^{(n_{k})},\nu^{(n_{k})}},V_{t}^{\mu^{(n_{k})},\nu^{(n_{k})}}) - \Lambda_{1}[\nu^{(n_{k})}](\mu^{(n_{k})}) \right) \right] \\ = \tilde{E} \left[ e^{\theta \sum_{t=0}^{\sigma_{0}^{(\infty)}-1} \left( r_{1}(X_{t}^{\mu^{(\infty)},\nu^{(\infty)}},U_{t}^{\mu^{(\infty)},\nu^{(\infty)}},V_{t}^{\mu^{(\infty)},\nu^{(\infty)}}) - \Lambda \right) \right].$$
(52)

Now from (35) we get that for any  $0 < \epsilon \le ||r_1||_{\infty}$ ,

$$\tilde{E}\left[e^{\theta\sum_{t=0}^{\sigma_{0}^{(n)}-1}\left(r_{1}(X_{t}^{\mu^{(n)},\nu^{(n)}},U_{t}^{\mu^{(n)},\nu^{(n)}},V_{t}^{\mu^{(n)},\nu^{(n)}})-\Lambda_{1}[\nu^{(n)}](\mu^{(n)})+\epsilon\right)}\right] \geq 1$$

$$\geq \tilde{E}\left[e^{\theta\sum_{t=0}^{\sigma_{0}^{(n)}-1}\left(r_{1}(X_{t}^{\mu^{(n)},\nu^{(n)}},U_{t}^{\mu^{(n)},\nu^{(n)}},V_{t}^{\mu^{(n)},\nu^{(n)}})-\Lambda_{1}[\nu^{(n)}](\mu^{(n)})\right)}\right].$$
(53)

Hence, taking  $\lim_{n_k\to\infty}$  on both sides of (53), we have by (52) and (53)

$$\tilde{E}\left[e^{\theta\sum_{t=0}^{\sigma_{0}^{(\infty)}-1}\left(r_{1}(X_{t}^{\mu^{(\infty)},\nu^{(\infty)}},U_{t}^{\mu^{(\infty)},\nu^{(\infty)}},V_{t}^{\mu^{(\infty)},\nu^{(\infty)}})-\Lambda+\epsilon\right)}\right] \geq 1 \\
\geq \tilde{E}\left[e^{\theta\sum_{t=0}^{\sigma_{0}^{(\infty)}-1}\left(r_{1}(X_{t}^{\mu^{(\infty)},\nu^{(\infty)}},U_{t}^{\mu^{(\infty)},\nu^{(\infty)}},V_{t}^{\mu^{(\infty)},\nu^{(\infty)}})-\Lambda\right)}\right],$$
(54)

for some  $\Lambda$  satisfying (45) and all  $0 < \epsilon \le ||r_1||_{\infty}$ . This implies, by (35), that  $\Lambda = \Lambda_1[\nu^{(\infty)}](\mu^{(\infty)})$ . Hence  $\Lambda_1[\nu](\mu)$  is jointly continuous under  $(\mu, \nu)$ . The continuity of  $\Lambda_1^*[\nu]$  with respect to  $\nu$  now follows from (45) since  $S_1$  is compact.

REMARK 4. Note that by (45) above the quantities  $\Lambda_1[\nu](\mu)$  and  $\Lambda_2[\mu](\nu)$  defined in (35) and (36) respectively are uniformly bounded for all  $\theta \in (0, \Theta]$ . This fact shall be implicitly used in the proof of Theorem 3.

Now we state and prove two useful lemmata.

LEMMA 1. Under (A1)-(A2) and for any  $(\mu, \nu) \in S_1 \times S_2$ ,

1. The multiplicative Poisson inequalities holds for all  $k \in X$ , i.e.,

$$\int_{U} \int_{V} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} h_{1}^{*}[\nu](j)q(j|k,u,v) \right) w_{1}^{*}[\nu](k)(du)\nu [dv|k] \leq e^{\theta \Lambda_{1}^{*}[\nu]} h_{1}^{*}[\nu](k), \tag{55}$$

$$\int_{U} \int_{V} e^{\theta r_{2}(k,u,v)} \left( \sum_{j \in X} h_{2}^{*}[\mu](j)q(j|k,u,v) \right) \mu\left[du|k\right] w_{2}^{*}[\mu](k)(dv) \le e^{\theta \Lambda_{2}^{*}[\mu]} h_{2}^{*}[\mu](k).$$
(56)

 $2. \ h_1^*[\nu](k) < \infty, \ h_2^*[\mu](k) < \infty, \ \forall k \in X.$ 

**Proof:** We prove it for player I when player II uses  $\nu \in S_2$ . The other case is analogous. For each  $k \neq 0$ ,

$$\begin{split} & e^{-\theta\Lambda_{1}^{*}[\nu]} \int_{V} \int_{U} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} h_{1}^{*}[\nu](j)q(j|k,u,v) \right) w_{1}^{*}[\nu](k)(du)\nu \left[ dv|k \right] = \\ & \inf_{\xi \in \mathcal{P}(U)} \int_{V} \int_{U} e^{\theta (r_{1}(k,u,v) - \Lambda_{1}^{*}[\nu])} \left( \sum_{j \in X} q(j|k,u,v) \left\{ \inf_{\mu \in S_{1}} E_{j}^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\tau_{0}-1} \left( r_{1}(X_{t},U_{t},V_{t}) - \Lambda_{1}^{*}[\nu] \right)} \right] \right\} \right) \xi(du)\nu \left[ dv|k \right] \\ & = h_{1}^{*}[\nu](k) \text{ (by one-step backward recursion),} \end{split}$$

while for k = 0, we have

$$\int_{V} \int_{U} \left( \sum_{j \in X} h_{1}^{*}[\nu](j)q(j|0,u,v) \right) w_{1}^{*}[\nu](0)(du)\nu[dv|0] = \inf_{\mu \in \mathcal{S}_{1}} E_{0}^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\sigma_{0}-1} (r_{1}(X_{t},U_{t},V_{t}) - \Lambda_{1}^{*}[\nu])} \right] \leq 1,$$
(57)

where the inequality follows from (35). It follows that (55) holds. From (55) it follows exactly as in the proof of Lemma 15.2.2(i) of Meyn and Tweedie [31] that the set  $S \stackrel{def}{=} \{k \in X : h_1^*[\nu](k) < \infty\}$ absorbing, i.e.,  $\sum_{j \in S} \int_V \int_U q(j|k, u, v) w_1^*[\nu](k) (du) \nu [dv|k] = 1$  for all  $k \in S$ . Since it is non-empty as  $0 \in S$  and the chain  $\{X_t\}$  is irreducible under the stationary strategy  $(w_1^*[\nu], \nu)$  by assumption (A1), we get by Proposition 4.2.3 of Meyn and Tweedie [31] that it is full, i.e. S = X establishing part 2.

Note that, under the additional assumption (A3), we can actually show a tighter result than part 2 of Lemma 1 as given below.

LEMMA 2. Under (A1) - (A3) and for any  $(\mu, \nu) \in S_1 \times S_2$ ,

$$\frac{1}{R_0 B_0} \le h_1^*[\nu](k) \le R_0 B_0, \ \frac{1}{R_0 B_0} \le h_2^*[\mu](k) \le R_0 B_0, \ \forall k \in X.$$
(58)

Moreover,  $h_1^*[\nu](k)$  (resp.  $h_2^*[\mu](k)$ ) is continuous in  $\nu$  (resp.  $\mu$ ) for all  $k \in X$ .

**Proof:** We prove it for  $h_1^*[\nu]$ . The proof is exactly similar for  $h_2^*[\mu]$ . Under the assumptions of this lemma, we have by (38) and part (a) of Proposition 3

$$0 < E_{k}^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\tau_{0}} (r_{1}(X_{t}, U_{t}, V_{t}) - \Lambda_{1}^{*}[\nu])} \right] = E_{k}^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\sigma_{0}} (r_{1}(X_{t}, U_{t}, V_{t}) - \Lambda_{1}^{*}[\nu])} \right] \leq E_{k}^{\mu,\nu} \left[ e^{2\theta ||r_{1}||_{\infty}(\sigma_{0}+1)} \right] \leq E_{k}^{\mu,\nu} \left[ e^{3\Theta ||r_{1}||_{\infty}(\sigma_{0}+1)} \right] \leq R_{0} \sup_{k \in X} E_{k}^{\mu,\nu} \left[ R_{0}^{\sigma_{0}} \right] \leq R_{0} B_{0}, \ \forall k \neq 0,$$
(59)

where the equality follows from Proposition 3.4.5(ii) of Meyn and Tweedie [31]. Hence we get

$$0 < h_1^*[\nu](k) = \inf_{\tilde{\mu} \in \mathcal{S}_1} E_k^{\tilde{\mu},\nu} \left[ e^{\theta \sum_{t=0}^{\tau_0} (r_1(X_t, U_t, V_t) - \Lambda_1^*[\nu])} \right] \le R_0 B_0, \ \forall k \neq 0.$$
(60)

Let  $\tilde{\mu}[\nu] \in S_1$  be any minimizer in (38). Such a minimizer always exists by (59), Proposition 1 and the compactness of  $S_1$ . Hence, for all  $k \neq 0$ , we have using Jensen's inequality and part (a) of Proposition 3

$$h_{1}^{*}[\nu](k) = E_{k}^{\tilde{\mu}[\nu],\nu} \left[ e^{\theta \sum_{t=0}^{\tau_{0}} (r_{1}(X_{t},U_{t},V_{t})-\Lambda_{1}^{*}[\nu])} \right] = E_{k}^{\tilde{\mu}[\nu],\nu} \left[ e^{\theta \sum_{t=0}^{\sigma_{0}} (r_{1}(X_{t},U_{t},V_{t})-\Lambda_{1}^{*}[\nu])} \right]$$

$$\geq E_{k}^{\tilde{\mu}[\nu],\nu} \left[ e^{-2\theta ||r_{1}||_{\infty}(\sigma_{0}+1)} \right] \geq E_{k}^{\tilde{\mu}[\nu],\nu} \left[ e^{-3\Theta ||r_{1}||_{\infty}(\sigma_{0}+1)} \right] \geq \frac{1}{R_{0}} E_{k}^{\tilde{\mu}[\nu],\nu} \left[ \frac{1}{R_{0}^{\sigma_{0}}} \right] \geq \frac{1}{R_{0}^{E_{k}^{\tilde{\mu}[\nu],\nu}[\sigma_{0}]+1}}$$

$$\geq \frac{1}{R_{0}E_{k}^{\tilde{\mu}[\nu],\nu} \left[ R_{0}^{\sigma_{0}} \right]} \geq \frac{1}{R_{0}\sup_{j\in X} E_{j}^{\tilde{\mu}[\nu],\nu} \left[ R_{0}^{\sigma_{0}} \right]} \geq \frac{1}{R_{0}B_{0}}.$$

$$(61)$$

Note that  $h_1^*[\nu](0) = \inf_{\xi \in \mathcal{P}(U)} \int_V \int_U e^{\theta r_1(0,u,v) - \theta \Lambda_1^*[\nu]} \xi(du) \nu[dv|0]$  obviously obeys the bound. The joint continuity of  $E_k^{\mu,\nu} \left[ e^{\theta \sum_{t=0}^{\tau_0} (r_1(X_t, U_t, V_t) - \Lambda_1^*[\nu])} \right]$  under  $(\mu, \nu)$  can be proved as in the proof of Proposition 4. Hence the continuity of  $h_1^*[\nu]$  in  $\nu$  follows from (59) and the compactness of  $\mathcal{S}_1$ .  $\Box$ 

Now we state and prove following theorem which is a key step for proving the existence of Nash equilibria.

THEOREM 3. Under (A1) - (A3) and for  $(\mu,\nu) \in S_1 \times S_2$ ,  $(\ln h_1^*[\nu], \hat{\lambda}_1[\nu])$  with  $h_1^*[\nu](0) = \int_V \int_U e^{\theta r_1(0,u,v) - \theta \Lambda_1^*[\nu]} w_1^*[\nu](0) (du)\nu[dv|0]$ , where  $w_1^*[\nu]$  is defined in (39), is the unique solution in  $B(X) \times \mathbb{R}_+$  to (26). Similarly,  $(\ln h_2^*[\mu], \hat{\lambda}_2[\mu])$  with  $h_2^*[\mu](0) = \int_U \int_V e^{\theta r_2(0,u,v) - \theta \Lambda_2^*[\mu]} \mu[du|0] w_2^*[\mu](0) (dv)$ , where  $w_2^*[\mu]$  is defined in (42), is the unique solution to (27) again in  $B(X) \times \mathbb{R}_+$ .

**Proof:** We shall prove this for player I when player II plays the strategy  $\nu$ . The other case is proved analogously. We first prove that  $(\ln h_1^*[\nu], \hat{\lambda}_1[\nu])$  is a solution to (26). By Lemma 1,  $h_1^*[\nu]$  as defined in (38) is finite. Consider  $w_1^*[\nu]$  defined in (39) with this  $h_1^*[\nu]$ . Let

$$f_1^*[\nu](k) \stackrel{def}{=} E_k^{w_1^*[\nu],\nu} \left[ e^{\theta \sum_{t=0}^{\tau_0} (r_1(X_t, U_t, V_t) - \Lambda_1[\nu](w_1^*[\nu]))} \right]$$
(62)

under the stationary strategy  $w_1^*[\nu]$ . Note that, for  $k \neq 0$ , we can get as in (60) of Lemma 2,

$$0 < f_1^*[\nu](k) \le R_0 B_0 < \infty.$$
(63)

It follows from the Markov property that

$$\int_{V} \int_{U} \left( \sum_{j \in X} f_{1}^{*}[\nu](j)q(j|k,u,v) \right) w_{1}^{*}[\nu](k)(du)\nu[dv|k] = E_{k}^{w_{1}^{*}[\nu],\nu} \left[ e^{\theta \sum_{t=1}^{\sigma_{0}} (r_{1}(X_{t},U_{t},V_{t}) - \Lambda_{1}[\nu](w_{1}^{*}[\nu]))} \right] \\
= \int_{V} \int_{U} e^{-\theta r_{1}(k,u,v) + \theta \Lambda_{1}[\nu](w_{1}^{*}[\nu])} f_{1}^{*}[\nu](k)w_{1}^{*}[\nu](k)(du)\nu[dv|k], \ k \neq 0.$$
(64)

Since,  $f_1^*[\nu](0) = \int_V \int_U e^{\theta r_1(0,u,v) - \theta \Lambda_1[\nu](w_1^*[\nu])} w_1^*[\nu](0)(du)\nu[dv|0]$ , it follows that the following multiplicative Poisson equation holds:

$$\int_{V} \int_{U} \left( \sum_{j \in X} f_{1}^{*}[\nu](j)q(j|k, u, v) \right) w_{1}^{*}[\nu](k)(du)\nu[dv|k]$$

$$= \int_{V} \int_{U} e^{-\theta r_{1}(k, u, v) + \theta \Lambda_{1}[\nu](w_{1}^{*}[\nu])} f_{1}^{*}[\nu](k)w_{1}^{*}[\nu](k)(du)\nu[dv|k], \forall k.$$
(65)

Since by (45) in Proposition 4  $\Lambda_1[\nu](\{w_1^*[\nu]\}) < \infty$  and by (63)  $0 < f_1^*[\nu](k) < \infty$ , under the transition kernel  $q(\cdot|k, u, v)$ , we can define the "twisted kernel"  $\check{q}_1(\cdot|k, w_1^*[\nu](k), \nu(k))$  as follows:

$$\breve{q}_{1}(j|k,w_{1}^{*}[\nu](k),\nu(k)) \stackrel{def}{=} \int_{U} \int_{V} \frac{e^{\theta r_{1}(k,u,v) - \theta \Lambda_{1}[\nu](w_{1}^{*}[\nu])}}{f_{1}^{*}[\nu](k)} f_{1}^{*}[\nu](j)q(j|k,u,v)w_{1}^{*}[\nu](k)(du)\nu[dv|k], \ j \in X.$$

$$(66)$$

Note that the kernel  $\check{q}_1$  is indeed stochastic i.e.,  $\sum_{j \in X} \check{q}_1(j|k, w_1^*[\nu](k), \nu(k)) = 1$ ,  $\forall k$  since (65) holds. Hence,  $\check{q}_1$  is the transition kernel for some Markov chain  $\{\check{X}_t\}$ . Since, by (45), for any  $-||r_1||_{\infty} \leq \Lambda < \Lambda_1[\nu](w_1^*[\nu]),$ 

$$0 < E_0^{w_1^*[\nu],\nu} \left[ e^{\theta \sum_{t=0}^{\sigma_0 - 1} (r_1(X_t, U_t, V_t) - \Lambda)} \right] \le E_0^{w_1^*[\nu],\nu} \left[ e^{3\Theta ||r_1||_{\infty} \sigma_0} \right] \le \sup_{k \in X} E_k^{w_1^*[\nu],\nu} \left[ R_0^{\sigma_0} \right] \le B_0 < \infty,$$

it follows from Theorem 4.1(i)-(ii) of Balaji and Meyn [2] that the chain  $\{\breve{X}_t\}$  under  $\breve{q}_1$  is geometrically recurrent. Hence it follows from Theorem 4.2(i) of Balaji and Meyn [2] that  $(f_1^*[\nu], \Lambda_1[\nu](\{w_1^*[\nu]\}))$  is a solution to the multiplicative Poisson equation:

$$e^{\theta\lambda}\psi(k) = \int_{U} \int_{V} e^{\theta r_{1}(k,u,v)} \left(\sum_{j \in X} \psi(j)q(j|k,u,v)\right) w_{1}^{*}[\nu](k)(du)\nu[dv|k].$$
(67)

It follows from (39), the fact that  $h_1^*[\nu] > 0$  by definition and (55) in Lemma 1 that, for all  $k \in X$ ,

$$\inf_{\xi \in \mathcal{P}(U)} \int_{V} \int_{U} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} h_{1}^{*}[\nu](j)q(j|k,u,v) \right) \xi(du)\nu[dv|k] \\
= \int_{V} \int_{U} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} h_{1}^{*}[\nu](j)q(j|k,u,v) \right) w_{1}^{*}[\nu](k)(du)\nu[dv|k] \\
\leq e^{\theta \Lambda_{1}^{*}[\nu]} h_{1}^{*}[\nu](k) \leq e^{\theta \Lambda_{1}[\nu](w_{1}^{*}[\nu])} h_{1}^{*}[\nu](k),$$
(68)

as  $\Lambda_1^*[\nu] \leq \Lambda_1[\nu](w_1^*[\nu])$  (see (35) and (37)). Hence, by Theorem 4.2(i)-(ii) of Balaji and Meyn [2], we obtain

$$\frac{h_1^*[\nu](k)}{h_1^*[\nu](0)} = \frac{f_1^*[\nu](k)}{f_1^*[\nu](0)},\tag{69}$$

and

$$\inf_{\xi \in \mathcal{P}(U)} \int_{U} \int_{V} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} h_{1}^{*}[\nu](j)q(j|k,u,v) \right) \xi(du)\nu[dv|k] \\
= e^{\theta \Lambda_{1}^{*}[\nu]} h_{1}^{*}[\nu](k) = e^{\theta \Lambda_{1}[\nu](w_{1}^{*}[\nu])} h_{1}^{*}[\nu](k), \ \forall k \in X.$$
(70)

Hence  $\Lambda_1^*[\nu] = \Lambda_1[\nu](w_1^*[\nu])$ . Since, by Lemma 2,  $0 < \sup_{k \in X} |\ln h_1^*[\nu](k)| \le \ln(R_0B_0) < \infty$  as by definition both  $R_0, B_0 > 1$ , uniqueness follows from Corollary 2.3 of Di Masi and Stettner [15] and the condition  $h_1^*[\nu](0) = f_1^*[\nu](0) = \inf_{\xi \in \mathcal{P}(U)} \int_U \int_V e^{\theta r_1(0,u,v) - \theta \Lambda_1^*[\nu]} \xi(du) \nu [dv|0]$ . It follows from Proposition 1.1 of Di Masi and Stettner [15] (see also Theorem 2.1 of Hernández-Hernández and Marcus [24]) that

$$\Lambda_1^*[\nu] = \hat{\lambda}_1[\nu] \tag{71}$$

and hence,

$$h_1^*[\nu](k) = E_k^{\{w_1^*[\nu]\},\nu} \left[ e^{\theta \sum_{t=0}^{\tau_0} (r_1(X_t, U_t, V_t) - \hat{\lambda}_1[\nu])} \right], \forall k \in X.$$
(72)

Similarly, it can be shown for player II that

$$\Lambda_2^*[\mu] = \hat{\lambda}_2[\mu] \tag{73}$$

and

$$h_{2}^{*}[\mu](k) = E_{k}^{\mu, \{w_{2}^{*}[\mu]\}} \left[ e^{\theta \sum_{t=0}^{\tau_{0}} (r_{2}(X_{t}, U_{t}, V_{t}) - \hat{\lambda}_{2}[\mu])} \right], \forall k \in X,$$
(74)

under the stationary strategy  $w_2^*[\mu]$  as defined in (42) with  $h_2^*[\mu]$  as defined in (41).

Hence, by (30) and (31), for any  $(\mu, \nu) \in S_1 \times S_2$ , there exists measurable (minimizing) selectors (see Beneš [9])

$$(\hat{\mu}[\nu], \hat{\nu}[\mu]) : X \mapsto \mathcal{P}(U) \times \mathcal{P}(V)$$

satisfying

$$\inf_{\xi \in \mathcal{P}(U)} \int_{V} \int_{U} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} e^{\ln h_{1}^{*}[\nu](j)} q(j|k,u,v) \right) \xi(du) \nu[dv|k] \\
= \int_{V} \int_{U} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} e^{\ln h_{1}^{*}[\nu](j)} q(j|k,u,v) \right) \hat{\mu}[\nu](k)(du) \nu[dv|k],$$
(75)

implying, by Theorem 3 and (26),

$$e^{\theta \hat{\lambda}_{1}[\nu] + \ln h_{1}^{*}[\nu](k)} = \int_{V} \int_{U} e^{\theta r_{1}(k,u,v)} \left( \sum_{j \in X} e^{\ln h_{1}^{*}[\nu](j)} q(j|k,u,v) \right) \hat{\mu}[\nu](k)(du)\nu[dv|k],$$
(76)

and similarly

$$\inf_{\chi \in \mathcal{P}(V)} \int_{U} \int_{V} e^{\theta r_{2}(k,u,v)} \left( \sum_{j \in X} e^{\ln h_{2}^{*}[\mu](j)} q(j|k,u,v) \right) \mu\left[du|k\right] \chi(dv) \\
= \int_{U} \int_{V} e^{\theta r_{2}(k,u,v)} \left( \sum_{j \in X} e^{\ln h_{2}^{*}[\mu](j)} \right) q(j|k,u,v) \mu\left[du|k\right] \hat{\nu}[\mu](k)(dv),$$
(77)

implying, by Theorem 3 and (27),

$$e^{\theta \hat{\lambda}_{2}[\mu] + \ln h_{2}^{*}[\mu](k)} = \int_{U} \int_{V} e^{\theta r_{2}(k,u,v)} \left( \sum_{j \in X} e^{\ln h_{2}^{*}[\mu](j)} q(j|k,u,v) \right) \mu\left[du|k\right] \hat{\nu}[\mu](k)(dv).$$
(78)

Using the minimizing selectors  $\hat{\mu}[\nu]$  and  $\hat{\nu}[\mu]$  in (75) and (77) as our optimal responses for given  $(\mu, \nu) \in S_1 \times S_2$ , we now prove the existence of Nash equilibria in the class of stationary Markov strategies for the ergodic cost criterion (2).

THEOREM 4. Given  $\theta \in (0, \Theta]$ , there exists a pair of Nash equilibrium strategy in  $S_1 \times S_2$  for the game corresponding to the cost criterion (2). **Proof:** Given  $(\mu, \nu) \in S_1 \times S_2$ ,  $\mathcal{H}$  as defined in (34) can be proved to be a map with non-empty, closed and convex values as in Theorem 1. Now, by Proposition 1,  $\mathcal{L}^{\mu,\nu}\{X_t, t \ge 0\}$  is jointly continuous in  $(\mu, \nu) \in S_1 \times S_2$ . Note that  $\hat{\lambda}_1[\nu]$  as obtained in (28) has a minimizing stationary strategy (30) and  $\hat{\lambda}_2[\mu]$  as obtained in (29) has a minimizing stationary strategy (31), i.e., the minimizations in (28) and (29) can be effectively considered over  $S_1$  and  $S_2$  (resp.) which are compact sets. Also,  $r_i$ , i = 1, 2 are bounded functions. Hence, by Berge Maximum Theorem (see Theorem 17.31 of Aliprantis and Border [7]),  $\hat{\lambda}_1[\nu]$  and  $\hat{\lambda}_2[\mu]$  are continuous in  $\nu$  and  $\mu$  respectively. Similarly, the continuity of  $h_1^*[\nu]$  (resp.  $h_2^*[\mu]$ ) follows from (72) (resp. (74)) and Lemma 2. Now consider a sequence  $\{(\mu_k, \nu_k)\}_{k\in\mathbb{N}}$  in  $S_1 \times S_2$  converging to  $(\mu_{\infty}, \nu_{\infty}) \in S_1 \times S_2$ . Let  $\mathcal{H}(\mu_k, \nu_k) \ni (\tilde{\mu}_k, \tilde{\nu}_k) \rightarrow (\tilde{\mu}_{\infty}, \tilde{\nu}_{\infty}) \in \mathcal{H}(\mu_{\infty}, \nu_{\infty})$ . Hence the map  $\mathcal{H}$  is upper semi-continuous. Then, by Theorem 1 of Fan [19], there exists a fixed point  $(\mu^*, \nu^*): X \mapsto \mathcal{P}(U) \times \mathcal{P}(V)$  of the map  $\mathcal{H}$  defined as

$$(\mu^*(k), \nu^*(k)) \stackrel{def}{=} (\hat{\mu}[\nu^*](k), \hat{\nu}[\mu^*](k))$$
(79)

where  $\hat{\mu}[\nu^*]$  (resp.  $\hat{\nu}[\mu^*]$ ) is any minimizing selector in (75) (resp. (77)). Then  $(\mu^*, \nu^*) \in S_1 \times S_2$  is a Nash equilibrium strategy for the cost criterion (2), i.e., for all  $k \in X$ 

$$\hat{\lambda}_{1}[\nu^{*}] = \beta_{1}^{\mu^{*},\nu^{*}}(\theta,k) \leq \beta_{1}^{\mu,\nu^{*}}(\theta,k), \ \forall \mu \in \Pi_{1},$$

and

$$\hat{\lambda}_{2}[\mu^{*}] = \beta_{2}^{\mu^{*},\nu^{*}}(\theta,k) \le \beta_{2}^{\mu^{*},\nu}(\theta,k), \ \forall \nu \in \Pi_{2}.$$

5. Conclusions. We have established Nash equilibria for nonzero-sum discrete-time stochastic games on a countable state space with risk-sensitive cost criteria. We have studied both discounted and ergodic cost criteria on the infinite time horizon. For discounted cost criterion, we have established Nash equilibria under fairly general conditions. For ergodic cost criterion, however, we have established our results under additional criteria of irreducibility (A1), stability (A2) and smallness of cost (A3) all of which are standard in the corresponding literature. A natural continuation of this work is to analyze such discrete-time games on general state spaces. Another direction is to investigate such games in continuous time in the set-up of stochastic differential games.

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