# Linear Convergence of Projection Algorithms 

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#### Abstract

Projection algorithms are well known for their simplicity and flexibility in solving feasibility problems. They are particularly important in practice due to minimal requirements for software implementation and maintenance. In this work, we study linear convergence of several projection algorithms for systems of finitely many closed sets. The results complement contemporary research on the same topic.


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## 1 Introduction

In this paper, $X$ is a Euclidean space with inner product $\langle\cdot, \cdot\rangle$ and induced norm $\|\cdot\|$. Throughout, we set $I:=\{1, \ldots, m\}$ and assume that $\left\{C_{i}\right\}_{i \in I}$ is a system of closed (possibly nonconvex) subsets of $X$. The notation used in the paper is fairly standard and follows [3]. The nonnegative integers are $\mathbb{N}$, the real numbers are $\mathbb{R}$, while $\mathbb{R}_{+}:=\{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_{++}:=\{x \in \mathbb{R} \mid x>0\}$. If $w \in X$ and $\rho \in \mathbb{R}_{+}$, then $\boldsymbol{B}(w ; \rho):=\{x \in X \mid\|x-w\| \leq \rho\}$ is the closed ball centered at $w$ with radius $\rho$. Given a subset $C$ of $X$, the affine hull of $C$ is denoted by aff $C$ and the orthogonal complement of $C$ is $C^{\perp}:=\{x \in X \mid \forall c \in C:\langle c, x\rangle=0\}$. The notation $T: X \rightrightarrows X$ means that $T$ is a set-valued operator from $X$ to $X$ and $\operatorname{Fix} T:=\{x \in X \mid x \in T x\}$ denotes the set of fixed points of $T$. As usual, Id represents the identity operator.

The paper is concerned with cyclic algorithms for solving the feasibility problem

$$
\begin{equation*}
\text { find a point } \quad x \in \bigcap_{i \in I} C_{i} \text {. } \tag{1}
\end{equation*}
$$

This problem has long been known for its importance in many applications. To describe cyclic algorithms for (1), we first associate each set $C_{i}$ with an operator $T_{i}: X \rightrightarrows X$ and adopt the following convention

$$
\begin{equation*}
\forall n \in \mathbb{N}, \forall i \in I: \quad C_{m n+i}:=C_{i} \quad \text { and } \quad T_{m n+i}:=T_{i} . \tag{2}
\end{equation*}
$$

[^0]Given a starting point $x_{0} \in X$, the cyclic algorithm with respect to the ordered tuple $\left(T_{i}\right)_{i \in I}$ generates sequences $\left(x_{n}\right)_{n \in \mathbb{N}}$ by

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad x_{n+1} \in T_{n+1} x_{n} . \tag{3}
\end{equation*}
$$

Each such sequence is called a cyclic sequence generated by $\left(T_{i}\right)_{i \in I}$. When $m=1$, we drop the subscripts and write $C:=C_{1}$ and $T:=T_{1}$. The recurrence (3) then reads as

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad x_{n+1} \in T x_{n}, \tag{4}
\end{equation*}
$$

and we say that the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ is generated by $T$. The corresponding operators include, but not limited to, projectors and their variants. Recall that for a set $C$, the distance function to $C$ is defined by

$$
\begin{equation*}
d_{C}: X \rightarrow \mathbb{R}: x \mapsto \inf _{c \in C}\|x-c\|, \tag{5}
\end{equation*}
$$

and the projector onto $C$ is defined by

$$
\begin{equation*}
P_{C}: X \rightrightarrows C: x \mapsto \underset{c \in C}{\operatorname{argmin}}\|x-c\|=\left\{c \in C \mid\|x-c\|=d_{C}(x)\right\} . \tag{6}
\end{equation*}
$$

In general, one expects the cyclic sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ or other acquired sequences converge to a solution of (1). In such case, we are interested in $R$-linear convergence of those sequences. Recall that a sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ is said to converge $R$-linearly to a point $\bar{x}$ with rate $\rho \in[0,1[$ if there exists a constant $\sigma \in \mathbb{R}_{+}$such that

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad\left\|x_{n}-\bar{x}\right\| \leq \sigma \rho^{n} \tag{7}
\end{equation*}
$$

Among the main contributions of the paper, under certain regularity assumptions on sets and system of sets, we show that:
(R1) The cyclic relaxed projections with at most one reflection, which includes the reflectionprojection algorithm [9], converge $R$-linearly locally (see Theorem 5.7 and Remark 5.13);
(R2) A refined $R$-linear rate is obtained for cyclic over-relaxed projections (see Theorem 5.8 and Corollary 5.10);
(R3) The cyclic semi-intrepid projections for injectable sets converge locally with $R$-linear rate (see Theorem 5.19);
Moreover, the linear convergence is global in the presence of convexity (see Corollaries 5.12 and 5.20). To the best of our knowledge, these results are new and have not been observed in the literature. In addition, we also present other new results involving Douglas-Rachford (DR) operators [18, 28]; see Theorems 5.21 and 5.25. Our work complements other studies on projection algorithms $[4,5,6,12,13,19,23,27,32,33]$.

The remainder of the paper is organized as follows. Section 2 contains basic concepts needed for our analysis. Section 3 then provides key components for $R$-linear convergence. In Section 4, we prove $R$-linear convergence for general cyclic algorithms. Finally, Section 5 presents applications to various cyclic algorithms including the cyclic relaxed projections, cyclic semi-intrepid projections, and cyclic generalized DR algorithm.

## 2 Preliminaries

Given a subset $C$ of $X$ and $x \in C$, the Fréchet normal cone to $C$ at $x$ [30, Definition 1.1(i)] is defined by

$$
\begin{equation*}
\widehat{N}_{C}(x):=\left\{u \in X \left\lvert\, \limsup _{y \rightarrow x, y \in C \backslash\{x\}} \frac{\langle u, y-x\rangle}{\|y-x\|} \leq 0\right.\right\}, \tag{8}
\end{equation*}
$$

the proximal normal cone to $C$ at $x$ (see [30, Section 2.5 .2 , D] and [34, Example 6.16]) is given by

$$
\begin{equation*}
N_{C}^{\operatorname{prox}}(x):=\left\{\lambda(z-x) \mid z \in P_{C}^{-1}(x), \lambda \in \mathbb{R}_{+}\right\}, \tag{9}
\end{equation*}
$$

and the limiting normal cone to $C$ at $x$ [30, Definition 1.1(ii)] can be given by ([30, Theorem 1.6])

$$
\begin{align*}
N_{C}(x) & :=\left\{u \in X \mid \exists x_{n} \rightarrow x, u_{n} \rightarrow u \text { with } x_{n} \in C, u_{n} \in \hat{N}_{C}\left(x_{n}\right)\right\}  \tag{10a}\\
& =\left\{u \in X \mid \exists x_{n} \rightarrow x, u_{n} \rightarrow u \text { with } x_{n} \in C, u_{n} \in N_{C}^{\text {prox }}\left(x_{n}\right)\right\} . \tag{10b}
\end{align*}
$$

As seen below, normal cones are used to describe superregularity for sets and strong regularity for systems of sets. We recall the superregularity concept, which was first introduced in [27] and later refined in $[10,11,23,32]$. Superregularity holds for a major class of sets including convex sets and sets with "smooth" boundary. This property plays an important role in analyzing linear convergence of projection methods, see, e.g., [ $10,11,23,27,32,33]$.

Definition 2.1 (superregularity of sets). Let $C$ be a nonempty subset of $X, w \in X, \varepsilon \in \mathbb{R}_{+}$, and $\delta \in \mathbb{R}_{++}$. We say that $C$ is $(\varepsilon, \delta)$-regular at $w$ if

$$
\left.\begin{array}{l}
x, y \in C \cap \boldsymbol{B}(w ; \delta),  \tag{11}\\
u \in N_{C}^{\text {prox }}(x)
\end{array}\right\} \Rightarrow\langle u, x-y\rangle \geq-\varepsilon\|u\| \cdot\|x-y\|,
$$

and $(\varepsilon, \infty)$-regular at $w$ if it is $(\varepsilon, \delta)$-regular for all $\delta \in \mathbb{R}_{++}$. The set $C$ is said to be superregular at $w$ if for all $\varepsilon \in \mathbb{R}_{++}$, there exists $\delta \in \mathbb{R}_{++}$such that $C$ is $(\varepsilon, \delta)$-regular at $w$. The system $\left\{C_{i}\right\}_{i \in I}$ is said to be superregular at $w$ if $C_{i}$ is superregular at $w$ for every $i \in I$.

Next, we recall two regularity concepts for systems of sets: linear regularity and strong regularity.
Definition 2.2 (linear regularity of set systems). Let $\kappa \in \mathbb{R}_{++}$. The system $\left\{C_{i}\right\}_{\in I}$ is said to be $\kappa$-linearly regular on a subset $U$ of $X$ if

$$
\begin{equation*}
\forall x \in U: \quad d_{C}(x) \leq \kappa \max _{i \in I} d_{C_{i}}(x), \quad \text { where } \quad C:=\bigcap_{i \in I} C_{i} . \tag{12}
\end{equation*}
$$

The constant $\kappa$ is called a linear regularity modulus of $\left\{C_{i}\right\}_{i \in I}$ on $U$. We say that $\left\{C_{i}\right\}_{i \in I}$ is linearly regular around $w \in X$ if there exist $\delta \in \mathbb{R}_{++}$and $\kappa \in \mathbb{R}_{++}$such that $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta)$.

Linear regularity for set systems has a long history and was first defined in convex settings, see, e.g., [1, Definition 5.6], [2, Definition 3], and [12, Section 5.2] for a brief summary on this property. Naturally, linear regularity was extended to system of closed sets, for instance, [23, Definition 3.5]; and was known as metric inequality in [25, Equation (15)], [31, Section 3], and [24, Section 5]; and as subtransversality in [26, Definition 1].

Definition 2.3 (strong regularity of set systems). The system $\left\{C_{i}\right\}_{\in I}$ is said to be strongly regular at $w \in \bigcap_{i \in I} C_{i}$ if

$$
\begin{equation*}
\sum_{i \in I} u_{i}=0 \text { and } u_{i} \in N_{C_{i}}(w) \Rightarrow \quad \forall i \in I: u_{i}=0 . \tag{13}
\end{equation*}
$$

In the case $I=\{1,2\}$, condition (13) can be rewritten as

$$
\begin{equation*}
N_{C_{1}}(w) \cap\left(-N_{C_{2}}(w)\right)=\{0\} . \tag{14}
\end{equation*}
$$

Strong regularity of systems is also known as normal qualification condition in [30, Definition 3.2], as $C Q$ condition in [10, Definition 6.6], and as transversality in [26, Definition 2]. To clear the confusion it may cause, we will show that strong regularity in Definition 2.3 is equivalent to the ones in [25, Definition 1(vi)] and in [23, Definition 3.2]. In view of [25, Proposition 2, Proposition 10(ii), and Corollary 2], it suffices to prove the following result.

Proposition 2.4 (characterization of strong regularity). The system $\left\{C_{i}\right\}_{i \in I}$ is strongly regular at $w \in \bigcap_{i \in I} C_{i}$ if and only if there exist $\zeta \in \mathbb{R}_{++}$and $\delta \in \mathbb{R}_{++}$such that

$$
\begin{equation*}
\forall i \in I, \forall x_{i} \in C_{i} \cap \boldsymbol{B}(w ; \delta), \forall u_{i} \in \widehat{N}_{C_{i}}\left(x_{i}\right): \quad\left\|\sum_{i \in I} u_{i}\right\| \geq \zeta \sum_{i \in I}\left\|u_{i}\right\| . \tag{15}
\end{equation*}
$$

Proof. $(\Leftarrow)$ : Suppose that (15) holds and that $\sum_{i \in I} u_{i}=0$ with $u_{i} \in N_{C_{i}}(w)$. Then for every $i \in I$, by (10), there exist sequences $x_{i, n} \rightarrow w, u_{i, n} \rightarrow u_{i}$ with $x_{i, n} \in C_{i}$ and $u_{i, n} \in \widehat{N}_{C_{i}}\left(x_{i, n}\right)$. Since $x_{i, n} \rightarrow w$, we can assume without loss of generality that $x_{i, n} \in \boldsymbol{B}(w ; \delta)$ for all $n \in \mathbb{N}$. It follows that $x_{i, n} \in C_{i} \cap \boldsymbol{B}(w ; \delta)$, and then by (15), we have $\left\|\sum_{i \in I} u_{i, n}\right\| \geq \zeta \sum_{i \in I}\left\|u_{i, n}\right\|$ for all $n \in \mathbb{N}$. Passing to the limit as $n \rightarrow \infty$, we get $\left\|\sum_{i \in I} u_{i}\right\| \geq \zeta \sum_{i \in I}\left\|u_{i}\right\|$. Combining with the assumption $\sum_{i \in I} u_{i}=0$, we derive $u_{i}=0$ for every $i \in I$.
$(\Rightarrow)$ : Suppose to the contrary that (15) is not true. Then there exist sequences $\zeta_{n} \rightarrow 0^{+}$, $\delta_{n} \rightarrow 0^{+}, x_{i, n} \in C_{i} \cap \boldsymbol{B}\left(w ; \delta_{n}\right)$, and $u_{i, n} \in \widehat{N}_{C_{i}}\left(x_{i, n}\right)$ such that

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad\left\|\sum_{i \in I} u_{i, n}\right\|<\zeta_{n} \sum_{i \in I}\left\|u_{i, n}\right\| \quad \text { and } \quad \sum_{i \in I}\left\|u_{i, n}\right\|=1, \tag{16}
\end{equation*}
$$

where the latter is obtained by rescaling if necessary. Thus, for every $i \in I$, the sequence $\left(u_{i, n}\right)_{n \in \mathbb{N}}$ is bounded, and by extracting subsequences, we can assume that $u_{i, n} \rightarrow u_{i}$. Since $x_{i, n} \rightarrow w$ and $x_{i, n} \in C_{i}$, it follows from (10) that $u_{i} \in N_{C_{i}}(w)$. Letting $n \rightarrow \infty$ in (16), we obtain $\left\|\sum_{i \in I} u_{i}\right\|=0$ and $\sum_{i \in I}\left\|u_{i}\right\|=1$, which contradicts the strong regularity. Thus (15) holds.

We end this section with connections between linear regularity and strong regularity.
Fact 2.5. ([25, Theorem 1]) If the system $\left\{C_{i}\right\}_{i \in I}$ is strongly regular at $w \in \bigcap_{i \in I} C_{i}$, then it is linearly regular around $w$.
Remark 2.6 (strong regularity of subsystems). By definition, if the system $\left\{C_{i}\right\}_{i \in I}$ is strongly regular at $w$, then so is each of its subsystems. However, even when each proper subsystem $\left\{C_{i}\right\}_{i \in J}$ with $J \varsubsetneqq I$ is strongly regular and the entire system $\left\{C_{i}\right\}_{i \in I}$ is linearly regular, it does not imply that $\left\{C_{i}\right\}_{i \in I}$ is strongly regular. For example, in $\mathbb{R}^{2}$, consider $C_{1}=\{(\xi, \zeta) \mid \xi+\zeta \leq 0\}, C_{2}=$ $\{(\xi, \zeta) \mid \xi-\zeta \leq 0\}, C_{3}=\{(\xi, \zeta) \mid \xi \geq 0\}$, and $w=(0,0) \in C_{1} \cap C_{2} \cap C_{3}$. Then one can check that $\left\{C_{i}\right\}_{i \in J}$ with $J \varsubsetneqq\{1,2,3\}$ is strongly regular at $w$, and $\left\{C_{1}, C_{2}, C_{3}\right\}$ is linearly regular around $w$, but $\left\{C_{1}, C_{2}, C_{3}\right\}$ is not strongly regular at $w$.

## 3 Quasi Fejér monotonicity and quasi coercivity

The following quasi Fejér monotonicity concept generalizes the Fejér monotonicity for sequences and operators, see, e.g., [3, Definition 5.1] and [17, Definition 2.1.15].
Definition 3.1 (quasi firm Fejér monotonicity). Let $C$ and $U$ be nonempty subsets of $X$, let $\gamma \in\left[1,+\infty\left[\right.\right.$, and let $\beta \in \mathbb{R}_{+}$. A set-valued operator $T: X \rightrightarrows X$ is said to be $(C, \gamma, \beta)$-quasi firmly Fejér monotone on $U$ if

$$
\begin{equation*}
\forall x \in U, \forall x_{+} \in T x, \forall \bar{x} \in C: \quad\left\|x_{+}-\bar{x}\right\|^{2}+\beta\left\|x-x_{+}\right\|^{2} \leq \gamma\|x-\bar{x}\|^{2} . \tag{17}
\end{equation*}
$$

We say that $T$ is $(C, \gamma)$-quasi firmly Fejér monotone on $U$ if $\beta=1$, i.e.,

$$
\begin{equation*}
\forall x \in U, \forall x_{+} \in T x, \forall \bar{x} \in C: \quad\left\|x_{+}-\bar{x}\right\|^{2}+\left\|x-x_{+}\right\|^{2} \leq \gamma\|x-\bar{x}\|^{2}, \tag{18}
\end{equation*}
$$

and that $T$ is $(C, \gamma)$-quasi Fejér monotone on $U$ if $\beta=0$, i.e.,

$$
\begin{equation*}
\forall x \in U, \forall x_{+} \in T x, \forall \bar{x} \in C: \quad\left\|x_{+}-\bar{x}\right\| \leq \gamma^{1 / 2}\|x-\bar{x}\| . \tag{19}
\end{equation*}
$$

From the definition, we observe that
(i) ( $C, \gamma, \beta$ )-quasi firm Fejér monotonicity implies ( $C, \gamma$ )-quasi Fejér monotonicity, while $(C, 1)$ quasi Fejér monotonicity is exactly Fejér monotonicity with respect to $C$ in [17, Definition 2.1.15].
(ii) If $\gamma^{\prime} \geq \gamma \geq 1,0 \leq \beta^{\prime} \leq \beta, C^{\prime} \subseteq C$, and $U^{\prime} \subseteq U$, then $(C, \gamma, \beta)$-quasi firm Fejér monotonicity on $U$ implies $\left(C^{\prime}, \gamma^{\prime}, \beta^{\prime}\right)$-quasi firm Fejér monotonicity on $U^{\prime}$.
(iii) If $T$ is nonexpansive (see [3, Definition 4.1]), then $T$ is (Fix $T, 1$ )-quasi Fejér monotone on $X$.
(iv) If $T$ is $\lambda$-averaged (see [3, Definition 4.23]), then by [3, Proposition $4.25(\mathrm{iii})], T$ is (Fix $T, 1, \frac{1-\lambda}{\lambda}$ )-quasi firmly Fejér monotone on $X$. In particular, if $T$ is firmly nonexpansive, then $T$ is ( $\operatorname{Fix} T, 1$ )-quasi firmly Fejér monotone on $X$.

Quasi firm Fejér monotonicity is closely related to [23, Definition 2.3] and [29, Proposition 2.4(iii)]. Also, $(C, \gamma)$-quasi firm Fejér monotonicity is more restrictive than [33, Definition 2.7] since the latter requires only

$$
\begin{equation*}
\forall x \in U, \forall x_{+} \in T x, \forall \bar{x} \in P_{C} x: \quad\left\|x_{+}-\bar{x}\right\|^{2}+\left\|x-x_{+}\right\|^{2} \leq \gamma\|x-\bar{x}\|^{2} . \tag{20}
\end{equation*}
$$

Nevertheless, it turns out that quasi firm Fejér monotonicity still holds for a broad class of operators, e.g., relaxed projectors for superregular sets (see Proposition 3.5) and generalized Douglas-Rachford operators for systems of two superregular sets (see Proposition 3.7).

The next lemma shows the quasi firm Fejér monotonicity for averaged-type operators.
Lemma 3.2 (averaged quasi firmly Fejér monotone operators). Let $C$ and $U$ be nonempty subsets of $X, \gamma \in\left[1,+\infty\left[, \beta \in \mathbb{R}_{+}, \lambda \in\right] 0,1+\beta\right]$, and let $S: X \rightrightarrows X$ be a $(C, \gamma, \beta)$-quasi firmly Fejér monotone operator on $U$. Then $T:=(1-\lambda) \operatorname{Id}+\lambda S$ is $\left(C, \gamma^{\prime}, \beta^{\prime}\right)$-quasi firmly Fejér monotone on $U$ with

$$
\begin{equation*}
\gamma^{\prime}:=1-\lambda+\lambda \gamma \quad \text { and } \quad \beta^{\prime}:=\frac{1-\lambda+\beta}{\lambda} . \tag{21}
\end{equation*}
$$

Proof. Let $x \in U, x_{+} \in T x$, and $\bar{x} \in C$. Writing $x_{+}=(1-\lambda) x+\lambda s$ with $s \in S x$, we have $x_{+}-\bar{x}=(1-\lambda)(x-\bar{x})+\lambda(s-\bar{x})$ and $x-x_{+}=\lambda(x-s)$. So

$$
\begin{align*}
\left\|x_{+}-\bar{x}\right\|^{2} & =(1-\lambda)\|x-\bar{x}\|^{2}+\lambda\|s-\bar{x}\|^{2}-\lambda(1-\lambda)\|(x-\bar{x})-(s-\bar{x})\|^{2}  \tag{22a}\\
& =(1-\lambda)\|x-\bar{x}\|^{2}+\lambda\|s-\bar{x}\|^{2}-\lambda(1-\lambda)\|x-s\|^{2} . \tag{22b}
\end{align*}
$$

Using the $(C, \gamma, \beta)$-quasi firm Fejér monotonicity of $S$ on $U$, we continue (22) as

$$
\begin{align*}
\left\|x_{+}-\bar{x}\right\|^{2} & \leq(1-\lambda)\|x-\bar{x}\|^{2}+\lambda\left(\gamma\|x-\bar{x}\|^{2}-\beta\|x-s\|^{2}\right)-\lambda(1-\lambda)\|x-s\|^{2}  \tag{23a}\\
& =(1-\lambda+\lambda \gamma)\|x-\bar{x}\|^{2}-\lambda(1-\lambda+\beta)\|x-s\|^{2}  \tag{23b}\\
& =(1-\lambda+\lambda \gamma)\|x-\bar{x}\|^{2}-\frac{1-\lambda+\beta}{\lambda}\left\|x-x_{+}\right\|^{2} . \tag{23c}
\end{align*}
$$

This completes the proof.
Definition 3.3 (quasi coercivity). Let $C$ and $U$ be nonempty subsets of $X$ and let $\nu \in \mathbb{R}_{++}$. An operator $T: X \rightrightarrows X$ is said to be $(C, \nu)$-quasi coercive on $U$ if

$$
\begin{equation*}
\forall x \in U, \forall x_{+} \in T x: \quad\left\|x-x_{+}\right\| \geq \nu d_{C}(x) \tag{24}
\end{equation*}
$$

We say that $T$ is $C$-quasi coercive around $w \in X$ if there exist $\delta \in \mathbb{R}_{++}$and $\nu \in \mathbb{R}_{++}$such that $T$ is $(C, \nu)$-quasi coercive on $\boldsymbol{B}(w ; \delta)$.

Obviously, if $0<\nu^{\prime} \leq \nu, C^{\prime} \supseteq C$, and $U^{\prime} \subseteq U$, then $(C, \nu)$-quasi coercivity on $U$ implies $\left(C^{\prime}, \nu^{\prime}\right)$-quasi coercivity on $U^{\prime}$. Quasi coercivity follows and slightly extends the coercivity condition in [23, Lemma 3.1(b)] because the latter requires $C \subseteq \operatorname{Fix} T$ while the former does not. Quasi coercivity is also closely related to the linear regularity for operators in [12, Definition 2.1]. Indeed, when $C=\operatorname{Fix} T \neq \varnothing$, then $T$ is $(C, \nu)$-quasi coercive on $X$ if and only if it is linearly regular with constant $\frac{1}{\nu}$ in the sense of [12, Definition 2.1]. Again, under certain conditions, we will show that quasi coercivity holds for several class of projectors.

### 3.1 Relaxed projectors

In this section, we show the quasi firm Fejér monotonicity and quasi coercivity of relaxed projectors for superregular sets. Let $C$ be a nonempty closed subset of $X$ and let $\lambda \in \mathbb{R}_{+}$. The relaxed projector for $C$ with parameter $\lambda$ is defined by

$$
\begin{equation*}
P_{C}^{\lambda}:=(1-\lambda) \operatorname{Id}+\lambda P_{C} . \tag{25}
\end{equation*}
$$

We say that $P_{C}^{\lambda}$ is under-relaxed if $\lambda \leq 1$ and over-relaxed if $\lambda \geq 1$. Clearly, $P_{C}^{0}=\mathrm{Id}, P_{C}^{1}=P_{C}$, and $P_{C}^{2}=R_{C}:=2 P_{C}-\mathrm{Id}$ (the reflector across $C$ ). The following lemma will be used several times in our analysis.
Lemma 3.4. Let $w \in C$, let $\gamma \in\left[1,+\infty\left[\right.\right.$, and let $\delta \in \mathbb{R}_{++}$. Then the following hold:
(i) For all $x \in \boldsymbol{B}(w ; \delta / 2), P_{C}^{\lambda} x \subseteq \boldsymbol{B}(w ;(1+\lambda) \delta / 2)$. In particular, $P_{C}(\boldsymbol{B}(w ; \delta / 2)) \subseteq C \cap \boldsymbol{B}(w ; \delta)$.
(ii) If $T: X \rightrightarrows X$ is $(C \cap \boldsymbol{B}(w ; \delta), \gamma)$-quasi Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$, then

$$
\begin{align*}
\forall x \in \boldsymbol{B}(w ; \delta / 2): & T x \subseteq \boldsymbol{B}\left(w ; \gamma^{1 / 2} \delta / 2\right),  \tag{26a}\\
\forall x \in \boldsymbol{B}(w ; \delta / 2), \forall x_{+} \in T x: & d_{C}\left(x_{+}\right) \leq \gamma^{1 / 2} d_{C}(x) . \tag{26b}
\end{align*}
$$

Proof. (i): Let $x \in \boldsymbol{B}(w ; \delta / 2)$ and let $x_{+} \in P_{C}^{\lambda} x$. Writing $x_{+}=(1-\lambda) x+\lambda p$ for some $p \in P_{C} x$ and noting that $w \in C$, we have $\left\|x_{+}-x\right\|=\lambda\|p-x\|=\lambda d_{C}(x) \leq \lambda\|x-w\|$ and so

$$
\begin{equation*}
\left\|x_{+}-w\right\| \leq\left\|x_{+}-x\right\|+\|x-w\| \leq(1+\lambda)\|x-w\| \leq(1+\lambda) \delta / 2 . \tag{27}
\end{equation*}
$$

Therefore, $P_{C}^{\lambda} x \subseteq C \cap \boldsymbol{B}(w ;(1+\lambda) \delta / 2)$.
(ii): Let $x \in \boldsymbol{B}(w ; \delta / 2)$ and let $x_{+} \in T x$. By quasi Fejér monotonicity,

$$
\begin{equation*}
\forall \bar{x} \in C \cap \boldsymbol{B}(w ; \delta): \quad\left\|x_{+}-\bar{x}\right\| \leq \gamma^{1 / 2}\|x-\bar{x}\| \tag{28}
\end{equation*}
$$

Setting $\bar{x}=w$, we have $\left\|x_{+}-w\right\| \leq \gamma^{1 / 2}\|x-w\| \leq \gamma^{1 / 2} \delta / 2$. Hence, $T x \subseteq \boldsymbol{B}\left(w ; \gamma^{1 / 2} \delta / 2\right)$. Now let $p \in P_{C} x$. Then $p \in C \cap \boldsymbol{B}(w ; \delta)$ by (i). Applying (28) to $\bar{x}=p$ yields

$$
\begin{equation*}
d_{C}\left(x_{+}\right) \leq\left\|x_{+}-p\right\| \leq \gamma^{1 / 2}\|x-p\|=\gamma^{1 / 2} d_{C}(x) \tag{29}
\end{equation*}
$$

Proposition 3.5 (quasi firm Fejér monotonicity of relaxed projectors). Let $w \in C, \varepsilon \in$ $\left[0,1\left[, \delta \in \mathbb{R}_{++}\right.\right.$, and $\left.\left.\lambda \in\right] 0,2\right]$. Set

$$
\begin{equation*}
\Omega:=C \cap \boldsymbol{B}(w ; \delta), \quad \gamma:=1+\frac{\lambda \varepsilon}{1-\varepsilon}, \quad \text { and } \quad \beta:=\frac{2-\lambda}{\lambda} . \tag{30}
\end{equation*}
$$

Suppose that $C$ is $(\varepsilon, \delta)$-regular at $w$. Then $P_{C}^{\lambda}$ is $(\Omega, \gamma, \beta)$-quasi firmly Fejér monotone and, in particular, $R_{C}$ is $\left(\Omega, \frac{1+\varepsilon}{1-\varepsilon}\right)$-quasi Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$. Additionally, if $\varepsilon \in[0,1 / 3]$, then

$$
\begin{equation*}
\forall x \in \boldsymbol{B}(w ; \delta / 2): \quad P_{C}^{\lambda} x \subseteq \boldsymbol{B}(w ; \delta / \sqrt{2}) . \tag{31}
\end{equation*}
$$

Proof. Let $x \in \boldsymbol{B}(w ; \delta / 2)$ and let $p \in P_{C} x$. Then $p \in \Omega$ by Lemma 3.4(i). Since $C$ is $(\varepsilon, \delta)$-regular at $w$ and $x-p \in N_{C}^{\text {prox }}(p)$, we have

$$
\begin{equation*}
\forall \bar{x} \in \Omega: \quad\langle x-p, p-\bar{x}\rangle \geq-\varepsilon\|x-p\| \cdot\|p-\bar{x}\| \geq-\frac{\varepsilon}{2}\left(\|x-p\|^{2}+\|p-\bar{x}\|^{2}\right) . \tag{32}
\end{equation*}
$$

It then follows that

$$
\begin{align*}
\forall \bar{x} \in \Omega: \quad\|x-\bar{x}\|^{2} & =\|x-p\|^{2}+\|p-\bar{x}\|^{2}+2\langle x-p, p-\bar{x}\rangle  \tag{33a}\\
& \geq\|x-p\|^{2}+\|p-\bar{x}\|^{2}-\varepsilon\left(\|x-p\|^{2}+\|p-\bar{x}\|^{2}\right)  \tag{33b}\\
& =(1-\varepsilon)\left(\|x-p\|^{2}+\|p-\bar{x}\|^{2}\right) . \tag{33c}
\end{align*}
$$

So

$$
\begin{equation*}
\forall \bar{x} \in \Omega: \quad \frac{1}{1-\varepsilon}\|x-\bar{x}\|^{2} \geq\|x-p\|^{2}+\|p-\bar{x}\|^{2}, \tag{34}
\end{equation*}
$$

i.e., $P_{C}$ is $\left(\Omega, \frac{1}{1-\varepsilon}, 1\right)$-quasi firmly Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$. Now by Lemma 3.2, we conclude that $P_{C}^{\lambda}=(1-\lambda) \operatorname{Id}+\lambda P_{C}$ is $(\Omega, \gamma, \beta)$-quasi firmly Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$ with $\gamma$ and $\beta$ given by (30). For $\lambda=2$, we have that $\gamma=\frac{1+\varepsilon}{1-\varepsilon}, \beta=0$, and so $R_{C}=P_{C}^{2}$ is $\left(\Omega, \frac{1+\varepsilon}{1-\varepsilon}\right)$-quasi Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$.

Next assume that $\varepsilon \in[0,1 / 3]$. Then $\gamma=1-\lambda+\frac{\lambda}{1-\varepsilon} \leq 1+\frac{\lambda}{2} \leq 2$. By quasi Fejér monotonicity and Lemma 3.4(ii), $P_{C}^{\lambda} x \subseteq \boldsymbol{B}\left(w ; \gamma^{1 / 2} \delta / 2\right) \subseteq \boldsymbol{B}(w ; \delta / \sqrt{2})$.
Proposition 3.6 (quasi coercivity of relaxed projectors). If $\lambda \in \mathbb{R}_{++}$, then $P_{C}^{\lambda}$ is $(C, \lambda)$-quasi coercive on $X$.

Proof. Let $x \in X$ and let $x_{+} \in P_{C}^{\lambda} x$. Then $x_{+}=(1-\lambda) x+\lambda p$ for some $p \in P_{C} x$. So $\left\|x-x_{+}\right\|=$ $\lambda\|x-p\|=\lambda d_{C}(x)$.

### 3.2 Generalized Douglas-Rachford operators

In this section, we establish the quasi firm Fejér monotonicity and quasi coercivity of generalized Douglas-Rachford operators for systems of two superregular sets. Let $A$ and $B$ be closed subsets of $X$ such that $A \cap B \neq \varnothing$ and let $\lambda, \mu, \alpha \in \mathbb{R}_{++}$. The generalized Douglas-Rachford operator for $(A, B)$ with parameters $(\lambda, \mu, \alpha)$ is defined by

$$
\begin{equation*}
T_{\lambda, \mu}^{\alpha}:=(1-\alpha) \operatorname{Id}+\alpha P_{B}^{\mu} P_{A}^{\lambda} \tag{35}
\end{equation*}
$$

Note that $T_{1,1}^{1}=P_{B} P_{A}$ is the classical alternating projection operator [16] and that $T_{2,2}^{1 / 2}=$ $\frac{1}{2}\left(\operatorname{Id}+R_{B} R_{A}\right)$ is the classical DR operator [18, 28].
Proposition 3.7 (quasi firm Fejér monotonicity of generalized DR operators). Let $w \in$ $A \cap B, \varepsilon_{1} \in[0,1 / 3], \varepsilon_{2} \in\left[0,1\left[, \delta \in \mathbb{R}_{++}, \lambda, \mu \in\right] 0,2\right]$, and $\left.\left.\alpha \in\right] 0,1\right]$. Suppose that $A$ and $B$ are $\left(\varepsilon_{1}, \delta\right)$ - and $\left(\varepsilon_{2}, \sqrt{2} \delta\right)$-regular at $w$, respectively. Then $T_{\lambda, \mu}^{\alpha}$ is $(A \cap B \cap \boldsymbol{B}(w ; \delta), \gamma, \beta)$-quasi firmly Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$ with

$$
\begin{equation*}
\gamma:=1-\alpha+\alpha\left(1+\frac{\lambda \varepsilon_{1}}{1-\varepsilon_{1}}\right)\left(1+\frac{\mu \varepsilon_{2}}{1-\varepsilon_{2}}\right) \quad \text { and } \quad \beta:=\frac{1-\alpha}{\alpha} \text {. } \tag{36}
\end{equation*}
$$

Proof. Let $x \in \boldsymbol{B}(w ; \delta / 2)$, let $r \in P_{A}^{\lambda} x$, let $s \in P_{B}^{\mu} r$, and let $\bar{x} \in A \cap B \cap \boldsymbol{B}(w ; \delta)$. Then Proposition 3.5 applied to $P_{A}^{\lambda}$ yields

$$
\begin{equation*}
\|r-\bar{x}\| \leq \gamma_{1}^{1 / 2}\|x-\bar{x}\|, \quad \text { where } \quad \gamma_{1}:=1+\frac{\lambda \varepsilon_{1}}{1-\varepsilon_{1}} \tag{37}
\end{equation*}
$$

and also $r \in \boldsymbol{B}(w ; \delta / \sqrt{2})$. Next, Proposition 3.5 applied to $P_{B}^{\mu}$ yields

$$
\begin{equation*}
\|s-\bar{x}\| \leq \gamma_{2}^{1 / 2}\|r-\bar{x}\| \leq\left(\gamma_{1} \gamma_{2}\right)^{1 / 2}\|x-\bar{x}\|, \quad \text { where } \quad \gamma_{2}:=1+\frac{\mu \varepsilon_{2}}{1-\varepsilon_{2}} \tag{38}
\end{equation*}
$$

This proves $\left(A \cap B \cap \boldsymbol{B}(w ; \delta), \gamma_{1} \gamma_{2}\right)$-quasi Fejér monotonicity of $P_{B}^{\mu} P_{A}^{\lambda}$ on $\boldsymbol{B}(w ; \delta / 2)$. Now apply Lemma 3.2 to the operators $P_{B}^{\mu} P_{A}^{\lambda}$ and $T_{\lambda, \mu}^{\alpha}=(1-\alpha) \operatorname{Id}+\alpha P_{B}^{\mu} P_{A}^{\lambda}$.
Proposition 3.8 (quasi coercivity of generalized DR operators). Let $w \in A \cap B, \lambda, \mu \in] 0,2]$, and $\alpha \in \mathbb{R}_{++}$. Suppose that $A$ is superregular at $w$ and that $\{A, B\}$ is strongly regular at $w$. Then

$$
\begin{equation*}
\bar{\theta}:=\sup \left\{\langle u, v\rangle \mid u \in N_{A}(w) \cap \boldsymbol{B}(0 ; 1), v \in\left(-N_{B}(w)\right) \cap \boldsymbol{B}(0 ; 1)\right\}<1 \tag{39}
\end{equation*}
$$

and for all $\theta \in] \bar{\theta}, 1\left[\right.$, there exist $\delta \in \mathbb{R}_{++}$and $\kappa \in \mathbb{R}_{++}$such that $T_{\lambda, \mu}^{\alpha}$ is $(A \cap B, \nu)$-quasi coercive on $\boldsymbol{B}(w ; \delta / 2)$ with

$$
\begin{equation*}
\nu:=\frac{\alpha \sqrt{1-\theta}}{\kappa} \min \left\{\lambda, \frac{\mu}{\sqrt{1+\mu^{2}}}\right\} . \tag{40}
\end{equation*}
$$

Proof. Since $\{A, B\}$ is strongly regular at $w$, we have from [33, Lemma 2.3] that $\bar{\theta}<1$. Now let $\theta \in] \bar{\theta}, 1[$ and let $\varepsilon \in[0,1 / 3]$. Using Definition 2.1, Fact 2.5, and [33, Lemma 4.1], we can find $\delta \in \mathbb{R}_{++}$and $\kappa \in \mathbb{R}_{++}$such that $A$ is $(\varepsilon, \delta)$-regular at $w$, that

$$
\begin{equation*}
\forall x \in \boldsymbol{B}(w ; \delta / 2): \quad d_{A \cap B}(x) \leq \kappa \max \left\{d_{A}(x), d_{B}(x)\right\}, \tag{41}
\end{equation*}
$$

and that

$$
\left.\begin{array}{l}
a \in A \cap \boldsymbol{B}(w ; \delta), b \in B \cap \boldsymbol{B}(w ; \sqrt{2} \delta),  \tag{42}\\
u \in N_{A}^{\text {prox }}(a), v \in N_{B}^{\text {prox }}(b)
\end{array}\right\} \quad \Rightarrow \quad\langle u, v\rangle \geq-\theta\|u\|\|v\| .
$$

Let $x \in \boldsymbol{B}(w ; \delta / 2)$ and $x_{+} \in T_{\lambda, \mu}^{\alpha} x$. By definition, there exist

$$
\begin{cases}a \in P_{A} x, & r=(1-\lambda) x+\lambda a \in P_{A}^{\lambda} x,  \tag{43a}\\ b \in P_{B} r, & s=(1-\mu) r+\mu b \in P_{B}^{\mu} r\end{cases}
$$

such that $x_{+}=(1-\alpha) x+\alpha s$. Then

$$
\begin{equation*}
x-x_{+}=\alpha(x-s), \quad x-r=\lambda(x-a), \quad \text { and } \quad r-s=\mu(r-b) . \tag{44}
\end{equation*}
$$

By Lemma 3.4(i), $a \in P_{A} x \subseteq A \cap \boldsymbol{B}(w ; \delta)$. Since $\varepsilon \in[0,1 / 3]$, Proposition 3.5 yields $r \in \boldsymbol{B}(w ; \sqrt{2} \delta / 2)$. Using again Lemma 3.4(i), we get $b \in P_{B} r \subseteq B \cap \boldsymbol{B}(w ; \sqrt{2} \delta)$. Now since $x-r=\lambda(x-a) \in N_{A}^{\text {prox }}(a)$ and $r-s=\mu(r-b) \in N_{B}^{\text {prox }}(b)$, we use (42) and the arithmetic mean-geometric mean inequality to obtain

$$
\begin{equation*}
2\langle x-r, r-s\rangle \geq-2 \theta\|x-r\|\|r-s\| \geq-\theta\left(\|x-r\|^{2}+\|r-s\|^{2}\right) . \tag{45}
\end{equation*}
$$

So

$$
\begin{align*}
\left\|x-x_{+}\right\|^{2} & =\alpha^{2}\|x-s\|^{2}=\alpha^{2}\left(\|x-r\|^{2}+\|r-s\|^{2}+2\langle x-r, r-s\rangle\right)  \tag{46a}\\
& \geq(1-\theta) \alpha^{2}\left(\|x-r\|^{2}+\|r-s\|^{2}\right) . \tag{46b}
\end{align*}
$$

Furthermore,

$$
\begin{equation*}
\|x-r\|^{2}=\lambda^{2}\|x-a\|^{2}=\lambda^{2} d_{A}^{2}(x), \tag{47}
\end{equation*}
$$

and by the coordinate version of Cauchy-Schwarz inequality,

$$
\begin{align*}
\left(\mu^{2}+1\right)\left(\|x-r\|^{2}+\|r-s\|^{2}\right) & \geq(\mu\|x-r\|+\|r-s\|)^{2}  \tag{48a}\\
& =(\mu\|x-r\|+\mu\|r-b\|)^{2}  \tag{48b}\\
& \geq(\mu\|x-b\|)^{2} \geq \mu^{2} d_{B}^{2}(x) . \tag{48c}
\end{align*}
$$

Combining (41), (46), (47), and (48), we obtain

$$
\begin{align*}
\left\|x-x_{+}\right\| & \geq \alpha \sqrt{1-\theta} \min \left\{\lambda, \frac{\mu}{\sqrt{1+\mu^{2}}}\right\} \max \left\{d_{A}(x), d_{B}(x)\right\}  \tag{49a}\\
& \geq \frac{\alpha \sqrt{1-\theta}}{\kappa} \min \left\{\lambda, \frac{\mu}{\sqrt{1+\mu^{2}}}\right\} d_{A \cap B}(x)=\nu d_{A \cap B}(x), \tag{49b}
\end{align*}
$$

which completes the proof.

## 4 Linear convergence of cyclic algorithms

We start with an elementary result.
Lemma 4.1. Let $C$ be a closed subset of $X$, let $w \in C$, and let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $X$. Suppose that one of the following assumptions holds:
(i) There exist $\delta \in \mathbb{R}_{++}, \rho \in\left[0,1\left[\right.\right.$, and $\sigma \in \mathbb{R}_{++}$such that

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad x_{n} \in \boldsymbol{B}(w ; \delta) \Rightarrow d_{C}\left(x_{n+1}\right) \leq \rho d_{C}\left(x_{n}\right) \text { and }\left\|x_{n+1}-x_{n}\right\| \leq \sigma d_{C}\left(x_{n}\right) . \tag{50}
\end{equation*}
$$

(ii) The sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ is generated by an operator $T: X \rightrightarrows X$ and there exist $\delta, \sigma \in \mathbb{R}_{++}$and $\rho \in[0,1[$ such that

$$
\begin{equation*}
\forall x \in \boldsymbol{B}(w ; \delta), \forall x_{+} \in T x: \quad d_{C}\left(x_{+}\right) \leq \rho d_{C}(x) \text { and }\left\|x_{+}-x\right\| \leq \sigma d_{C}(x) \tag{51}
\end{equation*}
$$

Then if either $\left(x_{n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}(w ; \delta)$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta(1-\rho)}{\sigma+1-\rho}\right)$, there exists $\bar{x} \in C \cap \boldsymbol{B}(w ; \delta)$ such that

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad\left\|x_{n}-\bar{x}\right\| \leq \frac{\sigma d_{C}\left(x_{0}\right)}{1-\rho} \rho^{n} \tag{52}
\end{equation*}
$$

i.e., the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to a point in $C$ with rate $\rho$.

Proof. It suffices to prove the result for (i) because if (ii) holds, then (i) also holds for $\left(x_{n}\right)_{n \in \mathbb{N}}$. Suppose (i) holds, we distinguish two cases.

Case 1: $\left(x_{n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}(w ; \delta)$. Combining with (50), we have

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad d_{C}\left(x_{n+1}\right) \leq \rho d_{C}\left(x_{n}\right) \text { and }\left\|x_{n+1}-x_{n}\right\| \leq \sigma d_{C}\left(x_{n}\right), \tag{53}
\end{equation*}
$$

For each $n \in \mathbb{N}$, take $x_{n}^{*} \in P_{C} x_{n}$. On the one hand,

$$
\begin{equation*}
\left\|x_{n}-x_{n}^{*}\right\|=d_{C}\left(x_{n}\right) \leq \rho^{n} d\left(x_{0}\right) \rightarrow 0 \text { as } n \rightarrow+\infty \tag{54}
\end{equation*}
$$

On the other hand, for all $n \in \mathbb{N}$ and $k \in \mathbb{N} \backslash\{0\}$,

$$
\begin{equation*}
\left\|x_{n}-x_{n+k}\right\| \leq \sum_{i=n}^{n+k-1}\left\|x_{i+1}-x_{i}\right\| \leq \sum_{i=n}^{n+k-1} \sigma \rho^{i} d\left(x_{0}\right) \leq \frac{\sigma d_{C}\left(x_{0}\right)}{1-\rho} \rho^{n} \rightarrow 0 \text { as } n \rightarrow+\infty \tag{55}
\end{equation*}
$$

So $\left(x_{n}\right)$ is a Cauchy sequence. Therefore, $\left(x_{n}\right)_{n \in \mathbb{N}}$ and $\left(x_{n}^{*}\right)_{n \in \mathbb{N}}$ both converge to the same limit $\bar{x} \in C \cap \boldsymbol{B}(w ; \delta)$. We then obtain (52) by leting $k \rightarrow+\infty$ in (55).

Case 2: $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta(1-\rho)}{\sigma+1-\rho}\right)$. We show that this is an instance of Case 1 by proving

$$
\begin{equation*}
\left(x_{n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}(w ; \delta) . \tag{56}
\end{equation*}
$$

Clearly, $\left\|x_{0}-w\right\| \leq \frac{\delta(1-\rho)}{\sigma+1-\rho} \leq \delta$. So (56) holds for $n=0$. Suppose (56) holds for $0,1, \ldots, n-1$, we shall prove that it also holds for $n$. Indeed, the induction hypothesis and (50) yield

$$
\begin{equation*}
\forall i \in\{0,1, \ldots, n-1\}: \quad d_{C}\left(x_{i+1}\right) \leq \rho d_{C}\left(x_{i}\right) \text { and }\left\|x_{i+1}-x_{i}\right\| \leq \sigma d_{C}\left(x_{i}\right) . \tag{57}
\end{equation*}
$$

Noting that $d_{C}\left(x_{0}\right) \leq\left\|x_{0}-w\right\|$, we obtain

$$
\begin{align*}
\left\|x_{n}-w\right\| & \leq \sum_{i=0}^{n-1}\left\|x_{i+1}-x_{i}\right\|+\left\|x_{0}-w\right\| \leq \sigma \sum_{i=0}^{n-1} d_{C}\left(x_{i}\right)+\left\|x_{0}-w\right\|  \tag{58a}\\
& \leq \sigma \sum_{i=0}^{n-1} \rho^{i} d_{C}\left(x_{0}\right)+\left\|x_{0}-w\right\| \leq\left(\sigma \frac{1}{1-\rho}+1\right)\left\|x_{0}-w\right\| \leq \delta . \tag{58b}
\end{align*}
$$

Thus, (56) holds for $n$. By mathematical induction principle, (56) holds for all $n \in \mathbb{N}$. The conclusion now follows from Case 1 .
Corollary 4.2. ([33, Proposition 2.11]) Let $T: X \rightrightarrows X$ be an operator, let $C$ be a closed subset of $X$, let $w \in C$, and let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a sequence generated by $T$. Suppose that there exist $\delta \in \mathbb{R}_{++}$and $\rho \in[0,1[$ such that

$$
\begin{equation*}
\forall x \in \boldsymbol{B}(w ; \delta), \forall x_{+} \in T x, \forall p \in P_{C} x: \quad\left\|x_{+}-p\right\| \leq \rho\|x-p\|=\rho d_{C}(x) \tag{59}
\end{equation*}
$$

Then whenever $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta(1-\rho)}{2}\right)$, there exists $\bar{x} \in C \cap \boldsymbol{B}(w ; \delta)$ such that

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad\left\|x_{n}-\bar{x}\right\| \leq \frac{(1+\rho)\left\|x_{0}-w\right\|}{1-\rho} \rho^{n}, \tag{60}
\end{equation*}
$$

i.e., the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to a point in $C$ with rate $\rho$.

Proof. Let $x \in \boldsymbol{B}(w ; \delta)$, let $x_{+} \in T x$ and let $p \in P_{C} x$. By assumption,

$$
\begin{equation*}
d_{C}\left(x_{+}\right) \leq\left\|x_{+}-p\right\| \leq \rho\|x-p\|=\rho d_{C}(x), \tag{61}
\end{equation*}
$$

and also

$$
\begin{equation*}
\left\|x_{+}-x\right\| \leq\left\|x_{+}-p\right\|+\|x-p\| \leq(1+\rho)\|x-p\|=(1+\rho) d_{C}(x) . \tag{62}
\end{equation*}
$$

Now apply Lemma 4.1(ii) with $\sigma=1+\rho$ and note that $d_{C}\left(x_{0}\right) \leq\left\|x_{0}-w\right\|$.
The following result proves that if distance to the feasible set is reduced at least by a factor $\rho \in[0,1[$ after every fixed number of steps, then $R$-linear convergence is achieved.
Lemma 4.3 (linear reduction after $k$ steps). Let $C$ be a closed subset of $X$, let $w \in C$, and let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $X$. Let also $k \in \mathbb{N} \backslash\{0\}$, $\delta_{0} \in \mathbb{R}_{++}, \rho \in\left[0,1\left[, \Gamma \in \mathbb{R}_{+}\right.\right.$, and suppose that for every tuple $\left(z_{0}, z_{1}, \ldots, z_{k}\right):=\left(x_{k n}, x_{k n+1}, \ldots, x_{k n+k}\right)$ with $z_{0} \in \boldsymbol{B}\left(w ; \delta_{0}\right)$, we have

$$
\begin{align*}
& d_{C}\left(z_{k}\right) \leq \rho d_{C}\left(z_{0}\right) \quad \text { and }  \tag{63a}\\
& \forall i \in\{1, \ldots, k\}, \forall p \in C \cap \boldsymbol{B}\left(w ; 2 \delta_{0}\right): \quad\left\|z_{i}-p\right\| \leq \Gamma\left\|z_{0}-p\right\| . \tag{63b}
\end{align*}
$$

Then if either $\left(x_{k n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \delta_{0}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$, the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to a point in $C$ with rate $\rho^{1 / k}$. More specifically, there exists $\bar{x} \in C \cap \boldsymbol{B}\left(w ; \delta_{0}\right)$ such that

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad\left\|x_{n}-\bar{x}\right\| \leq \frac{\Gamma(1+\Gamma) d_{C}\left(x_{0}\right)}{1-\rho} \rho^{\left\lfloor\frac{n}{k}\right\rfloor}, \tag{64}
\end{equation*}
$$

where $\left\lfloor\frac{n}{k}\right\rfloor$ is the largest integer not exceeding $\frac{n}{k}$.
Proof. Consider the sequence $\left(y_{n}:=x_{k n}\right)_{n \in \mathbb{N}}$. Suppose $y_{n} \in \boldsymbol{B}\left(w ; \delta_{0}\right)$ and take $p \in P_{C} x_{k n}=$ $P_{C} y_{n} \subset P_{C}\left(\boldsymbol{B}\left(w ; \delta_{0}\right)\right) \subseteq C \cap \boldsymbol{B}\left(w ; 2 \delta_{0}\right)$ (see Lemma 3.4(i)). Then (63a) means $d_{C}\left(y_{n+1}\right) \leq \rho d_{C}\left(y_{n}\right)$ and (63b) yields

$$
\begin{equation*}
\left\|y_{n+1}-y_{n}\right\| \leq\left\|y_{n+1}-p\right\|+\left\|y_{n}-p\right\| \leq \Gamma d_{C}\left(y_{n}\right)+d_{C}\left(y_{n}\right)=(1+\Gamma) d_{C}\left(y_{n}\right) . \tag{65}
\end{equation*}
$$

So, by Lemma 4.1, if $\left(y_{n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \delta_{0}\right)$ or $y_{0} \in \boldsymbol{B}\left(w ; \frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$, the sequence $\left(y_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to some $\bar{x} \in C \cap \boldsymbol{B}\left(w ; \delta_{0}\right)$ and

$$
\begin{equation*}
\left\|y_{n}-\bar{x}\right\| \leq \frac{(1+\Gamma) d_{C}\left(y_{0}\right)}{1-\rho} \rho^{n}=\frac{(1+\Gamma) d_{C}\left(x_{0}\right)}{1-\rho} \rho^{n} . \tag{66}
\end{equation*}
$$

Now (63b) implies that for every $i \in\{1, \ldots, k\}$,

$$
\begin{equation*}
\left\|x_{k n+i}-\bar{x}\right\| \leq \Gamma\left\|x_{k n}-\bar{x}\right\|=\Gamma\left\|y_{n}-\bar{x}\right\| \leq \frac{\Gamma(1+\Gamma) d_{C}\left(x_{0}\right)}{1-\rho} \rho^{n} . \tag{67}
\end{equation*}
$$

Replacing $k n+i$ by $n$, we obtain

$$
\begin{equation*}
\left\|x_{n}-\bar{x}\right\| \leq \frac{\Gamma(1+\Gamma) d_{C}\left(x_{0}\right)}{1-\rho} \rho^{\left\lfloor\frac{n}{k}\right\rfloor} . \tag{68}
\end{equation*}
$$

Now if $\rho=0$, then $x_{n}=\bar{x}$ for all $n \geq 1$; and if $\rho>0$, then $\rho^{\left\lfloor\frac{n}{k}\right\rfloor} \leq \frac{1}{\rho} \cdot \rho^{\frac{n}{k}}$. The lemma is proved.
We next analyze the performance of $m$ steps of cyclic algorithms for quasi firmly Fejér monotone operators.

Lemma 4.4 (consecutive steps of cyclic algorithms). Let $w \in C:=\bigcap_{i \in I} C_{i}, \delta \in \mathbb{R}_{++}$, and $\nu \in] 0,1]$. For every $i \in I$, let $\gamma_{i} \in\left[1,+\infty\left[\right.\right.$ and $\beta_{i} \in \mathbb{R}_{++}$. Set $\Omega:=C \cap \boldsymbol{B}(w ; \delta), \Gamma:=\left(\gamma_{1} \cdots \gamma_{m}\right)^{1 / 2}$, and $\delta_{0}:=\frac{\delta}{2 \Gamma} \gamma_{m}^{1 / 2}$. Let $x_{0}, x_{1}, \ldots, x_{m}$ be $m+1$ consecutive points of the cyclic algorithm with respect to $\left(T_{i}\right)_{i \in I}$ such that

$$
\begin{equation*}
x_{0} \in \boldsymbol{B}\left(w ; \delta_{0}\right) \quad \text { and } \quad \forall i \in I: x_{i} \in T_{i} x_{i-1} . \tag{69}
\end{equation*}
$$

Then the following hold:
(i) If for every $i \in I, T_{i}$ is $\left(\Omega, \gamma_{i}\right)$-quasi Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$, then

$$
\begin{align*}
\forall i \in I \backslash\{m\}: & \left\|x_{i}-w\right\| \leq\left(\gamma_{1} \cdots \gamma_{i}\right)^{1 / 2} \delta_{0} \leq \frac{\delta}{2}  \tag{70a}\\
\forall i \in I, \quad \forall p \in \Omega: & \left\|x_{i}-p\right\| \leq \gamma_{i}^{1 / 2}\left\|x_{i-1}-p\right\| \leq\left(\gamma_{1} \cdots \gamma_{i}\right)^{1 / 2}\left\|x_{0}-p\right\| \leq \Gamma\left\|x_{0}-p\right\| \tag{70b}
\end{align*}
$$

(ii) If for every $i \in I, T_{i}$ is both $\left(\Omega, \gamma_{i}, \beta_{i}\right)$-quasi firmly Fejér monotone and $\left(C_{i}, \nu\right)$-quasi coercive on $\boldsymbol{B}(w ; \delta / 2)$, then

$$
\begin{equation*}
\forall p \in \Omega: \quad\left\|x_{m}-p\right\|^{2} \leq\left(\gamma_{1} \cdots \gamma_{m}\right)\left\|x_{0}-p\right\|^{2}-\beta \nu^{2} \max _{i \in I} d_{C_{i}}^{2}\left(x_{0}\right), \quad \text { where } \beta:=\left(\sum_{i \in I} \frac{1}{\beta_{i}}\right)^{-1} \tag{71}
\end{equation*}
$$

Proof. Let $p \in \Omega=C \cap \boldsymbol{B}(w ; \delta)$.
(i): First, we have $\left\|x_{0}-w\right\| \leq \delta_{0}=\frac{\delta}{2\left(\gamma_{1} \gamma_{2} \cdots \gamma_{m-1}\right)^{1 / 2}} \leq \frac{\delta}{2}$ since $\gamma_{i} \geq 1$ for every $i \in I$. The $\left(\Omega, \gamma_{1}\right)$-quasi Fejér monotonicity of $T_{1}$ on $\boldsymbol{B}(w ; \delta / 2)$ and Lemma 3.4(ii) then implies that

$$
\begin{equation*}
\left\|x_{1}-p\right\| \leq \gamma_{1}^{1 / 2}\left\|x_{0}-p\right\| \quad \text { and } \quad\left\|x_{1}-w\right\| \leq \gamma_{1}^{1 / 2} \delta_{0}=\frac{\delta}{2\left(\gamma_{2} \cdots \gamma_{m-1}\right)^{1 / 2}} \leq \frac{\delta}{2} \tag{72}
\end{equation*}
$$

Repeating the argument for $x_{1}, \ldots, x_{m-1}$, we get (70a) and the first part of (70b), from which the rest follows.
(ii): Since quasi firm Fejér monotonicity implies quasi Fejér monotonicity, (70a) holds due to (i), that is, $x_{0}, x_{1}, \ldots, x_{m-1} \in \boldsymbol{B}(w ; \delta / 2)$. Now since each $T_{i}$ is $\left(\Omega, \gamma_{i}, \beta_{i}\right)$-quasi firmly Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$, we derive that

$$
\begin{gather*}
\left\|x_{1}-p\right\|^{2}+\beta\left\|x_{0}-x_{1}\right\|^{2} \leq \gamma_{1}\left\|x_{0}-p\right\|^{2}  \tag{73a}\\
\left\|x_{2}-p\right\|^{2}+\beta\left\|x_{1}-x_{2}\right\|^{2} \leq \gamma_{2}\left\|x_{1}-p\right\|^{2}  \tag{73b}\\
\vdots  \tag{73c}\\
\left\|x_{m}-p\right\|^{2}+\beta\left\|x_{m-1}-x_{m}\right\|^{2} \leq \gamma_{m}\left\|x_{m-1}-p\right\|^{2}
\end{gather*}
$$

and so

$$
\begin{gather*}
\gamma_{2} \cdots \gamma_{m}\left\|x_{1}-p\right\|^{2}+\gamma_{2} \cdots \gamma_{m} \beta\left\|x_{0}-x_{1}\right\|^{2} \leq \gamma_{1} \gamma_{2} \cdots \gamma_{m}\left\|x_{0}-p\right\|^{2}  \tag{74a}\\
\gamma_{3} \cdots \gamma_{m}\left\|x_{2}-p\right\|^{2}+\gamma_{3} \cdots \gamma_{m} \beta\left\|x_{1}-x_{2}\right\|^{2} \leq \gamma_{2} \gamma_{3} \cdots \gamma_{m}\left\|x_{1}-p\right\|^{2},  \tag{74b}\\
\vdots  \tag{74d}\\
\gamma_{m}\left\|x_{m-1}-p\right\|^{2}+\gamma_{m} \beta\left\|x_{m-2}-x_{m-1}\right\|^{2} \leq \gamma_{m-1} \gamma_{m}\left\|x_{m-2}-p\right\|^{2}, \\
\left\|x_{m}-p\right\|^{2}+\beta\left\|x_{m-1}-x_{m}\right\|^{2} \leq \gamma_{m}\left\|x_{m-1}-p\right\|^{2} .
\end{gather*}
$$

Using the telescoping technique and the fact that $\gamma_{i} \geq 1$, we get

$$
\begin{equation*}
\left(\gamma_{1} \cdots \gamma_{m}\right)\left\|x_{0}-p\right\|^{2} \geq\left\|x_{m}-p\right\|^{2}+\sum_{j \in I} \beta_{j}\left\|x_{j-1}-x_{j}\right\|^{2} . \tag{75}
\end{equation*}
$$

Now the coordinate version of Cauchy-Schwarz inequality yields

$$
\begin{equation*}
\left(\gamma_{1} \cdots \gamma_{m}\right)\left\|x_{0}-p\right\|^{2} \geq\left\|x_{m}-p\right\|^{2}+\beta\left(\sum_{j \in I}\left\|x_{j-1}-x_{j}\right\|\right)^{2} . \tag{76}
\end{equation*}
$$

For every $i \in I, T_{i}$ is $\left(C_{i}, \nu_{i}\right)$-quasi coercive on $\boldsymbol{B}(w ; \delta / 2)$, so $\left\|x_{i-1}-x_{i}\right\| \geq \nu d_{C_{i}}\left(x_{i-1}\right)$. Hence,

$$
\begin{align*}
\forall i \in I: \quad \sum_{j \in I}\left\|x_{j-1}-x_{j}\right\| & \geq\left\|x_{0}-x_{i-1}\right\|+\left\|x_{i-1}-x_{i}\right\|  \tag{77a}\\
& \geq\left\|x_{0}-x_{i-1}\right\|+\nu d_{C_{i}}\left(x_{i-1}\right)  \tag{77b}\\
& \left.\geq \nu\left(\left\|x_{0}-x_{i-1}\right\|+d_{C_{i}}\left(x_{i-1}\right)\right) \quad \text { (because } 1 \geq \nu \geq 0\right)  \tag{77c}\\
& \geq \nu d_{C_{i}}\left(x_{0}\right), \tag{77d}
\end{align*}
$$

which yields

$$
\begin{equation*}
\sum_{j \in I}\left\|x_{j-1}-x_{j}\right\| \geq \nu \max _{i \in I} d_{C_{i}}\left(x_{0}\right) . \tag{78}
\end{equation*}
$$

Combining with (76), we obtain (71).
The following theorems are cornerstones in our convergence analysis. In the sequel, we denote $[\rho]_{+}:=\max \{0, \rho\}$ for $\rho \in \mathbb{R}$.
Theorem 4.5 (cyclic sequence of quasi firmly Fejér monotone operators). Let $w \in C:=$ $\bigcap_{i \in I} C_{i}, \delta \in \mathbb{R}_{++}$, and $\left.\left.\nu \in\right] 0,1\right]$. For every $i \in I$, let $\gamma_{i} \in\left[1,+\infty\left[\right.\right.$ and $\beta_{i} \in \mathbb{R}_{++}$. Set $\Omega:=$ $C \cap \boldsymbol{B}(w ; \delta)$ and let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a cyclic sequence generated by $\left(T_{i}\right)_{i \in I}$. Suppose that
(a) $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$ for some $\kappa \in \mathbb{R}_{++}$.
(b) For every $i \in I, T_{i}$ is $\left(\Omega, \gamma_{i}, \beta_{i}\right)$-quasi firmly Fejér monotone and $\left(C_{i}, \nu\right)$-quasi coercive on $\boldsymbol{B}(w ; \delta / 2)$.
Set $\Gamma:=\left(\gamma_{1} \cdots \gamma_{m}\right)^{1 / 2}$ and $\delta_{0}:=\frac{\delta}{2 \Gamma} \gamma_{m}^{1 / 2}$. Then

$$
\begin{equation*}
\forall x_{0} \in \boldsymbol{B}\left(w ; \delta_{0}\right): \quad d_{C}\left(x_{m}\right) \leq \rho d_{C}\left(x_{0}\right), \quad \text { where } \rho:=\left[\Gamma^{2}-\frac{\nu^{2}}{\kappa^{2}}\left(\sum_{i \in I} \frac{1}{\beta_{i}}\right)^{-1}\right]_{+}^{1 / 2} . \tag{79}
\end{equation*}
$$

Consequently, if $\rho<1$ and either $\left(x_{m n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \delta_{0}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$, then $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to some point $\bar{x} \in C$ with rate $\rho^{1 / m}$.

Proof. Let $x_{0} \in \boldsymbol{B}\left(w ; \delta_{0}\right) \subseteq \boldsymbol{B}(w ; \delta / 2)$. Since $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$,

$$
\begin{equation*}
\max _{i \in I} d_{C_{i}}\left(x_{0}\right) \geq \frac{1}{\kappa} d_{C}\left(x_{0}\right) . \tag{80}
\end{equation*}
$$

Setting $\beta:=\left(\sum_{i \in I} \frac{1}{\beta_{i}}\right)^{-1}$, Lemma 4.4(ii) then implies that

$$
\begin{align*}
\forall p \in \Omega: \quad\left\|x_{m}-p\right\|^{2} & \leq\left(\gamma_{1} \gamma_{2} \cdots \gamma_{m}\right)\left\|x_{0}-p\right\|^{2}-\beta \nu^{2} \max _{i \in I} d_{C_{i}}^{2}\left(x_{0}\right)  \tag{81a}\\
& \leq \Gamma^{2}\left\|x_{0}-p\right\|^{2}-\frac{\beta \nu^{2}}{\kappa^{2}} d_{C}^{2}\left(x_{0}\right) . \tag{81b}
\end{align*}
$$

Letting $p \in P_{C} x_{0}$ and noting from Lemma 3.4(i) that $P_{C} x_{0} \subseteq P_{C}(\boldsymbol{B}(w ; \delta / 2)) \subseteq \Omega$, we have

$$
\begin{equation*}
\left\|x_{m}-p\right\|^{2} \leq \Gamma^{2} d_{C}^{2}\left(x_{0}\right)-\frac{\beta \nu^{2}}{\kappa^{2}} d_{C}^{2}\left(x_{0}\right) \leq\left[\Gamma^{2}-\frac{\beta \nu^{2}}{\kappa^{2}}\right]_{+} d_{C}^{2}\left(x_{0}\right), \tag{82}
\end{equation*}
$$

which leads to (79).
Now assume $\rho<1$. Since $T_{i}$ is also ( $\Omega, \gamma_{i}$ )-quasi Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$ and $C \cap \boldsymbol{B}\left(w ; 2 \delta_{0}\right) \subseteq \Omega$, we obtain from (70b) in Lemma 4.4(i) that

$$
\begin{equation*}
\forall i \in I, \forall p \in C \cap \boldsymbol{B}\left(w ; 2 \delta_{0}\right): \quad\left\|x_{i}-p\right\| \leq \Gamma\left\|x_{0}-p\right\| . \tag{83}
\end{equation*}
$$

By combining with (79), for every tuple $\left(z_{0}, z_{1}, \ldots, z_{m}\right):=\left(x_{m n}, x_{m n+1}, \ldots, x_{m n+m}\right)$ with $z_{0} \in$ $\boldsymbol{B}\left(w ; \delta_{0}\right)$, one has

$$
\begin{align*}
& d_{C}\left(z_{m}\right) \leq \rho d_{C}\left(z_{0}\right) \quad \text { and }  \tag{84a}\\
& \forall i \in I, \forall p \in C \cap \boldsymbol{B}\left(w ; 2 \delta_{0}\right): \quad\left\|z_{i}-p\right\| \leq \Gamma\left\|z_{0}-p\right\|, \tag{84b}
\end{align*}
$$

which fulfills (63) with $k=m$. The result then follows from Lemma 4.3.
Remark 4.6. Regarding (82) in the proof of Theorem 4.5, we see that the term $\Gamma^{2}-\frac{\beta \nu^{2}}{\kappa^{2}}$ is necessarily nonnegative if $x_{0} \notin C$; however, no general conclusion about this term can be drawn otherwise. We therefore use the notation [.] $]_{+}$to ensure nonnegativity.

Now we prove linear convergence result for cyclic sequences when there is one quasi Fejér monotone operator. Clearly, we need at least two operators, i.e., $m=|I| \geq 2$. Here and in what follows, $|I|$ denotes the number of elements in the set $I$.
Theorem 4.7 (cyclic sequence with one quasi Fejér monotone operator). Let $w \in C:=$ $\left.\left.\bigcap_{i \in I} C_{i}, \delta \in \mathbb{R}_{++}, \nu \in\right] 0,1\right]$, and $\gamma_{i} \in\left[1,+\infty\left[\right.\right.$ for every $i \in I$. Set $\Omega:=C \cap \boldsymbol{B}(w ; \delta)$ and let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a cyclic sequence generated by $\left(T_{i}\right)_{i \in I}$. Suppose that
(a) $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$ for some $\kappa \in \mathbb{R}_{++}$.
(b) There is $j \in I$ such that for every $i \in I \backslash\{j\}$, $T_{i}$ is $\left(\Omega, \gamma_{i}, \beta_{i}\right)$-quasi firmly Fejér monotone and $\left(C_{i}, \nu\right)$-quasi coercive on $\boldsymbol{B}(w ; \delta / 2)$ for some $\beta_{i} \in \mathbb{R}_{++}$; while $T_{j}$ is $\left(\Omega, \gamma_{j}\right)$-quasi Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$ and $T_{j} x \subseteq C_{j}$ for all $x \in \boldsymbol{B}(w ; \delta / 2)$.
Set $\Gamma:=\left(\gamma_{1} \cdots \gamma_{m}\right)^{1 / 2}$ and $\delta_{0}:=\frac{\delta}{2 \Gamma} \gamma_{j}^{1 / 2}$. Then

$$
\begin{equation*}
\forall x_{0} \in \boldsymbol{B}\left(w ; \delta_{0}\right): \quad d_{C}\left(x_{m}\right) \leq \rho d_{C}\left(x_{0}\right), \quad \text { where } \rho:=\left[\Gamma^{2}-\frac{\gamma_{j} \nu^{2}}{\kappa^{2}}\left(\sum_{i \in I \backslash\{j\}} \frac{1}{\beta_{i}}\right)^{-1}\right]_{+}^{1 / 2} \tag{85}
\end{equation*}
$$

Consequently, if $\rho<1$ and either $\left(x_{m n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \delta_{0}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$, then $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to some point $\bar{x} \in C$ with rate $\rho^{1 / m}$.

Proof. It suffices to consider only the case $j=1$ because other cases are identical up to relabeling. Set $\delta_{1}:=\frac{\delta}{2\left(\gamma_{2} \cdots \gamma_{m-1}\right)^{1 / 2}} \leq \frac{\delta}{2}$ and $\beta:=\left(\sum_{i \in I \backslash\{1\}} \frac{1}{\beta_{i}}\right)^{-1}$. We first claim that

$$
\begin{equation*}
\forall x_{1} \in C_{1} \cap \boldsymbol{B}\left(w ; \delta_{1}\right): \quad d_{C}\left(x_{m}\right) \leq\left[\gamma_{2} \cdots \gamma_{m}-\frac{\beta \nu^{2}}{\kappa^{2}}\right]_{+}^{1 / 2} d_{C}\left(x_{1}\right) \tag{86}
\end{equation*}
$$

On the one hand, applying Lemma $4.4(\mathrm{ii})$ to the system $\left(C_{i}\right)_{i \in I \backslash\{1\}}$ and $m$ consecutive points $x_{1}, \ldots, x_{m}$ with $x_{1} \in \boldsymbol{B}\left(w ; \delta_{1}\right)$, we deduce that

$$
\begin{equation*}
\forall p \in \Omega: \quad\left\|x_{m}-p\right\|^{2} \leq\left(\gamma_{2} \cdots \gamma_{m}\right)\left\|x_{1}-p\right\|^{2}-\beta \nu^{2} \max _{i \in I \backslash\{1\}} d_{C_{i}}^{2}\left(x_{1}\right) \tag{87}
\end{equation*}
$$

On the other hand, since $d_{C_{1}}\left(x_{1}\right)=0$, the linear regularity of $\left\{C_{i}\right\}_{i \in I}$ yields

$$
\begin{equation*}
\max _{i \in I \backslash\{1\}} d_{C_{i}}\left(x_{1}\right)=\max _{i \in I} d_{C_{i}}\left(x_{1}\right) \geq \frac{1}{\kappa} d_{C}\left(x_{1}\right) . \tag{88}
\end{equation*}
$$

From (87) and (88), letting $p \in P_{C} x_{1} \subseteq C \cap \boldsymbol{B}(w ; \delta)=\Omega$ (see Lemma 3.4(i)), we obtain

$$
\begin{align*}
d_{C}^{2}\left(x_{m}\right) \leq\left\|x_{m}-p\right\|^{2} & \leq\left(\gamma_{2} \cdots \gamma_{m}\right)\left\|x_{1}-p\right\|^{2}-\beta \nu^{2} \max _{i \in I} d_{C_{i}}^{2}\left(x_{1}\right)  \tag{89a}\\
& \leq\left(\gamma_{2} \cdots \gamma_{m}\right) d_{C}^{2}\left(x_{1}\right)-\frac{\beta \nu^{2}}{\kappa^{2}} d_{C}^{2}\left(x_{1}\right)  \tag{89b}\\
& =\left[\gamma_{2} \cdots \gamma_{m}-\frac{\beta \nu^{2}}{\kappa^{2}}\right]_{+} d_{C}^{2}\left(x_{1}\right), \tag{89c}
\end{align*}
$$

which implies (86).
Now let $x_{0} \in \boldsymbol{B}\left(w ; \delta_{0}\right) \subseteq \boldsymbol{B}(w ; \delta / 2)$. Then $x_{1} \in T_{1} x_{0} \subseteq C_{1}$. By applying Lemma 3.4(ii) to $T_{1}$, we derive that $x_{1} \in \boldsymbol{B}\left(w ; \gamma_{1}^{1 / 2} \delta_{0}\right)=\boldsymbol{B}\left(w ; \delta_{1}\right)$ and $d_{C}\left(x_{1}\right) \leq \gamma_{1}^{1 / 2} d_{C}\left(x_{0}\right)$. Combining these with (86), we get (85). The rest of the proof is exactly the same as the second part of Theorem 4.5.

In the next result, we show that if the coercivity assumption is replaced by the assumption that the image of each operator $T_{i}$ lies in the corresponding set $C_{i}$, then linear reduction is obtained after $m-1$ steps (instead of $m$ steps). Thus, the rate of convergence is improved. This particular condition is satisfied for certain operators such as projectors and semi-intrepid projectors (see Section 5.2). Theorem 4.8 (refined linear convergence). Let $w \in C:=\bigcap_{i \in I} C_{i}$ and $\delta \in \mathbb{R}_{++}$. For every $i \in I$, let $\gamma_{i} \in\left[1,+\infty\left[\right.\right.$ and $\beta_{i} \in \mathbb{R}_{++}$. Set $\Omega:=C \cap \boldsymbol{B}(w ; \delta)$ and let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a cyclic sequence generated by $\left(T_{i}\right)_{i \in I}$. Suppose that
(a) $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$ for some $\kappa \in \mathbb{R}_{++}$.
(b) For every $i \in I, T_{i}$ is $\left(\Omega, \gamma_{i}, \beta_{i}\right)$-quasi firmly Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$.
(c) For every $i \in I$ and every $x \in \boldsymbol{B}(w ; \delta / 2), T_{i} x \subseteq C_{i}$.

Set $\Gamma:=\left(\frac{\gamma_{1} \cdots \gamma_{m}}{\min _{i \in I} \gamma_{i}}\right)^{1 / 2}, \delta_{0}:=\frac{\delta}{2 \Gamma}$, and $\rho:=\left[\Gamma^{2}-\frac{1}{\kappa^{2}}\left(\left(\sum_{i \in I} \frac{1}{\beta_{i}}\right)-\frac{1}{\max _{i \in I} \beta_{i}}\right)^{-1}\right]_{+}^{1 / 2}$. Then

$$
\begin{equation*}
\forall i \in I, \quad \forall x_{i} \in C_{i} \cap \boldsymbol{B}\left(w ; \delta_{0}\right): \quad d_{C}\left(x_{i+m-1}\right) \leq \rho d_{C}\left(x_{i}\right) . \tag{90}
\end{equation*}
$$

Consequently, if $\rho<1$ and either $\left(x_{(m-1) n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \gamma_{\max }^{-1 / 2} \delta_{0}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \gamma_{\max }^{-1 / 2} \cdot \frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$ where $\gamma_{\max }:=\max _{i \in I} \gamma_{i}$, then $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to some point $\bar{x} \in C$ with rate $\rho^{\frac{1}{m-1}}$.

Proof. In addition to convention (2), we also use $\gamma_{m n+i}:=\gamma_{i}$ for $n \in \mathbb{N}$ and $i \in I$. For every $i \in I$, it follows from (c) that

$$
\begin{equation*}
\forall x \in \boldsymbol{B}(w ; \delta / 2), \forall x_{+} \in T_{i} x: \quad\left\|x_{+}-x\right\| \geq d_{C_{i}}(x), \tag{91}
\end{equation*}
$$

so $T_{i}$ is ( $C_{i}, 1$ )-quasi coercive on $\boldsymbol{B}(w ; \delta / 2)$. Hence, all assumptions in Theorem 4.7 are fulfilled. Now let $i \in I$ and take $m$ consecutive points $\left(x_{i}, \ldots, x_{i+m-1}\right)$ of $\left(x_{n}\right)_{n \in \mathbb{N}}$ with $x_{i} \in C_{i} \cap \boldsymbol{B}\left(w ; \delta_{0}\right)$. Then

$$
\begin{equation*}
\left\|x_{i}-w\right\| \leq \delta_{0} \leq \frac{\delta}{2\left(\gamma_{i+1} \gamma_{i+2} \cdots \gamma_{i+m-1}\right)^{1 / 2}} \leq \frac{\delta}{2\left(\gamma_{i+1} \cdots \gamma_{i+m-2}\right)^{1 / 2}}=: \delta_{1} . \tag{92}
\end{equation*}
$$

First, noting that $\bar{\beta}_{i}:=\left(\sum_{j \in I \backslash\{i\}} \frac{1}{\beta_{j}}\right)^{-1} \geq \beta:=\left(\left(\sum_{i \in I} \frac{1}{\beta_{i}}\right)-\frac{1}{\max _{i \in I} \beta_{i}}\right)^{-1}$ and applying claim (86) in the proof of Theorem 4.7, we have

$$
\begin{equation*}
d_{C}\left(x_{i+m-1}\right) \leq\left[\frac{\gamma_{1} \cdots \gamma_{m}}{\gamma_{i}}-\frac{\bar{\beta}_{i}}{\kappa^{2}}\right]_{+}^{1 / 2} d_{C}\left(x_{i}\right) \leq\left[\Gamma^{2}-\frac{\beta}{\kappa^{2}}\right]_{+}^{1 / 2} d_{C}\left(x_{i}\right), \tag{93}
\end{equation*}
$$

which proves (90). Second, since $x_{i} \in \boldsymbol{B}\left(w ; \delta_{0}\right)$, we derive from the quasi Fejér monotonicity of $T_{i}$ 's and (70a) in Lemma 4.4(i) that $x_{i+m-2} \in \boldsymbol{B}(w ; \delta / 2)$, which together with (c) yields

$$
\begin{equation*}
x_{i+m-1} \in C_{i+m-1} . \tag{94}
\end{equation*}
$$

Third, it follows from (70b) in Lemma 4.4(i) that

$$
\begin{align*}
& \forall j \in\{1, \ldots, m-1\}, \forall p \in C \cap \boldsymbol{B}\left(w ; 2 \delta_{0}\right): \\
& \quad\left\|x_{i+j}-p\right\| \leq\left(\gamma_{i+1} \cdots \gamma_{i+m-1}\right)^{1 / 2}\left\|x_{i}-p\right\| \leq \Gamma\left\|x_{i}-p\right\| . \tag{95}
\end{align*}
$$

Taking $p \in P_{C} x_{i} \subseteq C \cap \boldsymbol{B}\left(w ; 2 \delta_{0}\right)$ (due to Lemma 3.4(i)), we obtain

$$
\begin{equation*}
\left\|x_{i+m-1}-x_{i}\right\| \leq\left\|x_{i}-p\right\|+\left\|x_{i+m-1}-p\right\| \leq(1+\Gamma)\left\|x_{i}-p\right\|=(1+\Gamma) d_{C}\left(x_{i}\right) . \tag{96}
\end{equation*}
$$

So by (93), (94), and (96), we have proved that

$$
\begin{align*}
& \forall i \in I, \forall x_{i} \in C_{i} \cap \boldsymbol{B}\left(w ; \delta_{0}\right): \\
& \quad x_{i+m-1} \in C_{i+m-1}, d_{C}\left(x_{i+m-1}\right) \leq \rho d_{C}\left(x_{i}\right) \quad \text { and } \quad\left\|x_{i+m-1}-x_{i}\right\| \leq(1+\Gamma) d_{C}\left(x_{i}\right) . \tag{97}
\end{align*}
$$

Now assume that $\rho<1$ and that either $\left(x_{(m-1) n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \gamma_{\max }^{-1 / 2} \delta_{0}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \gamma_{\max }^{-1 / 2}\right.$. $\left.\frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$. We claim that

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad x_{(m-1) n+1} \in C_{(m-1) n+1} \cap \boldsymbol{B}\left(w ; \delta_{0}\right) \tag{98}
\end{equation*}
$$

Indeed, if $\left(x_{(m-1) n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \gamma_{\max }^{-1 / 2} \delta_{0}\right) \subseteq \boldsymbol{B}(w ; \delta / 2)$, then (98) holds due to (c) and Lemma 3.4(ii). If $x_{0} \in \boldsymbol{B}\left(w ; \gamma_{\max }^{-1 / 2} \cdot \frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$, then by using again (c) and Lemma 3.4(ii), $x_{1} \in C_{1} \cap \boldsymbol{B}\left(w ; \frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$, and employing (97) and proceeding as in the proof of Lemma 4.1, we get (98).

Finally, we deduce from (95), (97), and (98) that for every tuple $\left(z_{0}, z_{1}, \ldots, z_{m-1}\right)$ := $\left(x_{(m-1) n+1}, x_{(m-1) n+2}, \ldots, x_{(m-1)(n+1)+1}\right)$,

$$
\begin{align*}
& d_{C}\left(z_{m-1}\right) \leq \rho d_{C}\left(z_{0}\right) \quad \text { and }  \tag{99a}\\
& \forall i \in I \backslash\{m\}, \forall p \in C \cap \boldsymbol{B}\left(w ; 2 \delta_{0}\right): \quad\left\|z_{i}-p\right\| \leq \Gamma\left\|z_{0}-p\right\|, \tag{99b}
\end{align*}
$$

which fulfills (63) with $k=m-1$. The proof is finished by applying Lemma 4.3.

## 5 Applications to projection algorithms

### 5.1 Cyclic relaxed projections

In this section, by specializing operators $T_{i}$ to relaxed projectors $P_{C_{i}}^{\lambda_{i}}$, we obtain linear convergence results for the cyclic relaxed projections, one of which is possibly a reflection across an injectable set. First, we give the definition for injectability.

Definition 5.1 (injectable set). Let $C$ be a nonempty closed subset of $X$ and let $\tau \in \mathbb{R}_{+}$. The set $C$ is said to be $\tau$-injectable on a subset $U$ of $X$ if

$$
\begin{equation*}
\forall x \in U, \forall p \in P_{C} x: \quad\left[p, p+\tau \frac{p-x}{\|p-x\|}\right] \subseteq C \tag{100}
\end{equation*}
$$

with the convention that $\frac{p-x}{\|p-x\|}=0$ if $p=x$. We say that $C$ is strictly injectable around $w \in X$ if there exist $\tau \in \mathbb{R}_{++}$and $\delta \in \mathbb{R}_{++}$such that $C$ is $\tau$-injectable on $\boldsymbol{B}(w ; \delta)$. When $C$ is $\tau$-injectable on $U=X$, we simply say that $C$ is $\tau$-injectable. When $C$ is $\tau$-injectable for all $\tau \in \mathbb{R}_{+}$, we say that $C$ is $\infty$-injectable.

Clearly, if $\tau>\tau^{\prime} \geq 0$, then $\tau$-injectability implies $\tau^{\prime}$-injectability. To give an example of injectable sets, we recall from [20, Section 3.2] that a closed convex cone $K$ of $X$ is obtuse if $-K^{\ominus} \subseteq K$, where $K^{\ominus}$ is the negative polar of $K$ defined by

$$
\begin{equation*}
K^{\ominus}:=\{y \in X \mid \forall x \in K:\langle x, y\rangle \leq 0\} \tag{101}
\end{equation*}
$$

The following result is a variant of [9, Lemma 2.1(v)].
Proposition 5.2. Let $C$ be a translation of an obtuse cone in $X$. Then

$$
\begin{equation*}
\forall \lambda \in\left[1,+\infty\left[, \quad \forall x \in X: \quad P_{C}^{\lambda} x \in C\right.\right. \tag{102}
\end{equation*}
$$

Consequently, $C$ is $\infty$-injectable.
Proof. By assumption, there exist a vector $c$ and an obtuse cone $K$ in $X$ such that $C=c+K$. First, we clearly have $C+K=c+K+K=c+K=C$ since $K$ is a convex cone.

Now let $x \in X$ and set $p=P_{C} x$, which is unique since $C$ is convex. It is easy to check that

$$
\begin{equation*}
p-x \in-N_{C}^{\text {prox }}(p) \subseteq-K^{\ominus} \subseteq K \tag{103}
\end{equation*}
$$

So, for every $\lambda \in[1,+\infty[$,

$$
\begin{equation*}
P_{C}^{\lambda} x=(1-\lambda) x+\lambda p=p+(\lambda-1)(p-x) \subseteq C+K=C . \tag{104}
\end{equation*}
$$

We therefore conclude that $C$ is $\infty$-injectable.
We now show that injectability is a generalization of the enlargement concept, which was first defined for convex sets in [7, Definition 2].
Definition 5.3 (enlargement of an arbitrary set). Given a nonempty closed subset $D$ of $X$ and $\tau \in \mathbb{R}_{+}$, the $\tau$-enlargement of $D$ is defined by the set

$$
\begin{equation*}
D_{[\tau]}:=\left\{x \in X \mid d_{D}(x) \leq \tau\right\}=D+\boldsymbol{B}(0 ; \tau) . \tag{105}
\end{equation*}
$$

It is clear that $D_{[0]}=D$ and $D_{[\tau]}$ is nonempty and closed.
Proposition 5.4. Let $\tau \in \mathbb{R}_{+}$. Then every $\tau$-enlargement is $2 \tau$-injectable. In particular, every ball with radius $\tau$ is $2 \tau$-injectable.

Proof. Let $C$ be a $\tau$-enlargement, say, $C=D+\boldsymbol{B}(0 ; \tau)$. Let $x \in X \backslash C$ and let $p \in P_{C} x$. There exists $q \in D$ such that $\|p-q\| \leq \tau$. It follows that $0<d_{C}(x) \leq\|x-q\| \leq\|x-p\|+\|p-q\| \leq d_{C}(x)+\tau$. We will show that the last two equalities happen, i.e.,

$$
\begin{equation*}
\|x-q\|=\|x-p\|+\|p-q\|=d_{C}(x)+\tau \tag{106}
\end{equation*}
$$

Suppose otherwise, then $\|x-q\|<d_{C}(x)+\tau$. Setting $z:=q+\tau \frac{x-q}{\|x-q\|}$, we have $z \in \boldsymbol{B}(q ; \tau) \subseteq C$ and $\|x-z\|=\|x-q\|-\tau<d_{C}(x)$, which is a contradiction. So (106) is true, which implies that $p$ lies in the segment $[x, q]$ and that $\|p-q\|=\tau$. From here, we derive that $p+2 \tau \frac{p-x}{\|p-x\|}=p+2 \tau \frac{q-p}{\|q-p\|}=$ $p+2(q-p)=2 q-p$ and also $[p-q, q-p] \subseteq \boldsymbol{B}(0 ; \tau)$. Hence,

$$
\begin{equation*}
\left[p, p+2 \tau \frac{p-x}{\|p-x\|}\right]=[p, 2 q-p]=q+[p-q, q-p] \subseteq q+\boldsymbol{B}(0 ; \tau) \subseteq C, \tag{107}
\end{equation*}
$$

and the conclusion follows.
Remark 5.5. The converse of Proposition 5.4 is not true. For example, consider a nontrivial obtuse cone $C$ in $\mathbb{R}^{2}$ that is strictly contained in a halfspace. Then, for every $\tau \in \mathbb{R}_{++}, C$ is $\tau$-injectable but is not a $\tau$-enlargement of any subset of $\mathbb{R}^{2}$.

Enlargements emerge in several applications. For example, the design problem in civil engineering discussed in [8] is modeled so that all constraints are represented in the form of enlargement sets. In this case, enlargements are exactly the original constraints of the feasibility problem. In general, one should not replace an original set by its enlargements since it may significantly change the solution of the feasibility problem. Yet there are certain cases where enlargements are actually useful. For instance, in [21], the image reconstruction problem is to solve a system of linear equations where constant coefficients may contain inevitable noise. Such systems may not have any exact solution. Therefore, it is reasonable to allow original equations to be only satisfied within a certain tolerance. This leads to a feasibility problem with enlargement sets. Here enlargements are replacements of the original constraints. In both examples, the injectability property is exploited to improve convergence.
Lemma 5.6. Let $C$ be a nonempty closed subset of $X$ and let $w \in C$. Suppose that $C$ is strictly injectable around $w$, i.e., there exist $\tau \in \mathbb{R}_{++}$and $\delta \in \mathbb{R}_{++}$such that $C$ is $\tau$-injectable on $\boldsymbol{B}(w ; \delta)$. Set $\delta^{\prime}:=\min \{\tau, \delta\}$. Then

$$
\begin{equation*}
\forall \lambda \in[1,2], \forall x \in \boldsymbol{B}\left(w ; \delta^{\prime}\right): \quad P_{C}^{\lambda} x \subseteq C . \tag{108}
\end{equation*}
$$

Proof. Let $\lambda \in[1,2]$, let $x \in \boldsymbol{B}\left(w ; \delta^{\prime}\right) \subseteq \boldsymbol{B}(w ; \delta)$, let $x_{+} \in P_{C}^{\lambda} x$, and write $x_{+}=(1-\lambda) x+\lambda p=$ $p+(\lambda-1)(p-x)$ for some $p \in P_{C} x$. Now assume that $p \neq x$, then $0<\|p-x\|=d_{C}(x) \leq\|x-w\| \leq$ $\delta^{\prime} \leq \tau$. Since $\lambda \in[1,2]$ we have $0 \leq \lambda-1 \leq 1 \leq \frac{\tau}{\|p-x\|}$. Combining with the $\tau$-injectability of $C$ on $\boldsymbol{B}(w ; \delta)$ yields

$$
\begin{equation*}
x_{+}=p+(\lambda-1)(p-x) \in\left[p, p+\tau \frac{p-x}{\|p-x\|}\right] \subseteq C, \tag{109}
\end{equation*}
$$

which finishes the proof.
We arrive at our main results on linear convergence of cyclic relaxed projections.
Theorem 5.7 (cyclic relaxed projections with at most one reflection). Let $w \in C:=$ $\bigcap_{i \in I} C_{i}, \varepsilon \in\left[0,1\left[\right.\right.$, and $\delta \in \mathbb{R}_{++}$. Suppose that
(a) $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$ for some $\kappa \in \mathbb{R}_{++}$.
(b) $\left\{C_{i}\right\}_{i \in I}$ is $(\varepsilon, \delta)$-regular at $w$.
(c) $\left.\left.\lambda_{i} \in\right] 0,2\right]$ for every $i \in I$ and there is at most one $\lambda_{j}$ equal to 2 with the corresponding set $C_{j}$ being $\tau$-injectable on $\boldsymbol{B}(w ; \delta / 2)$ for some $\tau \in \mathbb{R}_{++}$.
(d) Setting $\Gamma:=\prod_{i \in I}\left(1+\frac{\lambda_{i} \varepsilon}{1-\varepsilon}\right)^{1 / 2}, J:=\left\{j \in I \mid \lambda_{j}=2\right\}$, and $\nu:=\min _{i \in I \backslash J}\left\{1, \lambda_{i}\right\}$, it holds that

$$
\begin{equation*}
\rho:=\left[\Gamma^{2}-\frac{\nu^{2}}{\kappa^{2}}\left(\sum_{i \in I \backslash J} \frac{\lambda_{i}}{2-\lambda_{i}}\right)^{-1}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{|J|}\right]_{+}^{\frac{1}{2 m}}<1 . \tag{110}
\end{equation*}
$$

Then whenever the starting point is sufficiently close to $w$, the cyclic sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ generated by the relaxed projections $\left(P_{C_{i}}^{\lambda_{i}}\right)_{i \in I}$ converges $R$-linearly to a point $\bar{x} \in C$ with rate $\rho$. In particular, shrinking $\delta$ if necessary so that $\delta / 2 \leq \tau$, the linear convergence of $\left(x_{n}\right)_{n \in \mathbb{N}}$ is guaranteed provided that either $\left(x_{m n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \delta_{0}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$, where $\delta_{0}:=\frac{\delta}{2 \Gamma} \min _{i \in I}\left(1+\frac{\lambda_{i} \varepsilon}{1-\varepsilon}\right)^{1 / 2}$.

Proof. Set $\Omega:=C \cap \boldsymbol{B}(w ; \delta)$ and for every $i \in I$, set $\gamma_{i}:=1+\frac{\lambda_{i} \varepsilon}{1-\varepsilon}$ and $\beta_{i}:=\frac{2-\lambda_{i}}{\lambda_{i}}$. Then $\Gamma=\left(\gamma_{1} \cdots \gamma_{m}\right)^{1 / 2}$ and $\delta_{0}=\frac{\delta}{2 \Gamma} \min _{i \in I} \gamma_{i}^{1 / 2}$. On the one hand, for every $i \in I \backslash J$, Proposition 3.5 implies that $P_{C_{i}}^{\lambda_{i}}$ is $\left(C_{i} \cap \boldsymbol{B}(w ; \delta), \gamma_{i}, \beta_{i}\right)$-quasi firmly Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$. On the other hand, for every $i \in I$, Proposition 3.6 implies that $P_{C_{i}}^{\lambda_{i}}$ is $\left(C_{i}, \lambda_{i}\right)$ - and therefore $\left(C_{i}, \nu\right)$ - quasi coercive on $X$. We consider two cases.

Case 1: There is no $\lambda_{j}$ equal to 2, i.e, $J=\varnothing$. Noting that $\delta_{0} \leq \frac{\delta}{2 \Gamma} \gamma_{m}^{1 / 2}$, we then apply Theorem 4.5 to derive that if either $\left(x_{m n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \delta_{0}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$, the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly with rate

$$
\begin{equation*}
\rho=\left[\Gamma^{2}-\frac{\nu^{2}}{\kappa^{2}}\left(\sum_{i \in I} \frac{1}{\beta_{i}}\right)^{-1}\right]_{+}^{\frac{1}{2 m}}<1 . \tag{111}
\end{equation*}
$$

Case 2: There is only one $\lambda_{j}=2$, i.e, $J=\{j\}$. Using Lemma 5.6 and shrinking $\delta$ so that $\delta / 2 \leq \tau$, we have

$$
\begin{equation*}
\forall x \in \boldsymbol{B}(w ; \delta / 2): \quad P_{C_{j}}^{\lambda_{j}} x \subseteq C_{j} . \tag{112}
\end{equation*}
$$

It follows from Proposition 3.5 that $R_{C_{j}}=P_{C_{j}}^{\lambda_{j}}$ is $\left(C_{j}, \gamma_{j}\right)$-quasi Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$ with $\gamma_{j}=\frac{1+\varepsilon}{1-\varepsilon}$. Since $\delta_{0} \leq \frac{\delta}{2 \Gamma} \gamma_{j}^{1 / 2}$, Theorem 4.7 implies that if either $\left(x_{m n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \delta_{0}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta_{0}(1-\rho)}{2+\Gamma-\rho}\right)$, the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly with rate

$$
\begin{equation*}
\rho=\left[\Gamma^{2}-\frac{\nu^{2}}{\kappa^{2}}\left(\sum_{i \in I \backslash J} \frac{1}{\beta_{i}}\right)^{-1}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\right]_{+}^{\frac{1}{2 m}}<1 . \tag{113}
\end{equation*}
$$

Combining the two formulas for $\rho$, we obtain (110) and complete the proof.
In the following, we present the convergence result with refined linear rate for cyclic over-relaxed projections. In particular, if all sets are injectable, we will obtain linear reduction after every $m-1$ steps. Therefore, the upper bound for linear rate is reduced.
Theorem 5.8 (cyclic over-relaxed projections for injectable sets). Let $w \in C:=\bigcap_{i \in I} C_{i}$, $\varepsilon \in\left[0,1\left[, \delta \in \mathbb{R}_{++}\right.\right.$, and $\tau_{i} \in \mathbb{R}_{+}$for every $i \in I$. Suppose that
(a) $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$ for some $\kappa \in \mathbb{R}_{++}$.
(b) $\left\{C_{i}\right\}_{i \in I}$ is $(\varepsilon, \delta)$-regular at $w$.
(c) For every $i \in I, \lambda_{i} \in\left[1,2\left[\right.\right.$ and $C_{i}$ is $\tau_{i}$-injectable on $\boldsymbol{B}(w ; \delta / 2)$ with $\tau_{i}>0$ whenever $\lambda_{i}>1$.
(d) Setting $\Gamma:=\max _{j \in I} \prod_{i \in I \backslash\{j\}}\left(1+\frac{\lambda_{i} \varepsilon}{1-\varepsilon}\right)^{1 / 2}$, it holds that

$$
\begin{equation*}
\rho:=\left[\Gamma^{2}-\frac{1}{\kappa^{2}}\left(\left(\sum_{i \in I} \frac{\lambda_{i}}{2-\lambda_{i}}\right)-\min _{i \in I} \frac{\lambda_{i}}{2-\lambda_{i}}\right)^{-1}\right]_{+}^{\frac{1}{2(m-1)}}<1 \tag{114}
\end{equation*}
$$

Then whenever the starting point is sufficiently close to $w$, the cyclic sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ generated by the relaxed projections $\left(P_{C_{i}}^{\lambda_{i}}\right)_{i \in I}$ converges $R$-linearly to a point $\bar{x} \in C$ with rate $\rho$. In particular, shrinking $\delta$ if necessary so that $\delta / 2 \leq \min \left\{\tau_{i} \mid \lambda_{i}>1\right\}$, the linear convergence of $\left(x_{n}\right)_{n \in \mathbb{N}}$ is guaranteed provided that either $\left(x_{(m-1) n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}(w ; \bar{\delta})$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\bar{\delta}(1-\rho)}{2+\Gamma-\rho}\right)$, where $\bar{\delta}:=\frac{\delta}{2 \Gamma}\left(1+\max _{i \in I} \frac{\lambda_{i} \varepsilon}{1-\varepsilon}\right)^{-1 / 2}$.

Proof. We first shrink $\delta>0$ if necessary so that $\delta / 2 \leq \min \left\{\tau_{i} \mid \lambda_{i}>1\right\}$. For every $i \in I$, note that

$$
\begin{equation*}
\forall x \in \boldsymbol{B}(w ; \delta / 2): \quad P_{C_{i}}^{\lambda_{i}} x \subseteq C_{i} . \tag{115}
\end{equation*}
$$

Indeed, if $\lambda_{i}=1$ then (115) is automatic; and if $\lambda_{i}>1$, then (115) follows from Lemma 5.6.
Next, define $\gamma_{i}:=1+\frac{\lambda_{i} \varepsilon}{1-\varepsilon}$ and $\beta_{i}:=\frac{2-\lambda_{i}}{\lambda_{i}}$ for every $i \in I$. Then

$$
\begin{equation*}
\Gamma=\left(\frac{\gamma_{1} \cdots \gamma_{m}}{\min _{i \in I} \gamma_{i}}\right)^{1 / 2} \quad \text { and } \quad\left(\sum_{i \in I} \frac{\lambda_{i}}{2-\lambda_{i}}\right)-\min _{i \in I} \frac{\lambda_{i}}{2-\lambda_{i}}=\left(\sum_{i \in I} \frac{1}{\beta_{i}}\right)-\frac{1}{\max _{i \in I} \beta_{i}} . \tag{116}
\end{equation*}
$$

By Proposition 3.5, for every $i \in I, P_{C_{i}}^{\lambda_{i}}$ is $\left(C_{i} \cap \boldsymbol{B}(w ; \delta), \gamma_{i}, \beta_{i}\right)$-quasi firmly Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$. Now all assumptions in Theorem 4.8 are satisfied, hence the conclusion follows.
Remark 5.9 (refined linear rate). One can observe that, given the same constants $\varepsilon \in \mathbb{R}_{++}$, $\lambda_{i} \in\left[1,2\left[\right.\right.$ (which yields $\nu=1$ in Theorem 5.7), and $\kappa \in \mathbb{R}_{++}$, the (upper bound) rate $\rho$ in (114) is smaller than the one in (110). Thus, if all sets are injectable, we obtain a better upper bound for the linear rate.
Corollary 5.10 (refined linear convergence for cyclic projections). Let $w \in C:=\bigcap_{i \in I} C_{i}$, $\varepsilon \in\left[0,1\left[\right.\right.$, and $\delta \in \mathbb{R}_{++}$. Suppose that
(a) $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$ for some $\kappa \in \mathbb{R}_{++}$.
(b) $\left\{C_{i}\right\}_{i \in I}$ is $(\varepsilon, \delta)$-regular at $w$.
(c) It holds that

$$
\begin{equation*}
\rho:=\left[\frac{1}{(1-\varepsilon)^{m-1}}-\frac{1}{(m-1) \kappa^{2}}\right]_{+}^{\frac{1}{2(m-1)}}<1 . \tag{117}
\end{equation*}
$$

Then whenever the starting point is sufficiently close to $w$, the cyclic sequence generated by the classical projections $\left(P_{C_{i}}\right)_{i \in I}$ converges $R$-linearly to a point $\bar{x} \in C$ with rate $\rho$.

Proof. Apply Theorem 5.8 with $\lambda_{i}=1$ for every $i \in I$.
The next corollary shows that when $\left\{C_{i}\right\}_{i \in I}$ is a linearly regular system of superregular sets, the cyclic relaxed projections converge locally with linear rate.
Corollary 5.11 (cyclic relaxed projections for superregular sets). Let $w \in C:=\bigcap_{i \in I} C_{i}$ and let $\left.\left.\lambda_{i} \in\right] 0,2\right]$ for every $i \in I$, where there is at most one $\lambda_{i}$ equal to 2 with the corresponding $C_{i}$ being strictly injectable around $w$. Suppose that the system $\left\{C_{i}\right\}_{i \in I}$ is linearly regular around $w$ and superregular at $w$. Then when started at a point sufficiently close to $w$, the cyclic relaxed projection sequence generated by $\left(P_{C_{i}}^{\lambda_{i}}\right)_{i \in I}$ converges $R$-linearly to a point $\bar{x} \in C$.

Proof. Let $\varepsilon \in] 0,1\left[\right.$. By assumption, there exist $\kappa \in \mathbb{R}_{++}$and $\delta \in \mathbb{R}_{++}$such that $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$ and $(\varepsilon, \delta)$-regular at $w$. Borrowing notation from Theorem 5.7 and noting that $\frac{\beta \nu^{2}}{\kappa^{2}}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{|J|} \geq \frac{\beta \nu^{2}}{\kappa^{2}}>0$ and that $1+\frac{\lambda_{i} \varepsilon}{1-\varepsilon} \rightarrow 1^{+}$as $\varepsilon \rightarrow 0^{+}$, we choose $\varepsilon$ sufficiently small and shrink $\delta$ if necessary so that $\rho<1$. Finally, apply Theorem 5.7.

Now we turn our attention to the case of convexity in which global linear convergence is expected. Corollary 5.12 (global linear convergence of convex cyclic relaxed projections). Suppose that for every $i \in I, C_{i}$ is convex and that $\bigcap_{i \in I_{p}} C_{i} \cap \bigcap_{i \in I \backslash I_{p}}$ ri $C_{i} \neq \varnothing$, where $I_{p}:=$ $\left\{i \in I \mid C_{i}\right.$ is polyhedral $\}$. Let $\left.\left.\lambda_{i} \in\right] 0,2\right]$ for every $i \in I$ and suppose that there is at most one $\lambda_{i}$ equal to 2 with the corresponding $C_{i}$ being a translation of an obtuse cone in $X$. Then regardless of the starting point, the cyclic relaxed projection sequence generated by $\left(P_{C_{i}}^{\lambda_{i}}\right)_{i \in I}$ converges $R$-linearly to a point $\bar{x} \in C:=\bigcap_{i \in I} C_{i}$. In particular, for every starting point $x_{0} \in X$, the linear rate is

$$
\begin{equation*}
\rho:=\left[1-\frac{\nu^{2}}{\kappa^{2}}\left(\sum_{i \in I \backslash J} \frac{\lambda_{i}}{2-\lambda_{i}}\right)^{-1}\right]_{+}^{\frac{1}{2 m}} \tag{118}
\end{equation*}
$$

where $J:=\left\{i \in I \mid \lambda_{i}=2\right\}, \nu:=\min _{i \in I \backslash J}\left\{1, \lambda_{i}\right\}$, and $\kappa$ is a linear regularity modulus of $\left\{C_{i}\right\}_{i \in I}$ on $\boldsymbol{B}(w ; \delta / 2)$ for some $\delta \in \mathbb{R}_{++}$satisfying $\delta \geq 2 d_{C}\left(x_{0}\right)$.
Proof. Let $x_{0} \in X$, let $\delta \in \mathbb{R}_{++}$be such that $\delta \geq 2 d_{C}\left(x_{0}\right)$, and pick $w \in C$ such that $\delta \geq$ $2\left\|x_{0}-w\right\| \geq 2 d_{C}\left(x_{0}\right)$. Let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be the cyclic sequence generated by $\left(P_{C_{i}}^{\lambda_{i}}\right)_{i \in I}$ with starting point $x_{0}$. Employing [2, Corollary 5], there exists $\kappa \in \mathbb{R}_{++}$such that $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$. By convexity, $\left\{C_{i}\right\}_{i \in I}$ is $(0, \infty)$-regular at every point in $X$ (see [10, Remark $\left.8.2(\mathrm{v})\right]$ ), which combined with Proposition 3.5 implies that for every $i \in I, P_{C_{i}}^{\lambda_{i}}$ is $\left(C_{i}, 1\right)$-quasi Fejér monotone on $X$. In fact, $P_{C_{i}}^{\lambda_{i}}=\left(1-\frac{\lambda_{i}}{2}\right) \operatorname{Id}+\frac{\lambda_{i}}{2} R_{C_{i}}$ is even nonexpansive due to [3, Corollary 4.10 and Remark 4.24(i)].

By Proposition 5.2, the set $C_{i}$ corresponding to $\lambda_{i}=2$, if any, is $\infty$-injectable on $X$. We also see that $\rho<1$ and all assumptions in Theorem 5.7 are therefore satisfied with $\varepsilon=0$. Now since $x_{0} \in \boldsymbol{B}(w ; \delta / 2)$, Lemma 3.4(ii) and the $\left(C_{i}, 1\right)$-quasi Fejér monotonicity of $P_{C_{i}}^{\lambda_{i}}$ yield $\left(x_{n}\right)_{n \in \mathbb{N}} \subset$ $\boldsymbol{B}(w ; \delta / 2)$. Hence the proof is completed by applying Theorem 5.7.
Remark 5.13. When $\lambda_{1}=2, \lambda_{2}=\cdots=\lambda_{m}=1$ in Theorem 5.7 and Corollary 5.12, the cyclic relaxed projections is precisely the reflection-projection algorithm, whose global convergence was studied in [9] with the reflection across an obtuse cone. It is worth mentioning that our results are the first to conclude local and global $R$-linear convergence for the reflection-projection algorithm.

We finish this section by two examples showing that convergence may fail even in convex settings if there are more than one $\lambda_{i}$ equal to 2 or if the strict injectability of $C_{j}$ corresponding to $\lambda_{j}=2$ is violated.
Example 5.14 (failure of convergence when more than one $\lambda_{i}$ equal to 2). In $X=\mathbb{R}^{2}$, consider two convex sets $C_{1}=\mathbb{R}_{+}^{2}$ and $C_{2}=\left(-\mathbb{R}_{+}\right)^{2}$. Then $C_{1}$ and $C_{2}$ are obtuse cones and also polyhedral sets in $X$ with $C_{1} \cap C_{2}=\{(0,0)\} \neq \varnothing$, hence $\left\{C_{1}, C_{2}\right\}$ is linearly regular. It is easy to see that when started at a point $x_{0}=(\zeta, \xi) \in X \backslash\{(0,0)\}$, the sequence generated by $\left(R_{C_{1}}, R_{C_{2}}\right)$ does not converge since it cycles between two points $(|\zeta|,|\xi|)$ and $(-|\zeta|,-|\xi|)$.
Example 5.15 (failure of convergence if strict injectability is violated). Suppose $X=\mathbb{R}^{2}$, that $C_{1}=\mathbb{R} \times\{0\}$, and that $C_{2}=\{0\} \times \mathbb{R}$. Then $C_{1}$ and $C_{2}$ are polyhedral but not strictly injectable, and $C_{1} \cap C_{2}=\{(0,0)\} \neq \varnothing$. Take $x_{0}=(0, \xi)$ with $\xi \in \mathbb{R} \backslash\{0\}$, the sequence generated by $\left(R_{C_{1}}, P_{C_{2}}\right)$ cycles indefinitely between $x_{0}=(0, \xi), x_{1}=(0,-\xi), x_{2}=(0,-\xi)$ and $x_{3}=(0, \xi)$.

### 5.2 Cyclic semi-intrepid projections

Cyclic intrepid projections [7, 8] have found their applications in solving the feasibility problem (1), notably the road design problems [8]. The technique is to adjust the cyclic projections such
that for every projection $P_{C_{i}}$, one tries to be "more aggressive" by extrapolating into the set $C_{i}$ whenever possible. However, there is little incentive to "leave" the set $C_{i}$, therefore, the ratio is limited to which the extrapolated point remains within the set. This idea was first used in [21] for special polyhedra named "strips", i.e., intersections of two halfspaces with opposite normal vectors, see also [8, 22]; and was later generalized in [7] for enlargement sets. Motivated by this, we give the definition of semi-intrepid projectors.
Definition 5.16 (semi-intrepid projector to injectable sets). Let $\alpha \in[0,1]$, let $\tau \in \mathbb{R}_{+}$, let $C$ be a $\tau$-injectable set on a given set $U$ of $X$, and let $x \in X$. The $\alpha$-intrepid projection of $x$ into $C$ is defined by

$$
\begin{equation*}
P_{C}^{(\alpha, \tau)} x=\left\{\left.p+(p-x) \min \left\{\alpha, \frac{\tau}{\|p-x\|}\right\} \right\rvert\, p \in P_{C} x\right\} \tag{119}
\end{equation*}
$$

with the convention that $\frac{\tau}{\|p-x\|}=0$ if $p=x$.
We note that $P_{C}^{(0, \tau)}$ and $P_{C}^{(\alpha, 0)}$ are just the usual projector onto $C$ and that $P_{C}^{(1, \tau)}$ is the original intrepid projector [7, Definition 4], see also [8].
Proposition 5.17. Let $\tau \in \mathbb{R}_{+}$and let $C$ be a $\tau$-injectable set on a given set $U$ of $X$. Then

$$
\begin{equation*}
\forall \alpha \in[0,1], \quad \forall x \in U: \quad P_{C}^{(\alpha, \tau)} x \subseteq C \tag{120}
\end{equation*}
$$

Proof. The proof is straightforward from the definition.
Proposition 5.18 (quasi firm Fejér monotonicity of semi-intrepid operators). Let $\varepsilon \in \mathbb{R}_{+}$, $\delta \in \mathbb{R}_{++}, \tau \in \mathbb{R}_{+}$, and $\alpha \in[0,1]$. Let $C$ be a $\tau$-injectable set on $\boldsymbol{B}(w ; \delta / 2)$ and suppose that $C$ is $(\varepsilon, \delta)$-regular at $w \in C$. Then the semi-intrepid projector $P_{C}^{(\alpha, \tau)}$ is $\left(C \cap \boldsymbol{B}(w ; \delta), \frac{1+\alpha \varepsilon}{1-\varepsilon}, \frac{1-\alpha}{1+\alpha}\right)$-quasi firmly Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$.

Proof. Take $x \in \boldsymbol{B}(w ; \delta / 2)$ and $x_{+} \in P_{C}^{(\alpha, \tau)} x$. There exists $p \in P_{C} x$ such that

$$
\begin{equation*}
x_{+}=p+\alpha^{\prime}(p-x)=x+\left(1+\alpha^{\prime}\right)(p-x), \quad \text { where } \quad \alpha^{\prime}:=\min \left\{\alpha, \frac{\tau}{\|x-p\|}\right\} \tag{121}
\end{equation*}
$$

Then $x_{+}$is an image of the relaxed projection $P_{C}^{1+\alpha^{\prime}} x$ and, by Proposition 3.5,

$$
\begin{equation*}
\forall \bar{x} \in C \cap \boldsymbol{B}(w ; \delta): \quad\left\|x_{+}-\bar{x}\right\|^{2}+\frac{2-\left(1+\alpha^{\prime}\right)}{1+\alpha^{\prime}}\left\|x_{+}-x\right\|^{2} \leq\left(1+\frac{\left(1+\alpha^{\prime}\right) \varepsilon}{1-\varepsilon}\right)\|x-\bar{x}\|^{2} \tag{122}
\end{equation*}
$$

As $\alpha^{\prime} \leq \alpha$, one can check that $\frac{2-\left(1+\alpha^{\prime}\right)}{1+\alpha^{\prime}} \geq \frac{1-\alpha}{1+\alpha}$ and $1+\frac{\left(1+\alpha^{\prime}\right) \varepsilon}{1-\varepsilon} \leq \frac{1+\alpha \varepsilon}{1-\varepsilon}$. Hence,

$$
\begin{equation*}
\forall \bar{x} \in C \cap \boldsymbol{B}(w ; \delta): \quad\left\|x_{+}-\bar{x}\right\|^{2}+\frac{1-\alpha}{1+\alpha}\left\|x_{+}-x\right\|^{2} \leq \frac{1+\alpha \varepsilon}{1-\varepsilon}\|x-\bar{x}\|^{2}, \tag{123}
\end{equation*}
$$

and the proof is complete.
We now prove the $R$-linear convergence for the cyclic semi-intrepid projections, one of which is allowed to be the original intrepid projection $[7,8]$.
Theorem 5.19 (cyclic semi-intrepid projections). Let $\left.w \in C:=\bigcap_{i \in I} C_{i}, \varepsilon \in\right] 0,1[$, and $\delta \in \mathbb{R}_{++}$. For every $i \in I$, let $\tau_{i} \in \mathbb{R}_{+}$and $\alpha_{i} \in[0,1]$, where there is at most one $\alpha_{j}$ equal to 1 . Set $J:=\left\{j \in I \mid \alpha_{j}=1\right\}$ and

$$
\begin{equation*}
\Gamma:=\left(\frac{\gamma_{1} \cdots \gamma_{m}}{\min _{i \in I} \gamma_{i}^{1-|J|}}\right)^{\frac{1}{2}}, \text { where } \gamma_{i}:=\frac{1+\alpha_{i} \varepsilon}{1-\varepsilon} \text { for every } i \in I \tag{124}
\end{equation*}
$$

Suppose that
(a) $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$ for some $\kappa \in \mathbb{R}_{++}$.
(b) $\left\{C_{i}\right\}_{i \in I}$ is $(\varepsilon, \delta)$-regular at $w$.
(c) For every $i \in I, C_{i}$ is $\tau_{i}$-injectable on $\boldsymbol{B}(w ; \delta / 2)$.
(d) It holds that

$$
\begin{equation*}
\rho:=\left[\Gamma^{2}-\frac{1}{\kappa^{2}}\left(\sum_{i \in I \backslash J} \frac{1+\alpha_{i}}{1-\alpha_{i}}-(1-|J|) \min _{i \in I \backslash J} \frac{1+\alpha_{i}}{1-\alpha_{i}}\right)^{-1}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{|J|}\right]_{+}^{\frac{1}{2(m-1+|J|)}}<1 . \tag{125}
\end{equation*}
$$

Then whenever the starting point is sufficiently close to $w$, the cyclic sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ generated by semi-intrepid projections $\left(P_{C_{i}}^{\left(\alpha_{i}, \tau_{i}\right)}\right)_{i \in I}$ converges $R$-linearly to a point in $C$ with rate $\rho$. In particular, the linear convergence of $\left(x_{n}\right)_{n \in \mathbb{N}}$ is guaranteed provided that either $\left(x_{(m-1+|J|) n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \delta^{\prime}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta^{\prime}(1-\rho)}{2+\Gamma-\rho}\right)$, where $\delta^{\prime}:=\frac{\delta}{2 \Gamma} \min _{i \in I} \gamma_{i}^{1 / 2}\left(\max _{i \in I} \gamma_{i}^{1 / 2}\right)^{|J|-1}$.
Proof. According to Proposition 5.17, for every $i \in I$,

$$
\begin{equation*}
\forall x \in \boldsymbol{B}(w ; \delta / 2): \quad P_{C_{i}}^{\left(\alpha_{i}, \tau_{i}\right)} x \subseteq C_{i} \tag{126}
\end{equation*}
$$

and $P_{C_{i}}^{\left(\alpha_{i}, \tau_{i}\right)}$ is thus $\left(C_{i}, 1\right)$-quasi coercive on $\boldsymbol{B}(w ; \delta / 2)$. Next, we learn from Proposition 5.18 that, for $i \in I \backslash J, P_{C_{i}}^{\left(\alpha_{i}, \tau_{i}\right)}$ is $\left(C \cap \boldsymbol{B}(w ; \delta), \gamma_{i}, \frac{1-\alpha_{i}}{1+\alpha_{i}}\right)$-quasi firmly Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$ and that, for $j \in J, P_{C_{j}}^{\left(\alpha_{j}, \tau_{j}\right)}$ is $\left(C \cap \boldsymbol{B}(w ; \delta), \gamma_{j}\right)$-quasi Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$.

Case 1: $J=\{j\}$. In this case, $\Gamma=\left(\gamma_{1} \cdots \gamma_{m}\right)^{1 / 2}$ and $\delta^{\prime}=\frac{\delta}{2 \Gamma} \min _{i \in I} \gamma_{i}^{1 / 2} \leq \frac{\delta}{2 \Gamma} \gamma_{j}^{1 / 2}$. By Theorem 4.7, if either $\left(x_{m n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \delta^{\prime}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta^{\prime}(1-\rho)}{2+\Gamma-\rho}\right)$, then $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges with $R$-linear rate

$$
\begin{equation*}
\rho=\left[\Gamma^{2}-\frac{1}{\kappa^{2}}\left(\sum_{i \in I \backslash J} \frac{1+\alpha_{i}}{1-\alpha_{i}}\right)^{-1}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\right]_{+}^{\frac{1}{2 m}}<1 . \tag{127}
\end{equation*}
$$

Case 2: $J=\varnothing$. In this case, $\Gamma=\left(\frac{\gamma_{1} \cdots \gamma_{m}}{\min _{i \in I} \gamma_{i}}\right)^{1 / 2}$ and $\delta^{\prime}=\left(\max _{i \in I} \gamma_{i}\right)^{-1 / 2} \delta_{0}$ where $\delta_{0}:=\frac{\delta}{2 \Gamma}$. We get from Theorem 4.8 that if either $\left(x_{(m-1) n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}\left(w ; \delta^{\prime}\right)$ or $x_{0} \in \boldsymbol{B}\left(w ; \frac{\delta^{\prime}(1-\rho)}{2+\Gamma-\rho}\right)$, then $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges with $R$-linear rate

$$
\begin{equation*}
\rho=\left[\Gamma^{2}-\frac{1}{\kappa^{2}}\left(\left(\sum_{i \in I} \frac{1+\alpha_{i}}{1-\alpha_{i}}\right)-\min _{i \in I} \frac{1+\alpha_{i}}{1-\alpha_{i}}\right)^{-1}\right]_{+}^{\frac{1}{2(m-1)}}<1 . \tag{128}
\end{equation*}
$$

The result follows by combining two cases.
Corollary 5.20 (global linear convergence of convex cyclic semi-intrepid projections). Suppose that for every $i \in I, C_{i}$ is convex and that $\bigcap_{i \in I_{p}} C_{i} \cap \bigcap_{i \in I \backslash I_{p}}$ ri $C_{i} \neq \varnothing$, where $I_{p}:=$ $\left\{i \in I \mid C_{i}\right.$ is polyhedral $\}$. Suppose also that each $C_{i}$ is $\tau_{i}$-injectable for some $\tau_{i} \in \mathbb{R}_{+}$. Let $\alpha_{i} \in[0,1]$ for every $i \in I$ and assume there is at most one $\alpha_{j}$ equal to 1 . Then regardless of the starting point, the cyclic semi-intrepid projection sequence generated by $\left(P_{C_{i}}^{\left(\alpha_{i}, \tau_{i}\right)}\right)_{i \in I}$ converges $R$-linearly to a point $\bar{x} \in C:=\bigcap_{i \in I} C_{i}$. In particular, for every starting point $x_{0} \in X$, the linear rate is

$$
\begin{equation*}
\rho:=\left[1-\frac{1}{\kappa^{2}}\left(\left(\sum_{i \in I \backslash J} \frac{1+\alpha_{i}}{1-\alpha_{i}}\right)-(1-|J|) \min _{i \in I \backslash J} \frac{1+\alpha_{i}}{1-\alpha_{i}}\right)^{-1}\right]_{+}^{\frac{1}{2(m-1+\mid J)}} \tag{129}
\end{equation*}
$$

where $J:=\left\{i \in I \mid \alpha_{i}=1\right\}$ and $\kappa$ is a linear regularity modulus of $\left\{C_{i}\right\}_{i \in I}$ on $\boldsymbol{B}(w ; \delta / 2)$ for some $\delta \in \mathbb{R}_{++}$satisfying $\delta \geq 2 d_{C}\left(x_{0}\right)$.

Proof. Take $x_{0} \in X, \delta \geq 2 d_{C}\left(x_{0}\right)$, and choose $w \in C$ such that $\delta \geq 2\left\|x_{0}-w\right\| \geq 2 d_{C}\left(x_{0}\right)$. Then $x_{0} \in \boldsymbol{B}(w ; \delta / 2)$. Let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be the cyclic sequence generated by $\left(P_{C_{i}}^{\left(\alpha_{i}, \tau_{i}\right)}\right)_{i \in I}$ with starting point $x_{0}$. We observe from [2, Corollary 5] that $\left\{C_{i}\right\}_{i \in I}$ is $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$ for some $\kappa \in \mathbb{R}_{++}$ and from [10, Remark $8.2(\mathrm{v})]$ that $\left\{C_{i}\right\}_{i \in I}$ is $(0, \infty)$-regular at every point in $X$ (due to convexity). Note that $\rho<1$ in (129), so all assumptions in Theorem 5.19 are fulfilled with $\varepsilon=0$.

Next, since $\left\{C_{i}\right\}_{i \in I}$ is $(0, \infty)$-regular at every point in $X$, Proposition 5.18 implies that, for every $i \in I, P_{C_{i}}^{\left(\alpha_{i}, \lambda_{i}\right)}$ is ( $C_{i}, 1$-quasi Fejér monotone on $X$. Combining with $x_{0} \in \boldsymbol{B}(w ; \delta / 2)$ and Lemma 3.4(ii) gives $\left(x_{n}\right)_{n \in \mathbb{N}} \subset \boldsymbol{B}(w ; \delta / 2)$. Now apply Theorem 5.19.

### 5.3 Cyclic generalized Douglas-Rachford algorithm

In this section, we work with the index set $J:=\{1, \ldots, \ell\}$, where $\ell$ is a positive integer. For every $j \in J$, let $\left.\left.\lambda_{j}, \mu_{j} \in\right] 0,2\right]$, let $\left.\alpha_{j} \in\right] 0,1\left[\right.$, and let $s_{j}, t_{j} \in I$ such that $s_{j} \neq t_{j}$ and that

$$
\begin{equation*}
\left\{s_{j} \mid j \in J\right\} \cup\left\{t_{j} \mid j \in J\right\}=I . \tag{130}
\end{equation*}
$$

We consider the cyclic generalized Douglas-Rachford algorithm defined by $\left(T_{j}\right)_{j \in J}$, where

$$
\begin{equation*}
\forall j \in J: \quad T_{j}:=\left(1-\alpha_{j}\right) \operatorname{Id}+\alpha_{j} P_{C_{t_{j}}}^{\mu_{j}} P_{C_{s_{j}}}^{\lambda_{j}}, \tag{131}
\end{equation*}
$$

and shall prove that this algorithm also possesses $R$-linear convergence properties. It is worth noting that if each $T_{j}$ is a classical DR operator (i.e., $\alpha_{j}=1 / 2, \lambda_{j}=\mu_{j}=2$ ), then the cyclic generalized DR algorithm is the multiple-sets $D R$ algorithm [15]. The latter reduces to the cyclic $D R$ algorithm [14] when $\ell=m,\left(s_{j}, t_{j}\right)=(j, j+1)$ for $j=1, \ldots, m-1$, and $\left(s_{m}, t_{m}\right)=(m, 1)$; and to the cyclically anchored $D R$ algorithm [12] when $\ell=m-1,\left(s_{j}, t_{j}\right)=(1, j+1)$ for $j=1, \ldots, m-1$.
Theorem 5.21 (cyclic generalized DR algorithm). Let $w \in C:=\bigcap_{i \in I} C_{i}$. Suppose that the system $\left\{C_{i}\right\}_{i \in I}$ is superregular at $w$ and linearly regular around $w$ and that $\left\{C_{s_{j}}, C_{t_{j}}\right\}$ is strongly regular at $w$ for every $j \in J$. Then when started at a point sufficiently close to $w$, the cyclic generalized $D R$ sequence generated by $\left(T_{j}\right)_{j \in J}$ converges $R$-linearly to a point $\bar{x} \in C$.

Proof. Let $j \in J$ and let $\varepsilon \in] 0,1 / 3]$. Since $\left\{C_{i}\right\}_{i \in I}$ is superregular at $w$, there exists $\delta \in \mathbb{R}_{++}$such that $C_{i}$ is $(\varepsilon, \sqrt{2} \delta)$-regular at $w$ for every $i \in I$. Then $C_{s_{j}}$ and $C_{t_{j}}$ are $(\varepsilon, \delta)$ - and $(\varepsilon, \sqrt{2} \delta)$-regular at $w$, respectively. Using Proposition 3.7, $T_{j}$ is $\left(C_{s_{j}} \cap C_{t_{j}}, \gamma_{j}, \frac{1-\alpha_{j}}{\alpha_{j}}\right)$-quasi firmly Fejér monotone on $\boldsymbol{B}(w ; \delta / 2)$, where

$$
\begin{equation*}
\gamma_{j}:=1-\alpha_{j}+\alpha_{j}\left(1+\frac{\lambda_{j} \varepsilon}{1-\varepsilon}\right)\left(1+\frac{\mu_{j} \varepsilon}{1-\varepsilon}\right) \rightarrow 1^{+} \quad \text { as } \quad \varepsilon \rightarrow 0^{+} . \tag{132}
\end{equation*}
$$

Shrinking $\delta$ if necessary, we derive from Proposition 3.8 that $T_{j}$ is $\left(C_{s_{j}} \cap C_{t_{j}}, \nu_{j}\right)$-quasi coercive on $\boldsymbol{B}(w ; \delta / 2)$, where

$$
\begin{equation*}
\left.\nu_{j}:=\frac{\alpha_{j} \sqrt{1-\theta_{j}}}{\kappa_{j}} \min \left\{\lambda_{j}, \frac{\mu_{j}}{\sqrt{1+\mu_{j}^{2}}}\right\} \quad \text { for some } \kappa_{j} \in \mathbb{R}_{++} \text {and } \theta_{j} \in\right] 0,1[. \tag{133}
\end{equation*}
$$

Now by the linear regularity of $\left\{C_{i}\right\}_{i \in I}$, we again shrink $\delta$ if necessary and find $\kappa \in \mathbb{R}_{++}$such that

$$
\begin{equation*}
\forall x \in \boldsymbol{B}(w ; \delta / 2): \quad d_{C}(x) \leq \kappa \max _{i \in I} d_{C_{i}}(x) \tag{134}
\end{equation*}
$$

Since $C_{s_{j}} \cap C_{t_{j}} \subseteq C_{s_{j}}$ and $C_{s_{j}} \cap C_{t_{j}} \subseteq C_{t_{j}}$,

$$
\begin{equation*}
\forall j \in J, \forall x \in X: \quad \max \left\{d_{C_{s_{j}}}(x), d_{C_{t_{j}}}(x)\right\} \leq d_{C_{s_{j}} \cap C_{t_{j}}}(x) . \tag{135}
\end{equation*}
$$

Noting also from (130) that

$$
\begin{equation*}
\bigcap_{j \in J}\left(C_{s_{j}} \cap C_{t_{j}}\right)=\bigcap_{i \in I} C_{i}=C, \tag{136}
\end{equation*}
$$

we conclude that the system $\left\{C_{s_{j}} \cap C_{t_{j}}\right\}_{j \in J}$ is also $\kappa$-linearly regular on $\boldsymbol{B}(w ; \delta / 2)$.
Finally, set $\nu:=\min _{j \in J}\left\{1, \nu_{j}\right\}$. Due to (132), we can choose $\varepsilon$ sufficiently small so that

$$
\begin{equation*}
\rho:=\left[\gamma_{1} \cdots \gamma_{m}-\frac{\nu^{2}}{\kappa^{2}}\left(\sum_{j \in J} \frac{\alpha_{j}}{1-\alpha_{j}}\right)^{-1}\right]_{+}^{1 / 2}<1 . \tag{137}
\end{equation*}
$$

Thus, applying Theorem 4.5 to $\left(T_{j}\right)_{j \in J}$ and the corresponding sets $\left(C_{s_{j}} \cap C_{t_{j}}\right)_{j \in J}$, we obtain the $R$-linear convergence.

We recall from Remark 2.6 that the linear regularity of a system together with the strong regularity of its subsystems are less restrictive than the strong regularity of that system. This observation supports the use of our separate assumptions on linear regularity and strong regularity in Theorem 5.21.

In the case $m=2$, we obtain a generalization of [33, Theorem 4.3] which proves $R$-linear convergence of the classical DR algorithm for two sets. In fact, the classical DR algorithm also converges $R$-linearly in other settings where cyclic projections may not, more details can be found in $[4,5,6]$.
Corollary 5.22 (generalized DR algorithm). Let $A$ and $B$ be closed subsets of $X$ and $w \in A \cap B$. Let $\lambda, \mu \in] 0,2], \alpha \in] 0,1[$, and set

$$
\begin{equation*}
T:=(1-\alpha) \operatorname{Id}+\alpha P_{B}^{\mu} P_{A}^{\lambda} . \tag{138}
\end{equation*}
$$

Suppose that the system $\{A, B\}$ is superregular and strongly regular at $w$. Then when started at a point sufficiently close to $w$, the generalized $D R$ sequence generated by $T$ converges $R$-linearly to $a$ point $\bar{x} \in A \cap B$.

Proof. Note that strong regularity implies linear regularity (see Fact 2.5) and apply Theorem 5.21 with $m=2, \ell=1$, and $\left(s_{1}, t_{1}\right)=(1,2)$.

### 5.4 Affine reduction for generalized Douglas-Rachford sequences

In this section, we extend the affine reduction scheme in [33, Section 3] to generalized DouglasRachford sequences. Let $A$ and $B$ be nonempty closed subsets of $X$. For every $n \in \mathbb{N}$, let $\lambda_{n}, \mu_{n} \in$ $] 0,2]$, and $\left.\alpha_{n} \in\right] 0,1[$. A generalized DR sequence is given by

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad x_{n+1} \in\left(1-\alpha_{n}\right) x_{n}+\alpha_{n} P_{B}^{\mu_{n}} P_{A}^{\lambda_{n}} x_{n} . \tag{139}
\end{equation*}
$$

We start with the following extension of [33, Lemma 3.1] whose elementary proof is omitted.

Lemma 5.23. Let $C$ be a nonempty closed subset of $X$, let $L$ be an affine subspace of $X$ containing $C$, and let $\lambda \in \mathbb{R}_{+}$. Then the following hold:
(i) $\left(\operatorname{Id}-P_{L}\right) P_{C}^{\lambda}=(1-\lambda)\left(\operatorname{Id}-P_{L}\right)$.
(ii) $P_{L} P_{C}^{\lambda}=P_{C}^{\lambda} P_{L}$.

The idea behind affine reduction for DR is to show that the shadow of any generalized DR sequence on a certain affine subspace is again a generalized DR sequence. The next lemma provides more details.
Lemma 5.24 (shadows of generalized DR sequences). Let $L$ be an affine subspace of $X$ containing $A \cup B$ and define $y_{n}:=P_{L} x_{n}$ for $n \in \mathbb{N}$. Then the following hold:
(i) $\forall n \in \mathbb{N}: y_{n+1} \in\left(1-\alpha_{n}\right) y_{n}+\alpha_{n} P_{B}^{\mu_{n}} P_{A}^{\lambda_{n}} y_{n}$, i.e., $\left(y_{n}\right)_{n \in \mathbb{N}}$ is also a generalized $D R$ sequence.
(ii) $\forall n \in \mathbb{N}: x_{n+1}-y_{n+1}=\left(\left(1-\alpha_{n}\right)+\alpha_{n}\left(1-\lambda_{n}\right)\left(1-\mu_{n}\right)\right)\left(x_{n}-y_{n}\right)$.

Proof. Let $n \in \mathbb{N}$.
(i): Then there exist $r_{n} \in P_{A}^{\lambda_{n}} x_{n}$ and $s_{n} \in P_{B}^{\mu_{n}} r_{n}$ such that $x_{n+1}=\left(1-\alpha_{n}\right) x_{n}+\alpha_{n} s_{n}$. By Lemma 5.23(ii), $P_{L} s_{n} \in P_{L} P_{B}^{\mu_{n}} P_{A}^{\lambda_{n}} x_{n}=P_{B}^{\mu_{n}} P_{A}^{\lambda_{n}} P_{L} x_{n}=P_{B}^{\mu_{n}} P_{A}^{\lambda_{n}} y_{n}$. Since $P_{L}$ is an affine operator (see [3, Corollary $3.20(\mathrm{ii})]$ ), it follows that

$$
\begin{align*}
y_{n+1}=P_{L} x_{n+1}=P_{L}\left(\left(1-\alpha_{n}\right) x_{n}+\alpha_{n} s_{n}\right) & =\left(1-\alpha_{n}\right) P_{L} x_{n}+\alpha_{n} P_{L} s_{n}  \tag{140a}\\
& \in\left(1-\alpha_{n}\right) y_{n}+\alpha_{n} P_{B}^{\mu_{n}} P_{A}^{\lambda_{n}} y_{n} . \tag{140b}
\end{align*}
$$

Hence, $\left(y_{n}\right)_{n \in \mathbb{N}}$ is a generalized DR sequence starting at $y_{0}$.
(ii): Using Lemma 5.23(i), we have

$$
\begin{align*}
s_{n}-P_{L} s_{n}=\left(1-\mu_{n}\right)\left(r_{n}-P_{L} r_{n}\right) & =\left(1-\mu_{n}\right)\left(1-\lambda_{n}\right)\left(x_{n}-P_{L} x_{n}\right)  \tag{141a}\\
& =\left(1-\lambda_{n}\right)\left(1-\mu_{n}\right)\left(x_{n}-y_{n}\right), \tag{141b}
\end{align*}
$$

which implies that

$$
\begin{align*}
x_{n+1}-y_{n+1} & =\left[\left(1-\alpha_{n}\right) x_{n}+\alpha_{n} s_{n}\right]-\left[\left(1-\alpha_{n}\right) P_{L} x_{n}+\alpha_{n} P_{L} s_{n}\right]  \tag{142a}\\
& =\left(1-\alpha_{n}\right)\left(x_{n}-P_{L} x_{n}\right)+\alpha_{n}\left(s_{n}-P_{L} s_{n}\right)  \tag{142b}\\
& =\left(\left(1-\alpha_{n}\right)+\alpha_{n}\left(1-\lambda_{n}\right)\left(1-\mu_{n}\right)\right)\left(x_{n}-y_{n}\right) . \tag{142c}
\end{align*}
$$

The proof is complete.
In Corollary 5.22 , strong regularity of $\{A, B\}$ at $w \in A \cap B$ is not the most general condition for $R$-linear convergence of DR sequences. Indeed, it can be relaxed to affine-hull regularity in the sense that

$$
\begin{equation*}
N_{A}(w) \cap\left(-N_{B}(w)\right) \cap(L-w)=\{0\} \quad \text { with } \quad L:=\operatorname{aff}(A \cup B) . \tag{143}
\end{equation*}
$$

This condition has been observed in [33, Theorem 4.7] for the classical DR sequence ( $\lambda=\mu=2$ and $\alpha=1 / 2$ ). We now continue extending such result for generalized DR sequences. For simplicity of presentation, we consider only the case of constant parameters $\left(\lambda_{n}, \mu_{n}, \alpha_{n}\right) \equiv(\lambda, \mu, \alpha)$.
Theorem 5.25 (affine reduction for generalized DR sequences). Let $A$ and $B$ be closed subsets of $X$ such that $A \cap B \neq \varnothing, w \in A \cap B$, and $L:=\operatorname{aff}(A \cup B)$. Suppose that $\{A, B\}$ is superregular and affine-hull regular at $w$. Let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a generalized $D R$ sequence generated by $T:=(1-\alpha) \operatorname{Id}+P_{B}^{\mu} P_{A}^{\lambda}$ with $\left.\left.\lambda, \mu \in\right] 0,2\right]$ and $\left.\alpha \in\right] 0,1[$. Then the following hold:
(i) If $\lambda=\mu=2$, then, whenever $P_{L} x_{0}$ is sufficiently close to $w$, the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to a point $\bar{x} \in \operatorname{Fix} T$ with $P_{A} \bar{x}=P_{B} \bar{x} \in A \cap B$.
(ii) If either $\lambda<2$ or $\mu<2$, then, whenever $P_{L} x_{0}$ is sufficiently close to $w$, the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to a point $\bar{x} \in A \cap B$.

Proof. Define $y_{n}:=P_{L} x_{n}$ for $n \in \mathbb{N}$. By Lemma $5.24(\mathrm{i}),\left(y_{n}\right)_{n \in \mathbb{N}} \subset L$ is also a generalized DR sequence generalized by $T$. By restricting our consideration within the affine subspace $L$, affine-hull regularity (143) becomes strong regularity of $\{A, B\}$ within $L$. Thus, Corollary 5.22 yields that $\left(y_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to a point $\bar{y} \in A \cap B$ when $P_{L} x_{0}=y_{0}$ is sufficiently close to $w$.

Setting $\eta:=(1-\alpha)+\alpha(1-\lambda)(1-\mu)$, we have from Lemma $5.24(\mathrm{ii})$ that

$$
\begin{equation*}
\forall n \in \mathbb{N}: \quad x_{n}-y_{n}=\eta^{n}\left(x_{0}-y_{0}\right) \tag{144}
\end{equation*}
$$

(i): Assume $\lambda=\mu=2$. Then $\eta=1$ and, since $\left(y_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to $\bar{y} \in A \cap B$, (144) implies that $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges $R$-linearly to $\bar{x}:=\bar{y}+\left(x_{0}-y_{0}\right)$. Now by [3, Corollary 3.20(i)], $x_{0}-y_{0} \in(L-L)^{\perp}=(L-w)^{\perp}$, and by [10, Lemma 3.2], $P_{A} \bar{x}=P_{A} \bar{y}=\bar{y}$ and $P_{B} R_{A} \bar{x}=P_{B}(2 \bar{y}-\bar{x})=$ $P_{B}\left(\bar{y}+y_{0}-x_{0}\right)=P_{B} \bar{y}=\bar{y}$. It follows that $R_{B} R_{A} \bar{x}=\bar{x}$, which yields $T \bar{x}=(1-\alpha) \bar{x}+\alpha R_{B} R_{A} \bar{x}=\bar{x}$ and so $\bar{x} \in \operatorname{Fix} T$.
(ii): Assume either $\lambda<2$ or $\mu<2$. Then $\eta<1$ and, by (144), $x_{n}-y_{n}$ converges $R$-linearly to 0 . Hence, $x_{n}=y_{n}+\left(x_{n}-y_{n}\right)$ converges $R$-linearly to $\bar{x}=\bar{y} \in A \cap B$.
Remark 5.26. Theorem 5.25 (ii) has never been explored before even in convex settings where one would obtain global $R$-linear convergence to the intersection; while Theorem 5.25 (i) was proved in [33, Theorem 4.7] for the classical DR algorithm. With some care on the parameters, Theorem 5.25 can certainly be extended to the case of generalized DR iterations of the form (139) with $1<$ $\inf _{n \in \mathbb{N}}\left\{\lambda_{n}, \mu_{n}\right\} \leq \sup _{n \in \mathbb{N}}\left\{\lambda_{n}, \mu_{n}\right\} \leq 2$ and $0<\inf _{n \in \mathbb{N}} \alpha_{n} \leq \sup _{n \in \mathbb{N}} \alpha_{n}<1$.

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